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# An enlargement of filtration formula with applications to multiple non-ordered default times

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**Abstract** In this work, for a reference filtration $\mathbb{F}$, we develop a method for computing the semimartingale decomposition of $\mathbb{F}$-martingales in a specific type of enlargement of $\mathbb{F}$. As an application, we study the progressive enlargement of $\mathbb{F}$ with a sequence of non-ordered default times and we show how to deduce results concerning the first-to-default, *k-th*-to-default, *k*-out-of-*n*-to-default or the all-to-default events. In particular, using this method, we compute explicitly the semimartingale decomposition of $\mathbb{F}$-martingales under the absolute continuity condition of Jacod.

**Keywords** Enlargement of filtration · Non-ordered default times · Reduced form credit risk modeling · Semimartingale decomposition

**JEL classification** G14

**Mathematics Subject Classification (2010)** 60G07 · 91G40

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## 1 Introduction

This paper is motivated by the needs to model multiple non-ordered default times in credit risk and to perform explicit computations in the pricing of credit instruments. Needless to say, in general the default times of all the parties or entities involved in a financial contract are not necessarily ordered, however they are observed in an ordered fashion and financial decisions are usually made based on the ordered observations. Examples in the current literature include the works of Brigo et al. [4,5] and Crépey [8,9], where the authors considered the modeling of bilateral counterparty risk when the default event is triggered by the first default of the two parties involved in the contract, i.e., first-to-default. In the modeling of Basket Default Swaps, see for example Laurent and Gregory [20] and Bielecki et al. [3], where $n$-entities are considered in a credit basket, the default event can be triggered either by the first-to-default, *k-th*-to-default, *k*-out-of-*n*-to-default or the all-to-default event. By keeping these applications in mind, we develop in this paper a mathematical framework which can be used to aggregate the individual models of default for the investor and each counterparty or entity, in order to obtain properties of the first-to-default, *k-th*-to-default, *k*-out-of-*n*-to-default or all-to-default model.

The study of reduced form credit risk models with multiple defaults have recently been considered in [7,11,12] and [24] through the framework of progressive enlargement of a base filtration $\mathbb{F}$ with multiple random times. In [7] and [24], under Jacod's absolute continuity criteria, the authors focused on optimization and mean variance hedging problems in the progressive enlargement of $\mathbb{F}$ with non-ordered random times. By rewriting the non-ordered random times into ordered random times with random marks, the authors of [7] and [24] have effectively reduced the problem to the case of multiple ordered random times. On the other hand, the authors of [11] and [12] have focused on the pricing problem by computing, again under the Jacod's criteria, the conditional expectation and the stochastic intensity of the default times in the progressive enlargement of $\mathbb{F}$ with ordered and non-ordered random times. In this work, we focus on the computation of the semimartingale decomposition of $\mathbb{F}$-martingales in some enlargement of $\mathbb{F}$, for which the progressive enlargement with multiple non-ordered random times is a special case. We develop here, without relying on the Jacod's criteria or the specific structure of the progressive enlargement, an unified abstract framework for computing the semimartingale decomposition and the necessary computation methodologies which are well adapted to, and is fully explicit, in the case of non-ordered random times satisfying the Jacod's criteria.

In this paper, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of completion and right-continuity with $\mathcal{F}_\infty \subset \mathcal{F}$. We call any filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ on the same probability space an *enlargement* of the filtration $\mathbb{F}$ if for all $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F}$, and we write $\mathbb{F} \subset \mathbb{G}$. Mathematically, we consider the following condition, called *hypothesis* $(H')$ in the literature.

**Hypothesis** $(H')$**:** Every $\mathbb{F}$-local martingale is a $\mathbb{G}$-semimartingale.

For notational simplicity, if hypothesis $(H')$ is satisfied between $\mathbb{F}$ and $\mathbb{G}$ with $\mathbb{F} \subset \mathbb{G}$, then we write

$$\mathbb{F} \hookrightarrow^{H'} \mathbb{G}. \tag{1.1}$$

It is now well known from Jeulin [22, Theorem 2.1], that under *hypothesis* $(H')$, any $\mathbb{F}$-local martingale $X$ is a $\mathbb{G}$-special semimartingale and the notion of the *drift* of $X$ in $\mathbb{G}$ is well-defined. More precisely, there exists a unique $\mathbb{G}$-predictable process of finite variation $\Gamma(X)$ such that $X - \Gamma(X)$ is a $\mathbb{G}$-local martingale. The map $X \to \Gamma(X)$, which we call the *drift operator* (see [28]), defined on the space of all $\mathbb{F}$-local martingales, taking values into the space of $\mathbb{G}$-predictable processes with finite variation, constitutes the most important characteristic in theory of enlargement of filtration.

Generally speaking, in the study of hypothesis $(H')$, the difficulty of the problem does not lie in checking whether hypothesis $(H')$ is satisfied or not, but is to compute as explicitly as possible the drift operator as a function of known quantities in the smaller filtration. The form and the properties of the drift operator are central in various studies of the enlarged filtration $\mathbb{G}$. It was shown in Song [28], that the drift operator together with the raw structure condition are required in studying the no-arbitrage condition of the first kind in $\mathbb{G}$. In Jeanblanc and Song [19] the drift operator has been used to establish the martingale representation property in $\mathbb{G}$. In Hillairet and Jiao [14] the drift operator is directly linked to the pricing measure in a market with insider. It was shown, in Ankirchner et al. [2], that the difference of the value functions associated with maximisation of expected utility of the wealth for an informed and uninformed trader is determined by the form of the drift operator. Therefore, one of the goals and challenges in the theory of enlargement of filtrations and its applications in financial mathematics, is to express as explicitly as possible, the drift operator using the given input information. Examples of these calculations can be found in the setting of initial or progressive enlargement of filtration.

In the rest of the paper, we suppose that the enlargement $\mathbb{G}$ of the base filtration $\mathbb{F}$ satisfies the following assumption.

**Assumption 1.1** The enlargement $\mathbb{G}$ of $\mathbb{F}$ is such that there exist $k \in \mathbb{N}^+$ and a family of right-continuous filtrations $(\mathbb{F}^1, \dots, \mathbb{F}^k)$ satisfying the usual conditions and a partition $\mathcal{D} = \{D_1, \dots, D_k\}$ of $\Omega$ such that
$(i)$ for every $1 \le i \le k$ and $t \ge 0$, we have $D_i \in \mathcal{F}$ and $\mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i$,
$(ii)$ for every $1 \le i \le k$, we have $\mathbb{F} \subset \mathbb{F}^i$ and this filtration enlargement satisfies *hypothesis* $(H')$ with the $\mathbb{F}^i$-drift operator $\Gamma^i$.

The structure of $\mathbb{G}$ described in Assumption 1.1 arises naturally in the setting of reduced form credit risk models for multiple defaults times. We illustrate this using the following example.

*Example 1.2* Given a random time $\tau$, we denote by $\mathbb{F}^\tau$ the progressive enlargement of $\mathbb{F}$ with $\tau$, that is the smallest filtration containing $\mathbb{F}$, satisfying the usual conditions, and making $\tau$ a stopping time. Suppose now that we are given two random times $\tau_1$ and $\tau_2$ representing default times of two firms and assume $\mathbb{F}\overset{H'}{\hookrightarrow}\mathbb{F}^{\tau_i}$ for $i = 1, 2$ (see notation (1.1)). The enlarged filtration $\mathbb{G} := \mathbb{F}^{\tau_1\wedge\tau_2} \supset \mathbb{F}$ satisfies Assumption 1.1 with $(\mathbb{F}^i := \mathbb{F}^{\tau_i})_{i=1,2}$ and the partition $\{D_1 := \{\tau_1 < \tau_2\}, D_2 := \{\tau_2 \le \tau_1\}\}$. In this setting, the filtration $\mathbb{G}$ represents the information in $\mathbb{F}$ and the time where the first default $\tau_1 \wedge \tau_2$ occurs, but does not give information about which firm has defaulted. The filtration $\mathbb{F}^i$ represents the information in $\mathbb{F}$ and the time when the firm $i$ defaults and $\mathbb{F}^1 \vee \mathbb{F}^2$ contains information about the default times of each firm.

The aim of this paper is to study the *hypothesis* $(H')$ for the filtrations $\mathbb{F} \subset \mathbb{G}$ and to calculate the corresponding $\mathbb{G}$-drift operator, which we denote by $\Gamma$. To do this, we introduce the *direct sum filtration* $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t\ge 0}$.

