A geometric mechanism for mixed-mode bursting oscillations in a hybrid neuron model

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A geometric mechanism for MM(B)O

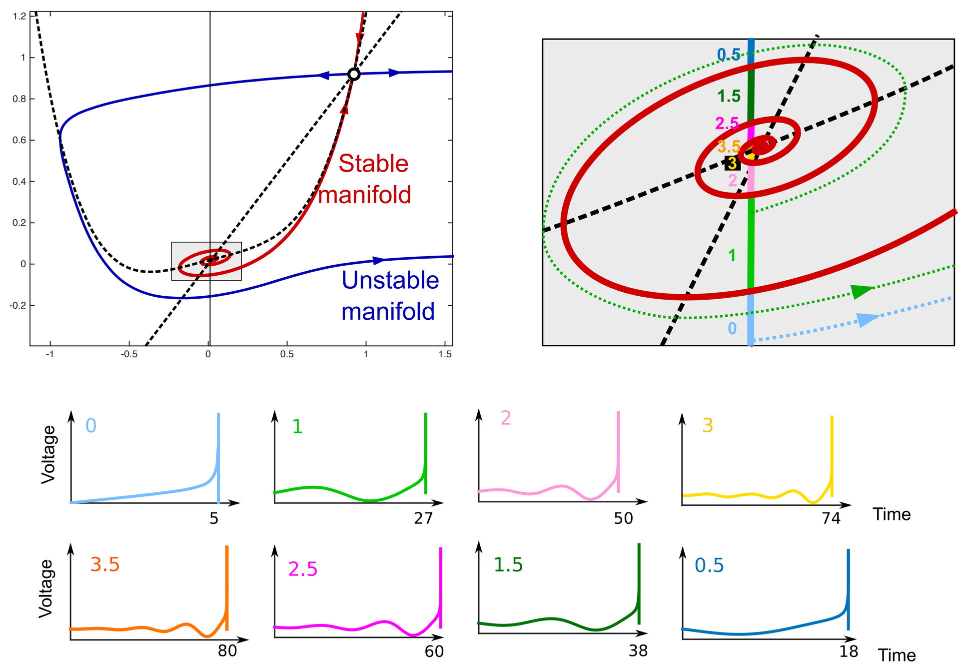
Nonlinear integrate-and-fire neuron models

Reset mechanism:

$$\begin{cases}\frac{dv}{dt} = F(v) - w + I\\ \frac{dw}{dt} = a(bv - w)\end{cases} \qquad v(t) \xrightarrow[t\to t^*]{} \infty \implies \begin{cases} v(t^*) = v_r\\ w(t^*) = \gamma\, w(t^{*-}) + d\end{cases}$$

- $a,\ b,\ I \in \mathbb{R}$: parameters of the vector field (I is the input current received by the neuron);
- $F \in \mathcal{C}^3(\mathbb{R})$: strictly convex, $\lim_{v\to-\infty} F'(v) < 0$, $\lim_{v\to\infty} F'(v) = \infty$ and $\lim_{v\to\infty} F(v)/v^{2+\varepsilon} \geq \alpha$.; in the simulations we consider the so-called quartic model with model $F(v) = v^4 + 2cv$ (with $c = 0.1$)
- $d > 0$, $\gamma \in (0, 1]$: parameters connected with the reset mechanism

The model can display complex dynamics including **Mixed-Mode Oscillations** and **Mixed-Mode Bursting Oscillations** (MM(B)O) that are sequences of spikes interspersed by small subthreshold oscillations.



