Adaptive algorithms for the numerical simulation of slow-faster oscillator networks

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Mathematical context

Slow-fast (SF) systems

\[ f(u, v, \sigma) = u - \varkappa u + \varkappa v + \sigma, \quad \dot{\sigma} = \gamma(u) \]

- slow (S) variables: \( u \rightarrow \mathbb{R}^N \); fast variables: \( v \rightarrow \mathbb{R}^N \); slow dynamics; \( \varkappa \) and \( \gamma \) assumed to be smooth.
- While \((u,v)\) lies in a \((\varkappa,\gamma)\) neighborhood of \((f(0,0),0)\), both \(u\) and \(v\) evolve slowly (\(O(\varkappa)\) speed).
- \((u,v)\) fast dynamics; \((f(0,0),0)\) speed.

Example: FitzHugh-Nagumo (FHN)

\[ \dot{x} = -y + x^2 - \lambda x \quad \dot{y} = \varepsilon (ux + ny + au) \]

Efficient adaptive time step for numerical simulations:

\[ \min\left(\Delta t_{\text{adapt}} \right) = \frac{\Delta t_{\min}}{f(x,u,v)} \]

Network of SF oscillators driven by global variables

- Fast variable \( x_j \): electrical activity;
- Slow variable \( y_j \): ionic activity;
- Slow variable \( Ca_j \): intracellular calcium level

Global variable \( \sigma \) can display slow and fast motions too.

Biological application

Intracellular calcium oscillations in GnRH neurons
Neuronal population located in the hypothalamus and responsible for the pulsatile secretion of GnRH (Gonadotropin Releasing Hormone) which drives the endocrine control of the reproductive function in mammals.

Pulsatile oscillations of intracellular calcium level \( i \):
- Proper amplitude and pulse frequency in each cell resulting in asynchrony between cells;
- Recurrent synchronization episodes with regular frequency.

Mathematical model (see [2])

Population of \( N \) cells.
For each cell \( j \in [1,N] \), model based on FHN:

- Fast variable \( s_j \): electrical activity;
- Slow variable \( x_j \): ionic activity;
- Slow variable \( Ca_j \): intracellular calcium level

Global variable \( \sigma \) representing the state of the network:
- Increases very slowly during the asynchronous phase;
- \( \sigma = \sigma_n \) change in the \( s_j \) dynamics: synchronization.
- Quickly decreases when the mean calcium level \( \sigma \) among cells is above a threshold.

Slow-fast splitting At a given point \((X,\sigma)\) of system (1) phase space, using the evaluations of \( f(X,\sigma) \), we distinguish the \( X \) cells in slow motion and the \( \sigma \) cells in fast motions:

- \( \sigma \) : \((m+1)N \) variables \( \sigma_j,Ca_j \) of cells in fast motion,
- \( X \) : \((m+1)N \) variables \( x_j,y_j,Ca_j \) of cells in slow motion.

With these new variables, system (1) writes:

\[ X_j = f_j(X,\sigma) \quad \text{with} \quad f_j(X,\sigma) = O(1) \]

\[ \sigma = f_\sigma(X,\sigma) \quad \text{with} \quad f_\sigma(X,\sigma) = O(\varkappa) \]

Associated differential operator:

\[ L = \frac{\partial}{\partial \sigma} \quad L_\sigma = \frac{\partial}{\partial \sigma} \quad \text{with} \quad L \sigma \frac{\partial}{\partial \sigma} \]

Decomposition of the vector field \((X,\sigma)\) and splitting:

- If \( \sigma \) is slow, i.e. \( H(X,\sigma) = O(\varkappa) \):

\[ S_{\text{BAS}} \sigma \quad L_\sigma = \frac{\partial}{\partial \sigma} \quad L = \frac{\partial}{\partial \sigma} \]

- If \( \sigma \) is fast, i.e. \( H(X,\sigma) = O(1) \):

\[ S_{\text{BAS}} \lambda \quad L_\sigma = \frac{\partial}{\partial \sigma} \quad L = \frac{\partial}{\partial \sigma} \]

Note: the splitting depends on which cells are in slow and fast motions at a given point.

Adaptive scheme

One step algorithm
To optimize the performances of the slow-fast splitting algorithm applied to model (2), we use RK4 on the coarse grid and RK2 on the fine grid as elementary scheme.

Consistency and performance

Model (2), \( N = 20, \varepsilon = 0.01 \)

References