**Definition 1.3** For any $t \ge 0$, we define the following auxiliary family of sets,

$$\widehat{\mathcal{F}}_t := \{A \in \mathcal{F} \,|\, \forall i,\ \exists A_t^i \in \mathcal{F}_t^i \text{ such that } A \cap D_i = A_t^i \cap D_i\}. \tag{1.2}$$

The direct sum filtration $\widehat{\mathbb{F}}$ can be thought of as the aggregation of the filtrations $\mathbb{F}^i$ for $i = 1, \ldots, k$, and it can be shown that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$. In general, the inclusion $\mathbb{G} \subset \widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{t\ge 0}$ is strict as for $i = 1, \ldots, k$, the sets $D_i$ are $\widehat{\mathcal{F}}_0$-measurable (hence $\widehat{\mathcal{F}}_t$-measurable) but not necessarily $\mathcal{G}_0$-measurable. We must point out that the filtration $\widehat{\mathbb{F}}$ is only introduced as a tool and instead of the filtration $\widehat{\mathbb{F}}$ one can work with any filtration $\widehat{\mathbb{F}}^*$ such that $\mathbb{G} \subset \widehat{\mathbb{F}}^* \subset \widehat{\mathbb{F}}$.

Before describing our main results and the structure of the paper, we introduce some notations. For a semi-martingale $X$ and a predictable process $H$, we denote by $H \centerdot X$ the stochastic integral of $H$ with respect to $X$, whenever it is well defined. Note that if the integrator $X$ is a process of finite variation then the integral can be understood in the Lebesgue-Stieltjes sense. The optional (resp. predictable) $\sigma$-algebra generated by a filtration $\mathbb{F}$ is denoted by $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{F})$). The $\mathbb{F}$-optional (resp. predictable) projection of an $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process $X$ is denoted by ${}^{o,\mathbb{F}}(X)$ (resp. ${}^{p,\mathbb{F}}(X)$) whenever it exists. For any process $X$ of locally integrable variation, the $\mathbb{F}$-dual optional (resp. predictable) projection of $X$ is denoted by $(X)^{o,\mathbb{F}}$ (resp. $(X)^{p,\mathbb{F}}$).

**The Main Result:** For the reader's convenience, we describe now our main result. Given an $\mathbb{F}$-martingale $M$, we consider for every $i = 1, \ldots, k$, the set $D_i$ and the processes

$$\begin{aligned} N^i &:= {}^{o,\mathbb{F}^i}(\mathbb{1}_{D_i}) \\ \widetilde{N}^i &:= {}^{o,\mathbb{G}}(\mathbb{1}_{D_i}) \\ \widehat{V}^i &:= \Gamma^i(M)\mathbb{1}_{D_i} + \frac{\mathbb{1}_{D_i}}{N^i_-} \centerdot \langle N^i, M\rangle^i \end{aligned} \tag{1.3}$$

where $\langle \cdot, \cdot\rangle^i$ is the $\mathbb{F}^i$-predictable bracket and $\Gamma^i(M)$ is the $\mathbb{F}^i$-drift of $M$. The processes $N^i$ and $\widetilde{N}^i$ are bounded and therefore are `BMO`-martingales in the filtrations $\mathbb{F}^i$ and $\mathbb{G}$ respectively. The process $\widehat{V}^i$ is $\widehat{\mathbb{F}}$-adapted and we will often write $\widehat{V}^i\mathbb{1}_{D_i}$ whenever there is a need to stress that the computations are done on the set $D_i$.

**Theorem 1.4** *Let $M$ be an $\mathbb{F}$-martingale such that for every $i = 1, \ldots, k$, it is an $\mathbb{F}^i$-semimartingale with $\mathbb{F}^i$-semimartingale decomposition $M = M^i + \Gamma^i(M)$, where $M^i$ is an $\mathbb{F}^i$-local martingale and $\Gamma^i(M)$ an $\mathbb{F}^i$-predictable process of finite variation. Then the $\mathbb{G}$-drift operator applied to $M$ is*

$$\Gamma(M) = \sum_{i=1}^{k} (\widehat{V}^i)^{p,\mathbb{G}},$$

*which is the $\mathbb{G}$-dual predictable projection of the $\widehat{\mathbb{F}}$-drift operator applied to $M$.*

*Proof* The proof follows from Theorem 4.1 and Lemma 4.2. □

*Remark 1.5* We show in Theorem 4.1 that the $\mathbb{G}$-drift operator $\Gamma$ can be expressed as a *'weighted average'* of the drift operators $(\Gamma^1, \ldots, \Gamma^k)$. More specifically, the $\mathbb{G}$-drift $\Gamma(M)$ takes the form

$$\Gamma(M) = \sum_{i=1}^{k} \widetilde{N}^i_- * \psi(\widehat{V}^i),$$

where the operations $\psi$ and $*$ are introduced in Lemma 3.2 and equality (3.5) to deal with the cases where $\widetilde{N}^i$ can hit zero. If one considers $\widetilde{N}^i_- * \psi(\widehat{V}^i)$ only up to the first time $\widetilde{N}^i$ hits zero, then the process $\widetilde{N}^i_- * \psi(\widehat{V}^i)$ is simply the stochastic integral of $\widetilde{N}^i_-$ with respect to the $\mathbb{G}$-predictable process of finite variation

$$\psi(\widehat{V}^i) = \frac{{}^{p,\mathbb{G}}(\widehat{V}^i \mathbb{1}_{D_i})}{{}^{p,\mathbb{G}}(\mathbb{1}_{D_i})}.$$

We then show in Lemma 4.2, that for every $i = 1, \ldots, k$, the process $\widetilde{N}^i_- * \psi(\widehat{V}^i)$ is equal to $(\widehat{V}^i)^{p,\mathbb{G}}$. To the best of our knowledge, this type of *'weighted average'* representation has not previously appeared in the literature.

We describe now the structure of the paper and highlight the technical difficulties. The $\mathbb{G}$-drift operator $\Gamma$, given in Theorem 1.4, is computed in two steps which we call *aggregation* and *projection*. In section 2, we perform the *aggregation* step and aggregate the given family of filtrations $(\mathbb{F}^1, \ldots, \mathbb{F}^k)$ into $\widehat{\mathbb{F}}$ and show that $\widehat{\mathbb{F}}$ is a filtration satisfying the usual conditions. By using Proposition 2.4, we establish in Theorem 2.5, an *aggregation* formula, which aggregates the drift operators $(\Gamma^1, \ldots, \Gamma^k)$ into the $\widehat{\mathbb{F}}$-drift operator $\widehat{\Gamma}$. Then by Stricker's theorem, hypothesis $(H')$ is satisfied between $\mathbb{F}$ and $\mathbb{G}$.

The computations performed in the *aggregation* step is similar to the classical works [26,33], as the $\widehat{\mathbb{F}}$-conditional expectations can be computed for every $i = 1, \ldots, k$ on $D_i$ using the $\mathbb{F}^i$-conditional expectation and the final result is obtained through summing over the partition $\mathcal{D} = \{D_1, \ldots, D_k\}$.

Given the result of Theorem 2.5, we perform next the *projection* step. The projection type arguments have been previously used in Jeanblanc and Le Cam [18], Callegaro et al. [6] and Kchia et al. [25], to study the relationship between the filtrations $\mathbb{F} \subset \mathbb{F}^{\tau} \subset \mathbb{F}^{\sigma(\tau)}$, where $\mathbb{F}^{\tau}$ (resp. $\mathbb{F}^{\sigma(\tau)}$) is the progressive (resp. initial) enlargement of $\mathbb{F}$ with a single random time $\tau$. However the situation we face here is much more difficult as one cannot exploit the structure between progressive and initial enlargement.

We point out that it is generally very difficult to obtain an explicit expression for the $\mathbb{G}$-drift operator from the $\widehat{\mathbb{F}}$-drift operator through computing the $\mathbb{G}$-optional projection or the $\mathbb{G}$-dual predictable projection. To see this, suppose that the $\widehat{\mathbb{F}}$-semimartingale decomposition of the $\mathbb{F}$-martingale $M$ is given by $Y + \widehat{\Gamma}(M)$, where $Y$ is a $\widehat{\mathbb{F}}$-local martingale and $\widehat{\Gamma}(M)$ is the $\widehat{\mathbb{F}}$-drift of $M$. To compute the $\mathbb{G}$-drift of $M$, we take the $\mathbb{G}$-optional projection of the $\mathbb{F}$-martingale $M$ to obtain

$$M = {}^{o,\mathbb{G}}(Y) + {}^{o,\mathbb{G}}(\widehat{\Gamma}(M)) - (\widehat{\Gamma}(M))^{p,\mathbb{G}} + (\widehat{\Gamma}(M))^{p,\mathbb{G}}.$$

It is clear from the above that, since ${}^{o,\mathbb{G}}(\widehat{\Gamma}(M)) - (\widehat{\Gamma}(M))^{p,\mathbb{G}}$ is a $\mathbb{G}$-local martingale, if the process ${}^{o,\mathbb{G}}(Y)$ is a $\mathbb{G}$-local martingale then $\Gamma(M) = (\widehat{\Gamma}(M))^{p,\mathbb{G}}$. However, the process $Y$ is only a $\widehat{\mathbb{F}}$-local martingale and it was shown in Stricker [30], and Föllmer and Protter [13] with explicit counter examples, that the optional projection of a local martingale onto a smaller filtration is not necessarily a local martingale in the smaller filtration.