The reset line intersects the stable manifold separating regions corresponding to a specific number of small oscillations (from 0 to 3.5). Parameters: quartic model with $a = 0.1$, $b = 1$, $I = 0.175$; initial conditions $v = 0.012$ and w chosen within the different intervals

Adaptation map

A spike train for a spiking solution $(V(t; v_r, w), W(t; v_r, w))$ starting with (v_r, w) at t_0 can be recovered via the sequence $\{w_n\}_{n\in\mathbb{N}}$ of the values of the adaptation variable exactly after the moment of the n-th spike, i.e. if $\{t_n\}_{n\in\mathbb{N}}$ is the sequence of spike times for this solution, then

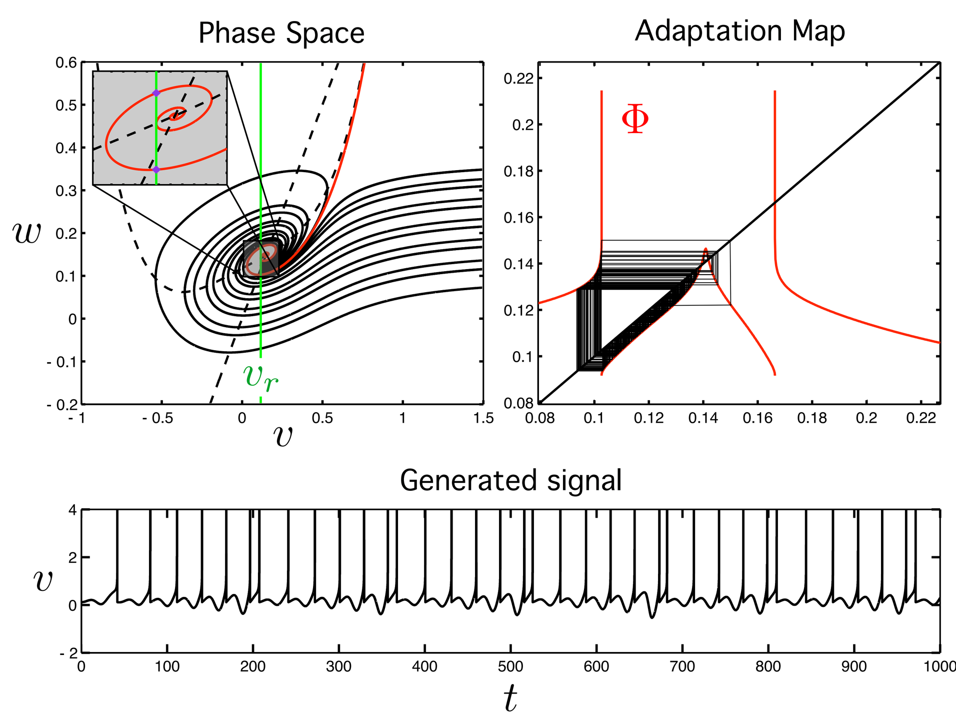
$$w_n := w(t_n) = \gamma w(t_n^-) + d.$$

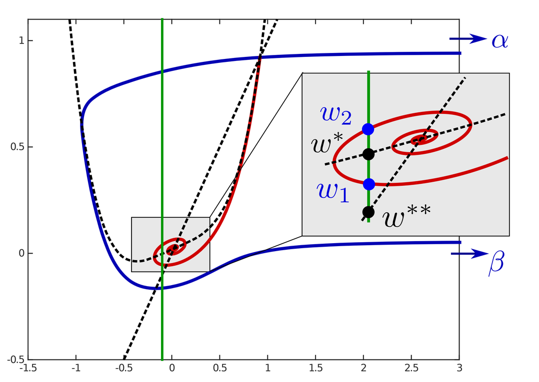
Definition: Adaptation map

The adaptation map Φ associates to a value of the adaptation variable w the value of the adaptation variable after reset:

$$\Phi(w) := \gamma W(t^*; v_r, w) + d,$$

Since $w_n = \Phi^n(t_0)$, the spike train can be qualitatively described via the dynamics of Φ, with fixed points of Φ corresponding to tonic, regular spiking and periodic orbits to bursts.