The main issue here is that a localizing sequence in $\widehat{\mathbb{F}}$ is not necessarily a localizing sequence in $\mathbb{G}$. To overcome the issue of localization and to show that the process ${}^{o,\mathbb{G}}(Y)$ is a $\mathbb{G}$-local martingale, one must study stopping times across two different filtrations. For this purpose, we developed independently in section 3, a set of self contained results showing that given two filtrations $\mathbb{K}$ and $\mathbb{H}$, if there exists a set $D \in \mathcal{F}$ such that

$$\mathcal{P}(\mathbb{K}) \cap D \subset \mathcal{P}(\mathbb{H}) \cap D$$

then for every $\mathbb{K}$-stopping time (resp. $\mathbb{K}$-predictable increasing process), one can find a $\mathbb{H}$-stopping time (resp. a $\mathbb{H}$-predictable increasing process) such that they are equal on the set $D$. This result in itself constitutes a new result in the general theory of stochastic processes and may have potential applications.

In section 4, we present our main theoretical result, where we combine the results of section 2 and section 3 to compute in Theorem 4.1 the $\mathbb{G}$-drift operator $\Gamma$, through computing the $\mathbb{G}$-optional projection and identifying it with the $\mathbb{G}$-dual predictable projection of $\widehat{\Gamma}$ in Lemma 4.2. We point out that by setting $\mathbb{F}^i = \mathbb{F}$ for all $i = 1, \dots, k$, we can retrieve from our framework, as a special case, the recently developed semimartingale decomposition result for thin times in Aksamit et al. [1] and the classical results of Meyer [26] and Yor [32] on discrete enlargement.

In section 5, we apply the general framework developed in section 2 to section 4 to study the dynamic of an $\mathbb{F}$-adapted martingale $M$ in $\mathbb{G}$, which is now the progressive enlargement of the reference $\mathbb{F}$ with the non-decreasing re-ordering of a family of default times. Under the additional assumption that the joint $\mathbb{F}$-conditional distribution of the given family of default times is absolutely continuous with respect to a non-atomic measure (Jacod's criteria), the semimartingale decomposition in $\mathbb{G}$ is fully explicit under mild integrability conditions. For the reader's convenience, we present only the main results and some of the more involved computations are postponed to the appendix.

## 2 The Direct Sum Filtration

In this section, given a triple $(\mathbb{F}, (\mathbb{F}^i)_{i=1,\dots,k}, \mathbb{G})$ satisfying Assumption 1.1 with respect to a given partition $\mathcal{D} = \{D_1, \dots, D_k\}$, we construct the *direct sum filtration* $\widehat{\mathbb{F}}$ and compute the $\widehat{\mathbb{F}}$-semimartingale decomposition of $\mathbb{F}$-local martingales. The $\widehat{\mathbb{F}}$-semimartingale decomposition formula derived in Theorem 2.5 extends the study of initial enlargement in Meyer [26] and Yor [32], where the authors enlarged the reference filtration $\mathbb{F}$ with a finite partition of $\Omega$. That is, in the particular case where we construct $\widehat{\mathbb{F}}$ by taking this finite partition of $\Omega$ and setting for all $i = 1, \dots, k$, $\mathbb{F}^i = \mathbb{F}$, we recover from Proposition 2.4 the semimartingale decomposition result given in [26] and [32].

**Lemma 2.1** *For every $t \geq 0$ and $i = 1, \dots, k$,*
*(i) the inclusion $D_i \subseteq \{\mathbb{P}\left(D_i \,|\, \mathcal{F}^i_t\right) > 0\}$ holds $\mathbb{P}$-a.s.*
*(ii) For any $\mathbb{P}$-integrable random variable $\eta$, one has*

$$\mathbb{1}_{D_i}\mathbb{E}_{\mathbb{P}}\big(\eta|\widehat{\mathcal{F}}_t\big) = \mathbb{1}_{D_i}\frac{\mathbb{E}_{\mathbb{P}}\left(\eta\mathbb{1}_{D_i} \,|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,|\, \mathcal{F}^i_t\right)}.$$

*Note that for every $i = 1, \dots, k$, the set $D_i$ belongs to $\widehat{\mathcal{F}}_0$.*

*Proof* Let $t \geq 0$ be fixed and $i = 1, \dots, k$.

(i) For $\Delta := \{\mathbb{P}(D_i|\mathcal{F}^i_t) > 0\}$, one has $\mathbb{E}(\mathbb{1}_{D_i}\mathbb{1}_{\Delta^c}) = 0$, which implies that $\mathbb{P}$-a.s. $D_i \subset \Delta$.

(ii) For $B \in \widehat{\mathcal{F}}_t$, by definition, there exists a set $B^i \in \mathcal{F}^i_t$ such that $B \cap D_i = B^i \cap D_i$. Then we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\eta\mathbb{1}_{D_i}\mathbb{1}_B) &= \mathbb{E}_{\mathbb{P}}(\eta\mathbb{1}_{D_i}\mathbb{1}_{B^i}) \\
&= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{D_i}\frac{\mathbb{E}_{\mathbb{P}}\left(\eta\mathbb{1}_{D_i} \,|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,|\, \mathcal{F}^i_t\right)}\mathbb{1}_{B^i}) \\
&= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{D_i}\frac{\mathbb{E}_{\mathbb{P}}\left(\eta\mathbb{1}_{D_i} \,|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,|\, \mathcal{F}^i_t\right)}\mathbb{1}_{B}).
\end{aligned}$$

□

The second part of the lemma can be viewed as the analogue of the key lemma used in credit risk.

**Lemma 2.2** *The family $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t\geq 0}$ is a right-continuous filtration and we have $\mathbb{G} \subset \widehat{\mathbb{F}}$.*

*Proof* It is not hard to check that $\widehat{\mathbb{F}}$ is a filtration and that $\mathbb{G}$ is a subfiltration of $\widehat{\mathbb{F}}$, therefore we only prove that $\widehat{\mathbb{F}}$ is right-continuous.

To do that, we fix $t \geq 0$ and we show that for every set $B \in \cap_{s>t}\widehat{\mathcal{F}}_s$, and for every $i = 1, \ldots, k$, there exists $B_i \in \mathcal{F}^i_t$ such that $B \cap D_i = B_i \cap D_i$. The set $B$ is $\widehat{\mathcal{F}}_q$-measurable for all rational number $q$ strictly greater than $t$, thus, for each rational $q > t$ and each $i = 1, \ldots, k$, there exists an $\mathcal{F}^i_q$-measurable set $B_{i,q}$, such that $B \cap D_i = B_{i,q} \cap D_i$. It is sufficient to set

$$B_i := \bigcap_{n\geq 0} \bigcup_{q\in(t,t+1/n]} B_{i,q},$$

which is $\cap_{s>t}\mathcal{F}^i_s$-measurable and therefore $\mathcal{F}^i_t$-measurable by right-continuity of the filtration $\mathbb{F}^i$. ☐

**Lemma 2.3** *For every $i = 1, \ldots, k$, we have*

$$\mathcal{P}(\widehat{\mathbb{F}}) \cap D_i = \mathcal{P}(\mathbb{F}^i) \cap D_i = \mathcal{P}(\mathbb{G}) \cap D_i.$$

*Proof* The fact that $\mathcal{P}(\widehat{\mathbb{F}}) \cap D_i = \mathcal{P}(\mathbb{F}^i) \cap D_i$ is a straightforward consequence of the definition of $\widehat{\mathbb{F}}$. On the other hand, the equality $\mathcal{P}(\mathbb{F}^i) \cap D_i = \mathcal{P}(\mathbb{G}) \cap D_i$ is due to (i) of Assumption 1.1. ☐

**Proposition 2.4** *Suppose that $M^i$ is an $\mathbb{F}^i$-local martingale, then for $N^i$ given in* (1.3),

$$M^i \mathbb{1}_{D_i} - \frac{\mathbb{1}_{D_i}}{N^i_-} \centerdot \langle N^i, M^i\rangle^i$$

*is an $\widehat{\mathbb{F}}$-local martingale.*

*Proof* We first note that the $\mathbb{F}^i$-predictable bracket $\langle M^i, N^i\rangle^i$ exists, since $N^i$ is a BMO-martingale. Let $(T_n)_{n\in\mathbb{N}^+}$ be a sequence of $\mathbb{F}^i$-stopping times converging to infinity, such that, for any $T_n$, the stopped process $(\langle M^i, N^i\rangle^i)^{T_n}$ is of integrable variation and $(M^i)^{T_n}$ is a uniformly integrable $\mathbb{F}^i$-martingale.