- $(w_i)_{i=1\cdots p}$ - intersections of the reset line $\{v = v_r\}$ with SMSFP
- p_1- the index such that $(w_i)_{i\leq p_1}$ are below the v-nullcline and $(w_i)_{i>p_1}$ are above
- $(I_i)_{i=0\cdots p+1}$ - intervals with endpoints w_i
- α, β - the value of w after a spike for an initial condition on the upper and, respectively, lower branch of UMSFP

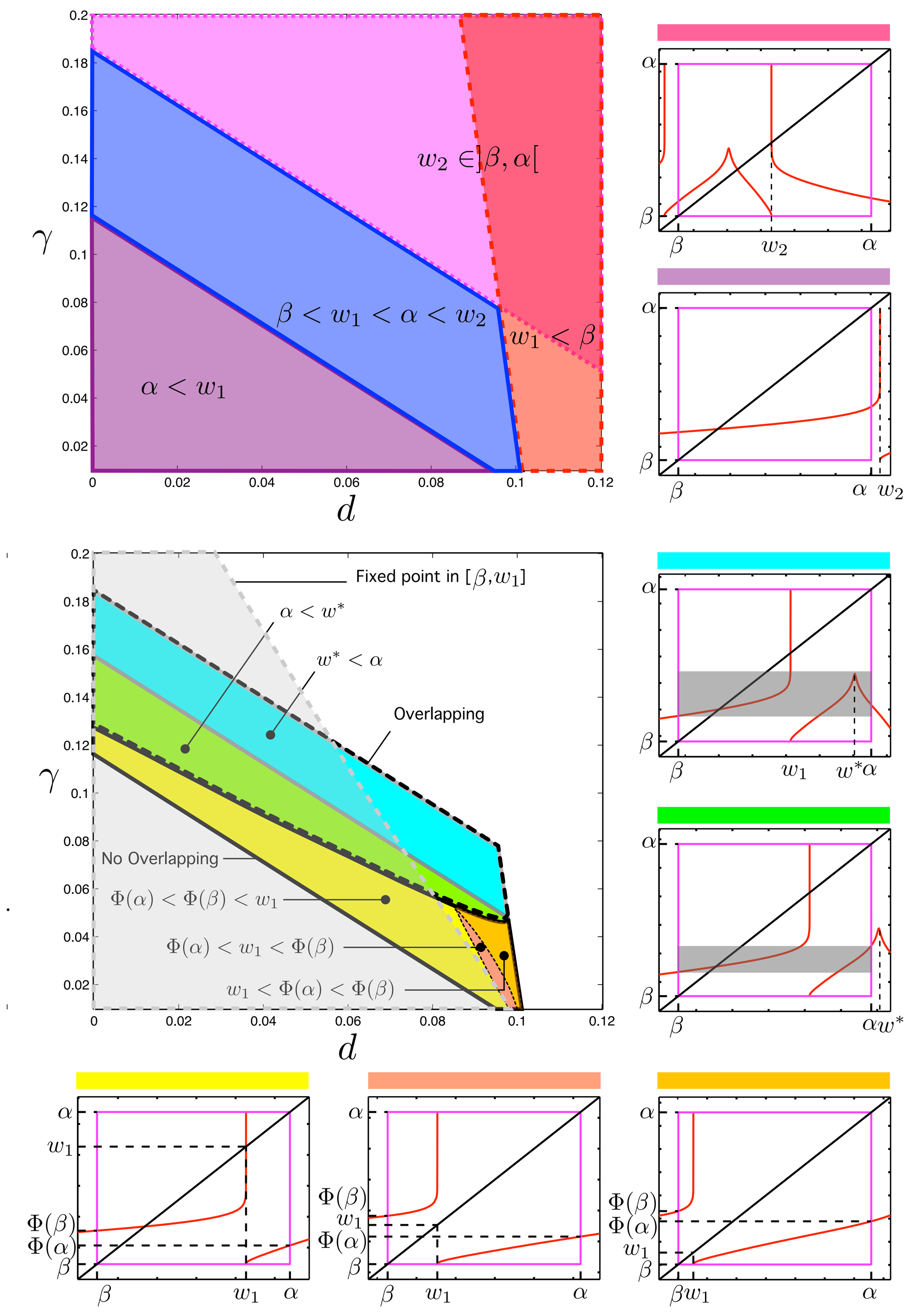
Theorem. The adaptation map has the following properties:

1. in any given interval I_i with $i \in \{1\cdots p+1\}$, the map is increasing for $w < w^*$ and decreasing for $w > w^*$
2. at the boundaries of the definition domain $\mathcal{D}$, $\{w_i; i = 1\cdots p\}$, the map has well-defined and distinct left and right limits:
$$\begin{cases}\lim_{w\to w_i^-}\Phi(w) = \alpha,\ \lim_{w\to w_i^+}\Phi(w) = \beta, & i \leq p_1\\ \lim_{w\to w_j^-}\Phi(w) = \beta,\ \lim_{w\to w_j^+}\Phi(w) = \alpha, & j > p_1\end{cases}$$
3. the derivative $\Phi'(w)$ diverges at the discontinuity points:
$$\begin{cases}\lim_{w\to w_i^\pm}\Phi'(w) = \infty & i \leq p_1\\ \lim_{w\to w_i^\pm}\Phi'(w) = -\infty & i > p_1\end{cases}$$
4. for $w < \min\{\frac{d}{1-\gamma}, w_1, w^{**}\}$ we have $\Phi(w) \geq \gamma w + d > w$

Mathematical context: Rotation Theory

Assume that the line $v = v_r$ has two intersections with SMSFP: w_1 and w_2, with $w_1 < w_2$. We distinguish the following cases:

I. $\beta < w_1 < \alpha < w_2$	II. $\alpha < w_* < w_2$	III. $\Phi(\beta) \geq \beta$
IV.a $\Phi(\alpha) \leq \Phi(\beta)$	IV.b $\Phi(\alpha) > \Phi(\beta)$	



Parameter values: $a = 0.1$, $b = 1$, $I = -3(a/4)^{(4/3)}(2a-1) + 0.1 \approx 0.1175$ and $v_r = 0.1158$.

Non-overlapping case: I., II., III. and IV.a

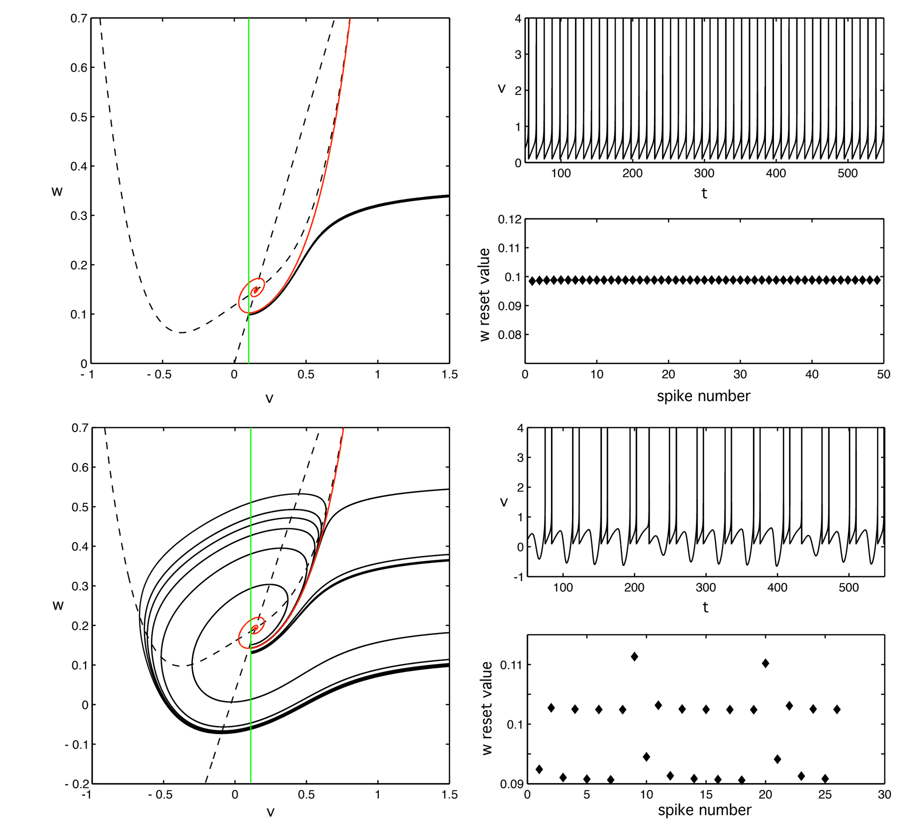
$\Phi : [\beta, \alpha] \to [\beta, \alpha]$ can be seen as a degree-one circle map with discontinuity at w_1. By $\Psi : \mathbb{R} \to \mathbb{R}$ denote the lift of Φ.