We define a sequence of $\mathbb{F}^i$-stopping times $(r_n)_{n\in\mathbb{N}^+}$ by setting for every $n \in \mathbb{N}^+$, $r_n := \inf\{t > 0 : N^i_t \leq 1/n\}$, where we set $\inf \emptyset = \{\infty\}$. Also, we set $S_{n,D_i} := (r_n \wedge T_n)\mathbb{1}_{D_i} + \infty\mathbb{1}_{D^c_i}$, which is a sequence of $\widehat{\mathbb{F}}$-stopping times such that $S_{n,D_i} \to \infty$ as $n \to \infty$. (It is not true that $N^i_0 = 1$, but on $D_i$, $N^i_0 > 0$.)

For any bounded elementary $\widehat{\mathbb{F}}$-predictable process $\widehat{\xi}$, by Lemma 2.3 there exists a bounded $\mathbb{F}^i$-predictable process $\xi^i$ such that for every $n \in \mathbb{N}^+$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}((\mathbb{1}_{D_i}\widehat{\xi} \centerdot (M^i)^{S_{n,D_i}})_\infty) &= \mathbb{E}_{\mathbb{P}}((\mathbb{1}_{D_i}\xi^i \centerdot (M^i)^{r_n\wedge T_n})_\infty)\\ &= \mathbb{E}_{\mathbb{P}}((\xi^i \mathbb{1}_{[\![0,r_n\wedge T_n]\!]} \centerdot \langle N^i, M^i\rangle^i)_\infty).\end{aligned}$$

Note that the second equality holds by first taking the $\mathcal{F}^i_\infty$-conditional expectation and then applying the integration by parts formula in $\mathbb{F}^i$. Then,

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}((\xi^i \mathbb{1}_{[\![0,r_n\wedge T_n]\!]} \centerdot \langle N^i, M^i\rangle^i)_\infty)\\ &\quad= \mathbb{E}_{\mathbb{P}}((\mathbb{1}_{D_i}\xi^i (N^i_-)^{-1}\mathbb{1}_{[\![0,r_n\wedge T_n]\!]} \centerdot \langle N^i, M^i\rangle^i)_\infty)\\ &\quad= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{D_i}(\widehat{\xi}\mathbb{1}_{D_i}(N^i_-)^{-1}\mathbb{1}_{[\![0,S_{n,D_i}]\!]} \centerdot \langle N^i, M^i\rangle^i)_\infty),\end{aligned}$$

where the first equality holds by the property of $\mathbb{F}^i$-dual predictable projection. This shows that

$$M^i \mathbb{1}_{D_i} - \frac{\mathbb{1}_{D_i}}{N^i_-} \centerdot \langle N^i, M^i \rangle^i$$

is an $\widehat{\mathbb{F}}$-local martingale. □

**Theorem 2.5** *Let $M$ be an $\mathbb{F}$-martingale as in Theorem 1.4 then the $\widehat{\mathbb{F}}$-drift operator applied to $M$ is*

$$\widehat{\Gamma}(M) = \sum_{i=1}^{k} \widehat{V}^i,$$

*where the processes $\widehat{V}^i$ are defined in* (1.3).

*Proof* For $i = 1, \ldots, k$, it is sufficient to apply Proposition 2.4 to the $\mathbb{F}^i$-local martingale $M - \Gamma^i(M)$ and note that $\langle N^i, \Gamma^i(M) \rangle^i = 0$, since $\Gamma^i(M)$ is a predictable process of finite variation. This shows that the process

$$\left(M - \Gamma^i(M)\right) \mathbb{1}_{D_i} - \frac{\mathbb{1}_{D_i}}{N^i_-} \centerdot \langle N^i, M - \Gamma^i(M) \rangle^i = (M - \widehat{V}^i) \mathbb{1}_{D_i}$$

is an $\widehat{\mathbb{F}}$-local martingale. The result then follows by summing over the elements $\{D_i\}_{i=1,\ldots,k}$ of the partition. □

## 3 Stopping Times and Increasing Processes: General Results

In this section, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with two filtrations $\mathbb{K} = (\mathcal{K}_t)_{t \geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ satisfying the usual conditions and for which there exists $D \in \mathcal{F}$ such that

$$\mathcal{P}(\mathbb{K}) \cap D \subset \mathcal{P}(\mathbb{H}) \cap D. \tag{3.1}$$

For the purpose of this paper, increasing processes are positive, càdlàg and finite valued unless specified otherwise. Since here we do not have a base filtration, we shall make precise, for any process and any stopping time, the choice of the filtration that we are dealing with.

We give two technical results in Lemma 3.1 and Lemma 3.2 linking increasing processes and stopping times in the filtrations $\mathbb{K}$ and $\mathbb{H}$. We show how one can associate with every $\mathbb{K}$-stopping time (resp. $\mathbb{K}$-increasing process) an $\mathbb{H}$-stopping time (resp. an $\mathbb{H}$-increasing process which can take infinite value) such that they are equal on the set $D$.

For the purpose of this section, we set for $n \in \mathbb{N}^+$,

$$\widetilde{N} = {}^{o,\mathbb{H}}(\mathbb{1}_D), \qquad R_n = \inf\{t \geq 0 : \widetilde{N}_t \leq 1/n\}$$

and $R := \sup_n R_n = \inf\{s : \widetilde{N}_s \widetilde{N}_{s-} = 0\}$.

**Lemma 3.1** *For any increasing sequence of* $\mathbb{K}$*-stopping times* $(T_n)_{n\in\mathbb{N}^+}$*, there exists an increasing sequence of* $\mathbb{H}$*-stopping times* $(S_n)_{n\in\mathbb{N}^+}$ *such that* $T_n 1\!\!1_D = S_n 1\!\!1_D$*. In addition, if* $\sup_n T_n = \infty$*, then*
*(i)* $S := \sup_n S_n$ *is greater or equal to* $R$*.*
*(ii)* $\cup_n\{R_n = R\} \subset \cup_n\{S_n \geq R\}$*.*

*Proof* Let $(T_n)_{n\in\mathbb{N}^+}$ be an increasing sequence of $\mathbb{K}$-stopping times. For every $n$, there exists an $\mathbb{H}$-predictable process $H_n$ such that $1\!\!1_{]\!]T_n,\infty[\![} 1\!\!1_D = H_n 1\!\!1_D$. Notice that by replacing $H_n$ with $H_n^+ \wedge 1$, we can suppose that $0 \leq H_n \leq 1$ and replacing $H_n$ with $\prod_{k=1}^n H_k$, we can suppose that $H_n \geq H_{n+1}$. Let $S_n := \inf\{t \geq 0 : H_n(t) = 1\}$, then $(S_n)_{n\in\mathbb{N}^+}$ is an increasing sequence of $\mathbb{H}$-stopping times such that $S_n = T_n$ on the set $D$.

(i) Suppose $\sup_n T_n = \infty$, then by taking the $\mathbb{H}$-predictable projection,

$$^{p,\mathbb{H}}(1\!\!1_{]\!]T_n,\infty[\![} 1\!\!1_D) = 1\!\!1_{]\!]S_n,\infty[\![} \widetilde{N}_-,$$

which implies that for every $\mathbb{H}$-predictable stopping time $T$

$$\mathbb{E}_{\mathbb{P}}(1\!\!1_{]\!]T_n,\infty[\![}(T) 1\!\!1_D 1\!\!1_{\{T<\infty\}}) = \mathbb{E}_{\mathbb{P}}(1\!\!1_{]\!]S_n,\infty[\![}(T)\widetilde{N}_{T-} 1\!\!1_{\{T<\infty\}}).$$

By applying the dominated convergence theorem to the above, we obtain for every $\mathbb{H}$-predictable stopping time $T$,

$$0 = \mathbb{E}_{\mathbb{P}}(\lim_{n\to\infty} 1\!\!1_{]\!]S_n,\infty[\![}(T)\widetilde{N}_{T-} 1\!\!1_{\{T<\infty\}}).$$

Using the fact that $\lim_{n\to\infty} 1\!\!1_{]\!]S_n,\infty[\![}\widetilde{N}_- = 1\!\!1_{\cap_n ]\!]S_n,\infty[\![}\widetilde{N}_-$, which is $\mathbb{H}$-predictable, we deduce from the section theorem (see for example Theorem 4.12 of He et al. [15]) that

$$1\!\!1_{\cap_n ]\!]S_n,\infty[\![}\widetilde{N}_- = 0.$$

For arbitrary $\epsilon > 0$, we have from the above that $\widetilde{N}_{(S+\epsilon)-} = 0$, where $S = \sup_n S_n$. The process $\widetilde{N}$ is a positive $\mathbb{H}$-supermartingale, this implies $S+\epsilon \geq R$ and therefore $S \geq R$, since $\epsilon$ is arbitrary.