Definition: Rotation number

$$\varrho(\Psi, w) := \lim_{n\to\infty}\frac{\Psi^n(w) - w}{n(\alpha - \beta)}$$

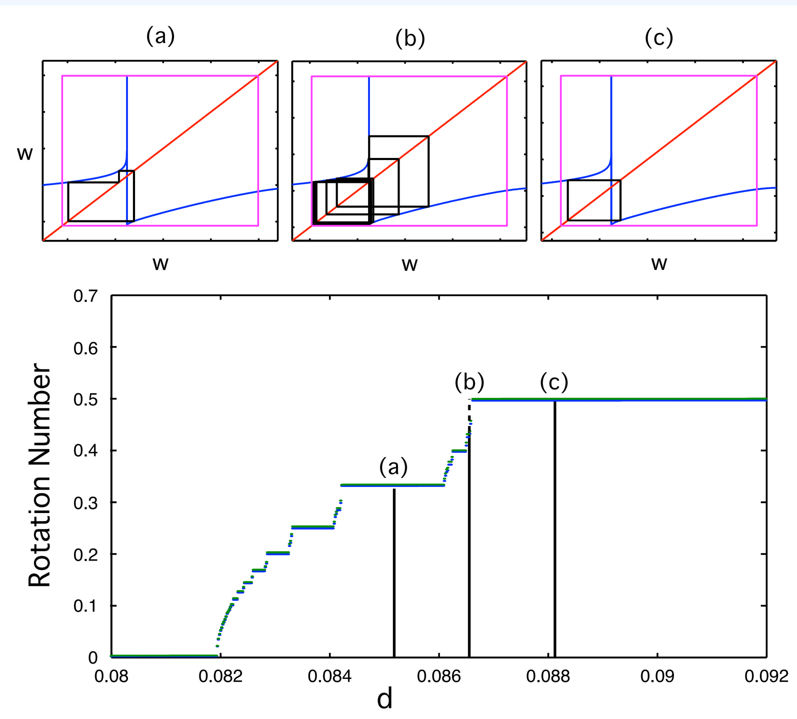
In the non-overlapping case the rotation number is well-defined and does not depend on w (cf.[2],[4]).

- $\varrho = 0 \mod 1 \implies$ tonic, regular spiking (for every initial condition $w_0 \in [\beta, \alpha] \setminus \{w_1\}$)
- $\varrho = p/q \in \mathbb{Q}\setminus\mathbb{Z} \implies$ MMBO (with periodicity of interspike-intervals and interspersing oscillations)
- $\varrho \in \mathbb{R}\setminus\mathbb{Q} \implies$ no periodic orbits and we observe *chaos*.



Parameter values: $v_r = 0.1$ and $\gamma = 0.05$. Top: $d = 0.08$; rotation number $\varrho = 0$.
Bottom: $d = 0.08657$; rotation number $\varrho = p/q$

Proposition. Let a, b, v_r, I, γ be fixed and consider $d \in [\lambda_1, \lambda_2]$. Then the mapping $\varrho : d \mapsto \varrho_d$ is continuous and if additionally for every $d \in [\lambda_1, \lambda_2]$ the adaptation map Φ_d satisfies $\Phi_d(\beta_{\lambda_1}) > \Phi_d(\alpha_{\lambda_2})$, then $\varrho : d \mapsto \varrho_d$ behaves like a *Devil's staircase*.



Parameter values: $v_r = 0.1$, $\gamma = 0.05$

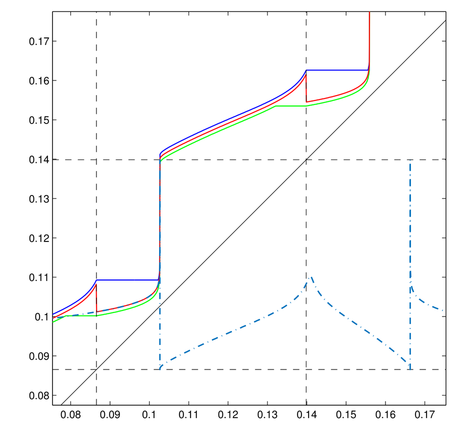
Overlapping case: I., III. and IV.b

Definition: Rotation interval $[a(\Psi), b(\Psi)]$

$$a(\Psi) := \inf_w \liminf_{n\to\infty}\frac{\Psi^n(w) - w}{n(\alpha - \beta)}, \qquad b(\Psi) := \sup_w \limsup_{n\to\infty}\frac{\Psi^n(w) - w}{n(\alpha - \beta)}$$

The analysis of Φ can be made via the results on *old heavy maps* ([3]) and with the use of enveloping maps Ψ_l and Ψ_r which provide the effective formula for the rotation interval.

$$\Psi_l(w) := \inf\{\Psi(z) : z \geq w\}$$
$$\Psi_r(w) := \sup\{\Psi(z) : z \leq w\}$$
$$a(\Psi) = \varrho(\Psi_l)$$
$$b(\Psi) = \varrho(\Psi_r)$$

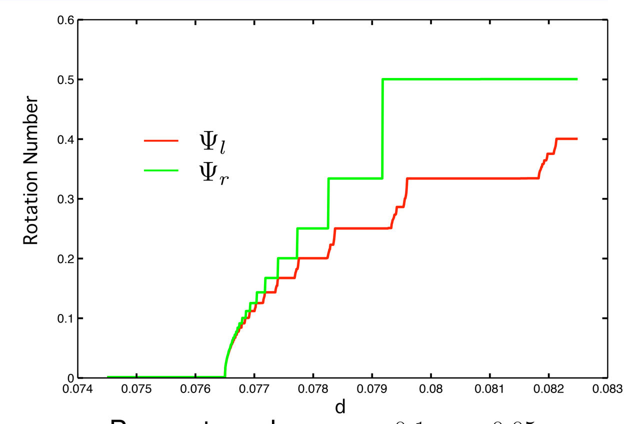


The nontrivial rotation interval corresponds to complex dynamics. In particular for every $p/q \in (a(\Psi), b(\Psi))$, there is a periodic orbit with period q.

Theorem [Condition for orbits of all periods]. Existence of a fixed point $w_f \in (\beta, w_1)$ and a periodic orbit with period $q > 1$ implies existence of periodic orbits with arbitrary periods $\tilde{q} > q$ and with MMBO. The same holds if $w_f \in (w_1, \alpha)$ provided that the q-periodic orbit is not of the type q/q (i.e. it admits points to the left and to the right of w_1).
In particular, whenever there is a fixed point $w_f \in (\beta, \alpha)$ and a periodic orbit of the type $1/2$, then there are periodic orbits of all periods, exhibiting MMBO.

Proposition. Suppose that for the fixed parameters v_R, a, b, γ and I and $d \in [\lambda_1, \lambda_2]$ the maps Φ_d are in the overlapping regime. Then the maps $d \mapsto a(\Psi_d)$ and $d \mapsto b(\Psi_d)$, assigning to d the endpoints of the rotation interval of Φ_d, are continuous.