(ii) Suppose there exists some $k$ such that $R_k = R$, then $\widetilde{N}_{R-} > 0$ and from $1\!\!1_{\cap_n\{S_n<R\}}\widetilde{N}_{R-} = 0$, one can then deduce that there exists $j$ such that $S_j \geq R$. □

The goal in the following is to study the measures associated with finite variation process considered in different filtrations.

**Lemma 3.2** *There exists a map* $\widehat{V} \longrightarrow \psi(\widehat{V})$ *from the space of* $\mathbb{K}$*-predictable increasing processes to the space of* $\mathbb{H}$*-predictable increasing processes, valued in* $\mathbb{R}_+ \cup \{+\infty\}$*, such that the following properties hold*
*(i)* $1\!\!1_D\widehat{V} = 1\!\!1_D\psi(\widehat{V})$ *and the support of the measure* $d\psi(\widehat{V})$ *is contained in the set* $\cup_n[\![0, R_n]\!]$*,*
*(ii) for any* $\mathbb{K}$*-stopping time* $\widehat{U}$*, there exists an* $\mathbb{H}$*-stopping time* $U$ *such that*

$$\psi(1\!\!1_{[\![0,\widehat{U}]\!]} \centerdot \widehat{V}) = 1\!\!1_{[\![0,U]\!]} \centerdot \psi(\widehat{V}),$$

*(iii) if the process* $\widehat{V}$ *is bounded, then* $\psi(\widehat{V})$ *is bounded by the same constant.*

*Remark 3.3* In Lemma 3.2, property $(ii)$ can be stated more generally as: for all non-negative bounded $\mathbb{K}$-predictable processes $K$,

$$\psi(K \centerdot \widehat{V}) = H \centerdot \psi(\widehat{V})$$

where $H$ is a non-negative $\mathbb{H}$-predictable process such that $H\mathbb{1}_D = K\mathbb{1}_D$, and if $\widehat{V}$ is $\mathbb{H}$-adapted then $\psi(\widehat{V}) = \widehat{V}$.

*Proof* Let $\widehat{V}$ be a $\mathbb{K}$-predictable increasing process. By condition (3.1), there exists an $\mathbb{H}$-predictable process $V$ such that $\widehat{V}\mathbb{1}_D = V\mathbb{1}_D$.

(i) Given a $\mathbb{K}$-localizing sequence $(T_n)_{n\in\mathbb{N}^+}$ such that $\widehat{V}^{T_n}$ is bounded, by Lemma 3.1 there exists an increasing sequence of $\mathbb{H}$-stopping times $(S_n)_{n\in\mathbb{N}^+}$ such that $\widehat{V}^{T_n}\mathbb{1}_D = V^{S_n}\mathbb{1}_D$. By taking the $\mathbb{H}$-optional projection, we have

$${}^{o,\mathbb{H}}(\widehat{V}^{T_n}\mathbb{1}_D) = {}^{o,\mathbb{H}}(V^{S_n}\mathbb{1}_D) = V^{S_n}\widetilde{N}\,.$$

From Theorem 47 in Dellacherie and Meyer [10], the process ${}^{o,\mathbb{H}}(\widehat{V}^{T_n}\mathbb{1}_D)$ is càdlàg, which implies that for all $n, k \in \mathbb{N}^+$, the process $V$ is càdlàg on $[\![\, 0, S_n \,]\!] \cap [\![\, 0, R_k \,]\!]$. Therefore the process $V$ is càdlàg on $\cup_k [\![\, 0, R_k \,]\!]$ due to the fact that $\sup_n S_n \geq R$.

For every $n \in \mathbb{N}^+$ and any pair of rationals $s \leq t \leq R_n$, we have $\mathbb{P}$-a.s

$$V_t\mathbb{1}_D = \widehat{V}_t\mathbb{1}_D \geq \widehat{V}_s\mathbb{1}_D = V_s\mathbb{1}_D$$

and by taking the $\mathcal{H}_{t-}$-conditional expectation and recalling that ${}^{p,\mathbb{H}}(\mathbb{1}_D) = \widetilde{N}_-$, we have $\mathbb{P}$-a.s, the inequality $V_t\widetilde{N}_{t-} \geq V_s\widetilde{N}_{t-}$. Using the fact that the process $V$ is càdlàg on $[\![\, 0, R_n \,]\!]$, we can first take the right-limit in $t$ of the two sides of this inequality to show that for all $t \in [\![\, 0, R_n \,]\!]$ and $s \in \mathbb{Q}_+ \cap [\![\, 0, R_n \,]\!]$ the inequality $V_t \geq V_s$ holds. Then by taking limit in $s$ we extend this inequality to all $s \leq t \leq R_n$.

From the above, we can deduce that the process $V$ is càdlàg and increasing on $\cup_n [\![\, 0, R_n \,]\!]$. Then, we define the increasing process $\psi(\widehat{V})$, which may take the value infinity by setting

$$\begin{aligned}\psi(\widehat{V}) := {} & V\mathbb{1}_{\cup_n [\![\, 0, R_n \,]\!]} + \lim_{s\uparrow R} V_s \mathbb{1}_{\{\widetilde{N}_{R-}=0, 0<R<\infty\}} \mathbb{1}_{[\![\, R, \infty [\![} \\ & + V_R \mathbb{1}_{]\!]\, R, \infty [\![} \mathbb{1}_{\{\widetilde{N}_{R-}>0, 0<R<\infty\}},\end{aligned} \tag{3.2}$$

which is $\mathbb{H}$-predictable since $\{\widetilde{N}_{R-} = 0, 0 < R < \infty\} \cap [\![\, R, \infty [\![$ is the intersection of the set $(\cup_n [\![\, 0, R_n \,]\!])^c$ and the complement of the set $]\!]\, R, \infty [\![ \,\cap \{\widetilde{N}_{R-} > 0, 0 < R < \infty\}$, which are both $\mathbb{H}$-predictable. From (3.2), we see that the support of $d\psi(\widehat{V})$ is contained in $\cup_n [\![\, 0, R_n \,]\!]$ and on the set $D$, we have $\sup_n R_n = R = \infty$ which implies $\widehat{V}\mathbb{1}_D = \psi(\widehat{V})\mathbb{1}_D$.

(ii) For any $\mathbb{K}$-stopping time $\widehat{U}$ and any $\mathbb{K}$-predictable increasing process $\widehat{V}$, we have from (i), the equality $\psi(\mathbb{1}_{[\![\, 0, \widehat{U} \,]\!]} \centerdot \widehat{V})\mathbb{1}_D = (\mathbb{1}_{[\![\, 0, \widehat{U} \,]\!]} \centerdot \widehat{V})\mathbb{1}_D$ and

$$(\mathbb{1}_{[\![\, 0, \widehat{U} \,]\!]} \centerdot \widehat{V})\mathbb{1}_D = \mathbb{1}_{[\![\, 0, U \,]\!]} \centerdot (\widehat{V}\mathbb{1}_D) = \mathbb{1}_{[\![\, 0, U \,]\!]} \centerdot (\psi(\widehat{V})\mathbb{1}_D) = (\mathbb{1}_{[\![\, 0, U \,]\!]} \centerdot \psi(\widehat{V}))\mathbb{1}_D\,.$$

The existence of the $\mathbb{H}$-stopping time $U$ in the first equality follows from Lemma 3.1, the second equality follows from (i) and the integral appearing in the second term is to be understood in the Lebesgue-Stieltjes sense. By taking the $\mathbb{H}$-predictable projection, we conclude that for every $n \in \mathbb{N}^+$, the processes $\mathbb{1}_{[\![0,U]\!]} \centerdot \psi(\widehat{V})$ and $\psi(\mathbb{1}_{[\![0,\widehat{U}]\!]} \centerdot \widehat{V})$ are equal on $[\![0, R_n]\!]$ and therefore on $\cup_n [\![0, R_n]\!]$. This implies that the processes $\mathbb{1}_{[\![0,U]\!]} \centerdot \psi(\widehat{V})$ and $\psi(\mathbb{1}_{[\![0,\widehat{U}]\!]} \centerdot \widehat{V})$ are equal everywhere, since they do not increase on the complement of $\cup_n [\![0, R_n]\!]$.

(iii) Note that for every $n \in \mathbb{N}^+$, the process $\widetilde{N}_-$ is strictly positive on $[\![0, R_n]\!]$. If the process $\widehat{V}$ is bounded by $C$, then we have $V \leq \frac{{}^{p,\mathbb{H}}(C\mathbb{1}_D)}{{}^{p,\mathbb{H}}(\mathbb{1}_D)} = C$ on every interval $[\![0, R_n]\!]$ and therefore on $\cup_n [\![0, R_n]\!]$. We deduce from (3.2) that on $\cup_n [\![0, R_n]\!]$ the process $\psi(\widehat{V})$ is bounded by $C$, which implies that $\psi(\widehat{V})$ is bounded by $C$ since the support of $d\psi(\widehat{V})$ is contained in $\cup_n [\![0, R_n]\!]$. □

The goal is now to extend the map $\psi$ to predictable finite variation processes. Our first idea was to define, starting from a $\mathbb{K}$-predictable process of finite variation $\widehat{V} = \widehat{V}_+ - \widehat{V}_-$, an $\mathbb{H}$-predictable process of finite variation by setting $\psi(\widehat{V})$ to be $\psi(\widehat{V}_+) - \psi(\widehat{V}_-)$. However, this is problematic since from (3.2) we see that in general the processes $\psi(\widehat{V}_\pm)$ can take the value infinity at the same time. Therefore one can not make use of the usual definitions.