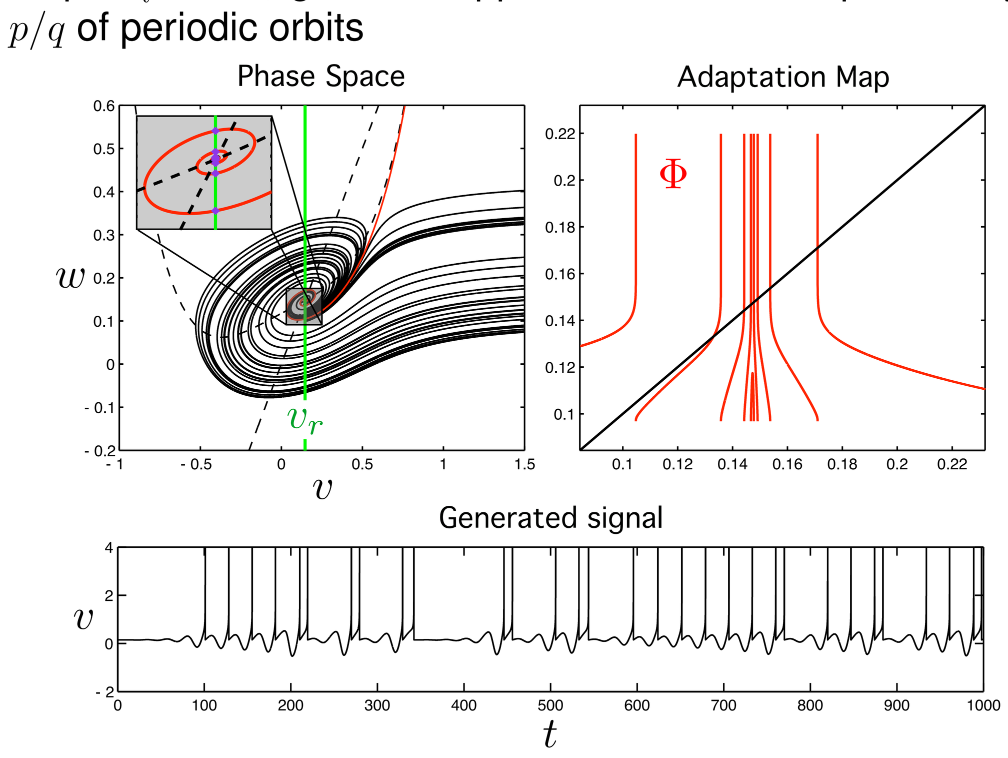
Moreover, in general we can expect that the maps $d \mapsto a(\Psi_d)$ and $d \mapsto b(\Psi_d)$ will also behave like a Devil's staircase.



Parameter values: $v_r = 0.1$, $\gamma = 0.05$

Conclusions and perspectives

- We are able to predict the output properties using geometrical analysis
- In the overlapping and non-overlapping cases existing mathematical tools of rotation theory provide complete description of the dynamics of Φ
- In the remaining cases (e.g. of both positive and negative jumps) one can obtain weaker results on the dynamics of Φ; in particular the rotation interval computed via the enveloping maps Ψ_l and Ψ_r gives the upper-estimate for the possible types p/q of periodic orbits



- For multiple discontinuity points the dynamics is even more complex and harder to be completely classified. However, some rigorous results can be obtained via the theory of piece-wise continuous piece-wise monotone maps.
- Consider forcing of the IF system through variable I. A simple starting point is a square signal for $I(t)$: the performed analysis can be generalized using a stroboscopic map.
- Tackle the problematic of 3D vector field appearing with two recovery variables. In this case we have $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. The general mechanism for generating MMBO is the same, yet leading to richer behaviors due to the geometric structure of the flow.

References

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