In order to treat the problem mentioned in the above, we follow the idea of Jacod [16, Chapter IV, Section 4], and we use the concept of dominated measures to define a measure associated with a process which is the difference of two increasing unbounded processes. Following the assumption and notation of Lemma 3.2, one can associate with a given $\mathbb{K}$-predictable process $\widehat{V}$ of finite variation ($\widehat{V} = \widehat{V}_+ - \widehat{V}_-$), a pair of $\mathbb{H}$-predictable increasing processes $\psi(\widehat{V}_+)$ and $\psi(\widehat{V}_-)$. We define an auxiliary finite random measure $dm$ on $\Omega \times \mathcal{B}(\mathbb{R}_+)$ by setting

$$dm := (1 + \psi(\widehat{V}_+) + \psi(\widehat{V}_-))^{-2} d(\psi(\widehat{V}_+) + \psi(\widehat{V}_-)). \tag{3.3}$$

Since the processes $\psi(\widehat{V}_+)$ and $\psi(\widehat{V}_-)$ can only take value infinity at the same time, we deduce that $d\psi(\widehat{V}_\pm)$ is absolutely continuous with respect to $dm$, which is absolutely continuous with respect to $d(\psi(\widehat{V}_+) + \psi(\widehat{V}_-))$. By (i) of Lemma 3.2, this implies that the support of $dm$ is contained in $\cup_n [\![0, R_n]\!]$. Let us denote by $q^\pm$ the Radon-Nikodým density of $d\psi(\widehat{V}_\pm)$ with respect to $dm$ and introduce the following set of $\Omega \times \mathcal{B}(\mathbb{R}_+)$-measurable functions,

$$\{f : \forall t > 0, \int_{(0,t]} |f_s||q_s^+ - q_s^-|\, dm_s < \infty\}. \tag{3.4}$$

We define a linear operation $*$ on the set (3.4), which maps any $f$ in the set (3.4) to a process by setting for every $t \geq 0$,

$$f * \psi(\widehat{V})_t := \int_{(0,t]} f_s (q_s^+ - q_s^-)\, dm_s. \tag{3.5}$$

We denote the set defined in (3.4) by $\mathcal{L}^1(\psi(\widehat{V}))$ and an $\Omega \times \mathcal{B}(\mathbb{R})$-measurable function $f$ is said to be $\psi(\widehat{V})$ integrable if $f \in \mathcal{L}^1(\psi(\widehat{V}))$. One should point out that the measure $dm$ is introduced to avoid the problem that $\psi(\widehat{V}_\pm)$ can take the value infinity, and that the set defined in (3.4) and the map defined in (3.5) are essentially independent of the choice of $dm$.

## 4 The $\mathbb{G}$-Semimartingale Decomposition

In this subsection, we place ourselves in the setting of section 2, and with the technical results from section 3, we are ready to derive the $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingales. We first summarize the results and notations.

We recall that for each $i = 1, \dots, k$, the $\mathbb{F}^i$-semimartingale decomposition of an $\mathbb{F}$-martingale $M$ is given by $M = M^i + \Gamma^i(M)$, where $M^i$ is an $\mathbb{F}^i$-local martingale and $\Gamma^i(M)$ an $\mathbb{F}^i$-predictable process of finite variation. Then, by applying Proposition 2.4 to $M^i = M - \Gamma^i(M)$, for every $i = 1, \dots, k$, the process

$$\widehat{M^i} := (M - \widehat{V}^i)\mathbb{1}_{D_i} \tag{4.1}$$

where $\widehat{V}^i$ is defined in (1.3), is an $\widehat{\mathbb{F}}$-local martingale. From Lemma 2.3, for every $i = 1, \dots, k$, we have $\mathcal{P}(\widehat{\mathbb{F}}) \cap D_i \subset \mathcal{P}(\mathbb{G}) \cap D_i$ and one can apply Lemma 3.2 with the set $D_i$, the filtrations $\widehat{\mathbb{F}}$ and $\mathbb{G}$ and define the linear operation $*$ as in (3.5) on the space of $\psi(\widehat{V}^i)$ integrable functions given by

$$\mathcal{L}^1(\psi(\widehat{V}^i)) := \{f : \int_{\mathbb{R}_+} |f_s||q^{i,+}_s - q^{i,-}_s|dm^i_s < \infty\},$$

where the measure $dm^i$ is constructed from the $\widehat{\mathbb{F}}$-predictable process $\widehat{V}^i$ as shown in (3.3) and $q^{i,\pm}$ is the density of $d\psi(\widehat{V}^i_\pm)$ with respect to $dm^i$.

For an arbitrary filtration $\mathbb{K}$, we write $X \overset{\mathbb{K}-\text{mart}}{=} Y$, if $X - Y$ is a $\mathbb{K}$-local martingale. Let us recall that for every $i = 1, \dots, k$ the processes $\widehat{V}^i$ and $\widetilde{N}^i$ are defined in (1.3).

**Theorem 4.1** *Let $M$ be an $\mathbb{F}$-martingale as in Theorem 1.4, then for every $i = 1, \dots, k$, the process $\widetilde{N}^i_-$ belongs to the set $\mathcal{L}^1(\psi(\widehat{V}^i))$ defined in (3.4) and the $\mathbb{G}$-drift operator applied to $M$ is*

$$\Gamma(M) = \sum_{i=1}^{k} \widetilde{N}^i_- * \psi(\widehat{V}^i)$$

*where the operations $\psi$ and $*$ are defined in Lemma 3.2 and equality (3.5) respectively.*

*Proof* By Theorem 2.5, the process $M$ is an $\widehat{\mathbb{F}}$-semimartingale and therefore a $\mathbb{G}$-semimartingale by the result of Stricker [30, Proposition 1.1]. To see that

$M$ is a $\mathbb{G}$-special semimartingale, we note that there exists a sequence of $\mathbb{F}$-stopping times (therefore $\mathbb{G}$-stopping times) which reduces $M$ to martingale in the Hardy space $\mathcal{H}^1$, and one can check directly, that $M$ is an $\mathbb{G}$-special semimartingale by using special semimartingale criteria such as Theorem 8.6 in [15]. The aim in the rest of the proof is to compute explicitly the $\mathbb{G}$-semimartingale decomposition of $M$.

Before proceeding, we notice that since $\sum_{i=1}^k \widetilde{N}^i = 1$, it is sufficient to show that for any fixed $i = 1, \dots, k$, the process $\widetilde{N}^i_-$ is in $\mathcal{L}^1(\psi(\widehat{V}^i))$ and $M\widetilde{N}^i - \widetilde{N}^i_- * \psi(\widehat{V}^i)$ is a $\mathbb{G}$-local martingale.

The process $M\widetilde{N}^i$ is a $\mathbb{G}$-special semimartingale since $M$ is a $\mathbb{G}$-special semimartingale and $\widetilde{N}^i$ is a bounded $\mathbb{G}$-martingale. Let us denote by $B^i$ the unique $\mathbb{G}$-predictable process of finite variation in the $\mathbb{G}$-semimartingale decomposition of $M\widetilde{N}^i$. For every $n \in \mathbb{N}^+$, we define the $\mathbb{G}$-stopping time $R_n = \inf\{t \geq 0, \widetilde{N}^i_t \leq 1/n\}$ and we introduce the $\mathbb{G}$-stopping time $R = \sup_n R_n = \inf\{t \geq 0, \widetilde{N}^i_{t-}\widetilde{N}^i_t = 0\}$. The method of the proof is to show the equality between the process $B^i$ and $\widetilde{N}^i_- * \psi(\widehat{V}^i)$. To do that, it is sufficient to show that the two processes coincide on the sets $\cup_n [\![ 0, R_n ]\!]$ and $[\![ 0, R ]\!] \setminus \cup_n [\![ 0, R_n ]\!]$: indeed the processes $B^i$ and $\widetilde{N}^i_- * \psi(\widehat{V}^i)$ are constant on $]\!] R, \infty [\![$, as $M_t \widetilde{N}^i_t = 0$ for $t > R$ and the support of the measure $dm^i$ is in the complement of $]\!] R, \infty [\![$ by (i) of Lemma 3.2.

On the set $\cup_n [\![ 0, R_n ]\!]$, let $(T_n)_{n\in\mathbb{N}^+}$ be a localizing sequence of $\widehat{\mathbb{F}}$-stopping times such that the process $\widehat{M}^i$ defined in (4.1) stopped at $T_n$, i.e., $(\widehat{M}^i)^{T_n}$ is an $\widehat{\mathbb{F}}$-martingale and $(\widehat{V}^i)^{T_n}$ is of bounded total variation. Then

$$
\begin{aligned}
(\widehat{M}^i)^{T_n} &= M^{T_n}\mathbb{1}_{D_i} - \mathbb{1}_{[\![ 0,T_n ]\!]} \centerdot \widehat{V}^i_+ \mathbb{1}_{D_i} + \mathbb{1}_{[\![ 0,T_n ]\!]} \centerdot \widehat{V}^i_- \mathbb{1}_{D_i} \\
&= M^{S_n}\mathbb{1}_{D_i} - \mathbb{1}_{[\![ 0,S_n ]\!]} \centerdot \psi(\widehat{V}^i_+)\mathbb{1}_{D_i} + \mathbb{1}_{[\![ 0,S_n ]\!]} \centerdot \psi(\widehat{V}^i_-)\mathbb{1}_{D_i}
\end{aligned} \tag{4.2}
$$

where the second equality holds from (i) of Lemma 3.2 with the existence of the sequence of $\mathbb{G}$-stopping times $(S_n)_{n\in\mathbb{N}^+}$ given by Lemma 3.1. Since for every $n \in \mathbb{N}^+$, the $\widehat{\mathbb{F}}$-adapted processes $\mathbb{1}_{[\![ 0,T_n ]\!]} \centerdot \widehat{V}^i_\pm$ are bounded, by (ii) and (iii) of Lemma 3.2, we can conclude that the processes $\mathbb{1}_{[\![ 0,S_n ]\!]} \centerdot \psi(\widehat{V}^i_\pm)$ are bounded $\mathbb{G}$-predictable processes. Together with the property that the Lebesgue-Stieltjes integral and the stochastic integral coincide when all quantities are finite, we obtain that

$$
\mathbb{1}_{[\![ 0,S_n ]\!]} \centerdot \psi(\widehat{V}^i_+) - \mathbb{1}_{[\![ 0,S_n ]\!]} \centerdot \psi(\widehat{V}^i_-) = \mathbb{1}_{[\![ 0,S_n ]\!]} * \psi(\widehat{V}^i),
$$

where $*$ is defined in (3.5). By taking the $\mathbb{G}$-optional projection, we obtain

$$
M^{S_n}\widetilde{N}^i \overset{\mathbb{G}-\text{mart}}{=} \widetilde{N}^i \left(\mathbb{1}_{[\![ 0,S_n ]\!]} * \psi(\widehat{V}^i)\right) \overset{\mathbb{G}-\text{mart}}{=} \widetilde{N}^i_- \centerdot \left(\mathbb{1}_{[\![ 0,S_n ]\!]} * \psi(\widehat{V}^i)\right)
$$

where the second equality follows from the integration by parts formula and Yœurp's lemma. For each $n \in \mathbb{N}^+$, from uniqueness of the $\mathbb{G}$-semimartingale decomposition, we deduce that

$$
\mathbb{1}_{[\![ 0,S_n ]\!]} \centerdot B^i = \widetilde{N}^i_- \centerdot \left(\mathbb{1}_{[\![ 0,S_n ]\!]} * \psi(\widehat{V}^i)\right) = (\widetilde{N}^i_- \mathbb{1}_{[\![ 0,S_n ]\!]}) * \psi(\widehat{V}^i),
$$

where the second equality follows again from the property that the Lebesgue-Stieltjes integral and the stochastic integral coincide when all quantities are finite. Then for any fixed $t \geq 0$ and $H_s := \mathbb{1}_{\{q_s^{i,+} - q_s^{i,-} < 0\}} - \mathbb{1}_{\{q_s^{i,+} - q_s^{i,-} \geq 0\}}$,

$$\int_{(0,t]} H_s \mathbb{1}_{\{s \leq S_n\}} dB_s^i = \int_{(0,t]} \mathbb{1}_{\{s \leq S_n\}} \widetilde{N}_-^i |q_s^{i,+} - q_s^{i,-}| \, dm_s^i.$$

To take the limit in $n$, one applies the dominated convergence theorem to the left-hand side and the Beppo-Levi (monotone convergence) theorem to the right. One then concludes that $\widetilde{N}_-^i \mathbb{1}_{\cup_n [\![ 0, S_n ]\!]}$ is $\psi(\widehat{V}^i)$ integrable as the limit on the left-hand side is finite. This implies that $\widetilde{N}_-^i$ is $\psi(\widehat{V}^i)$ integrable as the support of $dm^i$ is contained in $\cup_n [\![ 0, R_n ]\!]$ which is contained in $\cup_n [\![ 0, S_n ]\!]$ by (i) of Lemma 3.1.

We have shown that $\widetilde{N}_-^i \in \mathcal{L}^1(\psi(\widehat{V}^i))$ and therefore $\widetilde{N}_-^i * \psi(\widehat{V}^i)$ is a process such that for all $n \in \mathbb{N}^+$, we have $(\widetilde{N}_-^i * \psi(\widehat{V}^i))^{S_n} = (B^i)^{S_n}$. This implies $\widetilde{N}_-^i * \psi(\widehat{V}^i) = B^i$ on $\cup_n [\![ 0, S_n ]\!]$ and therefore on $\cup_n [\![ 0, R_n ]\!]$.

On the set $[\![ 0, R ]\!] \setminus \cup_n [\![ 0, R_n ]\!]$, one only needs to pay attention to the set $F = \{\forall n, R_n < R\} = \{\widetilde{N}_{R-}^i = 0\}$, this is because on the complement $F^c$, the set $[\![ 0, R ]\!] \setminus \cup_n [\![ 0, R_n ]\!]$ is empty. From Lemma 3.29 in He et al. [15], $R_F = R\mathbb{1}_F + \infty \mathbb{1}_{F^c}$ is a $\mathbb{G}$-predictable stopping time and on $F$, we have $[\![ 0, R ]\!] \setminus \cup_n [\![ 0, R_n ]\!] = [\![ R_F ]\!]$. From the fact that $\Delta(M\widetilde{N}^i)_{R_F} = 0$, Lemma 8.8 of [15] shows that $|\Delta B_{R_F}^i| = 0$. On the other hand, from (3.2), we deduce that the measure $dm^i$, which is absolutely continuous with respect to $\psi(\widehat{V}_+^i) + \psi(\widehat{V}_-^i)$ has no mass at $[\![ R_F ]\!]$ and therefore $\widetilde{N}_-^i * \psi(\widehat{V}^i)$ does not jump at $[\![ R_F ]\!]$. This implies that the jumps of the processes $B^i$ and $\widetilde{N}_-^i * \psi(\widehat{V}^i)$ coincide on $[\![ 0, R ]\!] \setminus \cup_n [\![ 0, R_n ]\!]$. □

We conclude this section with the following lemma, which is very useful when one wants to compute the $\mathbb{G}$-drift in practice.

**Lemma 4.2** *For $1 \leq i \leq k$, the process $\widetilde{N}_-^i * \psi(\widehat{V}^i)$ is the $\mathbb{G}$-dual predictable projection of $\widehat{V}^i$ and $N_-^i \centerdot V^i$ is the $\mathbb{F}^i$-drift of the $\mathbb{F}^i$-special semimartingale $N^i M$ multiplied by $\mathbb{1}_{D_i}$.*

*Proof* Similarly to the proof of Theorem 4.1, for a fixed $i = 1, \ldots, k$ and every $n \in \mathbb{N}^+$, we define the $\mathbb{G}$-stopping time $R_n = \inf\{t \geq 0, \widetilde{N}_t^i \leq 1/n\}$ and the $\mathbb{G}$-stopping time $R = \sup_n R_n = \inf\{t \geq 0, \widetilde{N}_{t-}^i \widetilde{N}_t^i = 0\}$.

Let $(T_n)_{n \in \mathbb{N}^+}$ be a localizing sequence of $\widehat{\mathbb{F}}$-stopping times (and $(S_n)_{n \in \mathbb{N}^+}$ the corresponding $\mathbb{G}$-stopping times on $D_i$ from Lemma 3.1) such that the process $(\widehat{M}^i)^{T_n}$ is an $\widehat{\mathbb{F}}$-martingale (the process $\widehat{M}^i$ is defined in (4.1)) and $(\widehat{V}^i)^{T_n}$ is of bounded total variation.

The method of the proof is to identify the process $(\widehat{V}^i)^{p,\mathbb{G}}$ with $\widetilde{N}^i_- * \psi(\widehat{V}^i)$. By taking the $\mathbb{G}$-optional projection in (4.2), we obtain for every $i = 1, \ldots, k$,

$$\begin{aligned} M^{S_n}\widetilde{N}^i \overset{\mathbb{G}-\text{mart}}{=} {}^{o,\mathbb{G}}(\mathbb{1}_{[\![0,T_n]\!]} \centerdot \widehat{V}^i \mathbb{1}_{D_i}) &\overset{\mathbb{G}-\text{mart}}{=} (\mathbb{1}_{[\![0,T_n]\!]} \centerdot \widehat{V}^i \mathbb{1}_{D_i})^{p,\mathbb{G}} \\ &= (\mathbb{1}_{[\![0,S_n]\!]} \centerdot \widehat{V}^i \mathbb{1}_{D_i})^{p,\mathbb{G}} \\ &= \mathbb{1}_{[\![0,S_n]\!]} \centerdot (\widehat{V}^i \mathbb{1}_{D_i})^{p,\mathbb{G}}, \end{aligned}$$

where the second and last equality follow from Corollary 5.31 and Corollary 5.24 in [15] respectively. From the result of Theorem 4.1 and the uniqueness of the $\mathbb{G}$-special semimartingale decomposition, the process $\widetilde{N}^i_- * \psi(\widehat{V}^i)$ is equal to $(\widehat{V}^i)^{p,\mathbb{G}}$ on $[\![0, S_n]\!]$ for all $n$ and therefore on $\cup_n [\![0, R_n]\!]$. On the complement $(\cup_n [\![0, R_n]\!])^c$, for every $C \in \mathcal{F}$ and every bounded $\mathbb{G}$-predictable process $\xi$, by duality we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_C(\xi \mathbb{1}_{(\cup_n [\![0,R_n]\!])^c} \centerdot (\widehat{V}^i \mathbb{1}_{D_i})^{p,\mathbb{G}})) = \mathbb{E}_{\mathbb{P}}(({}^{p,\mathbb{G}}(\mathbb{1}_C)\xi \mathbb{1}_{\cap_n ]\!]R_n,\infty[\![} \centerdot \widehat{V}^i \mathbb{1}_{D_i})) = 0$$

where the last equality holds, since on the set $D_i$, $\sup_n R_n = \infty$. This implies $(\widehat{V}^i)^{p,\mathbb{G}}$ also coincide with $\widetilde{N}^i_- * \psi(\widehat{V}^i)$ on $(\cup_n [\![0, R_n]\!])^c$ and therefore everywhere.

For every $i = 1, \ldots, k$, to compute the $\mathbb{F}^i$-drift of the $\mathbb{F}^i$-special semimartingale $N^i M$, it is sufficient to apply the $\mathbb{F}^i$-integration by parts formula to $N^i M = N^i(M - \Gamma^i(M)) + N^i \Gamma^i(M)$ to obtain

$$N^i M \overset{\mathbb{F}^i-\text{mart}}{=} \langle N^i, M - \Gamma^i(M)\rangle^i + N^i_- \centerdot \Gamma^i(M) \overset{\mathbb{F}^i-\text{mart}}{=} \langle N^i, M\rangle^i + N^i_- \centerdot \Gamma^i(M),$$

where the second equality follows from Yœurp's lemma. One can now conclude from the uniqueness of the $\mathbb{F}^i$-special semimartingale decomposition. □

*Remark 4.3* Rather than computing $\Gamma^i(M)$ and $\langle N^i, M\rangle^i$, Lemma 4.2 allows one to obtain the process $\widehat{V}^i$ through computing $N^i$ and the $\mathbb{F}^i$-drift of $N^i M$. This result is later used in Theorem 5.21.

## 5 Application to Multiple Random Times

Our goal in this section is to apply the methodology developed in the previous sections to credit risk modeling with multiple random times. In particular, we show that the abstract framework can be applied to the case where the default event is triggered by the first-to-default, *k-th*-to-default, *k*-out-of-*n*-to-default or the all-to-default event.

### 5.1 Enlargements with random times and their re-ordering

Let $\boldsymbol{\xi}$ be a $[0, \infty]$-valued random function defined on a non empty set $J \subset \mathbb{N}^+$, that is $\boldsymbol{\xi} = (\xi_j : j \in J)$ and for $s \geq 0$, we set $\boldsymbol{\xi} \wedge s := (\xi_j \wedge s : j \in J)$.

**Definition 5.1** The progressive enlargement of $\mathbb{F}$ with $\boldsymbol{\xi}$ is denoted by $\mathbb{F}^{\boldsymbol{\xi}} = (\mathcal{F}_t^{\boldsymbol{\xi}})_{t\geq 0}$; this is the smallest filtration satisfying the usual conditions containing $\mathbb{F}$ and making $\xi_j, j \in J$, stopping times. In other terms,

$$\mathcal{F}_t^{\boldsymbol{\xi}} := \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\boldsymbol{\xi} \wedge s)\right), \ t \in \mathbb{R}_+$$

**Definition 5.2** The initial enlargement of $\mathbb{F}$ with the family of random times $\boldsymbol{\xi} = (\xi_j : j \in J)$ is denoted by $\mathbb{F}^{\sigma(\boldsymbol{\xi})} := (\mathcal{F}_t^{\sigma(\boldsymbol{\xi})})_{t\geq 0}$; this is the smallest filtration containing $\mathbb{F}$, satisfying the usual conditions, such that the random function $\boldsymbol{\xi}$ is $\mathcal{F}_0^{\sigma(\boldsymbol{\xi})}$-measurable. One has

$$\mathcal{F}_t^{\sigma(\boldsymbol{\xi})} = \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\boldsymbol{\xi})\right), \ t \in \mathbb{R}_+.$$

In the following, we study the progressive enlargement of the reference filtration $\mathbb{F}$ with the non-decreasing re-ordering of a family of default times or random times denoted by $\mathbb{G}$. Under the assumption that the $\mathcal{F}_t$-conditional distribution of the random times are absolutely continuous with respect to a non-atomic measure, we show in Theorem 5.21, that the $\mathbb{G}$-drift of the $\mathbb{F}$-adapted martingale $M$ can be explicitly computed. In addition, the $\mathbb{G}$-drift can be represented as the *weighted average* of the drifts of $M$ in the progressive enlargements of $\mathbb{F}$ with every possible order for which the defaults can occur. We also find explicitly the weights, which can potentially be useful in practice.

**Lemma 5.3** *Suppose that $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$ are two $[0, \infty]$ valued random functions on the set $J$. If the measurable set $D$ is a subset of $\{\boldsymbol{\tau} = \boldsymbol{\xi}\}$, then for all $t \geq 0$, we have $\mathcal{F}_t^{\boldsymbol{\tau}} \cap D = \mathcal{F}_t^{\boldsymbol{\xi}} \cap D$.*

*Proof* By the symmetry of the problem, it is enough to show that for all $t \geq 0$, we have $\mathcal{F}_t^{\boldsymbol{\tau}} \cap D \subset \mathcal{F}_t^{\boldsymbol{\xi}} \cap D$. Fix $t \geq 0$, then for any $A \in \mathcal{F}_t^{\boldsymbol{\tau}}$ and all $n \geq 1$, there exists an $\mathcal{F}_{t+1/n} \otimes \mathcal{B}(\mathbb{R}^{\#J})$-measurable function $g_n$ such that $\mathbb{1}_A = g_n(\boldsymbol{\tau} \wedge (t + 1/n))$. From this we have

$$\mathbb{1}_A \mathbb{1}_D = g_n(\boldsymbol{\tau} \wedge (t + 1/n))\mathbb{1}_D = g_n(\boldsymbol{\xi} \wedge (t + 1/n))\mathbb{1}_D$$

and then

$$A \cap D = \{\lim_{n\to\infty} g_n(\boldsymbol{\xi} \wedge (t + 1/n)) = 1\} \cap D$$

whilst $\{\lim_{n\to\infty} g_n(\boldsymbol{\xi} \wedge (t + 1/n)) = 1\}$ is $\mathcal{F}_t^{\boldsymbol{\xi}}$-measurable by right continuity of the filtration $\mathbb{F}^{\boldsymbol{\xi}}$. □

To re-arrange the random times in non-decreasing order, we introduce an ordering method. For any function $\mathbf{x} = (x_j : j \in J)$ on $J \subset \mathbb{N}^+$, we consider a partial order $\prec$ on $(x_j : j \in J)$ defined by $x_i \prec x_j$ if $x_i < x_j$ or if $x_i = x_j$ and $i < j \in J$. We say that $x_i$ is smaller than $x_j$ if $x_i \prec x_j$ and for $k \in \{1, \ldots, \#J\}$, we denote by $x_{(k)}$, the $k$-th smallest element in the ordered set $((x_j : j \in J), \prec)$. We then set $\widetilde{\mathbf{x}} := (x_{(1)}, \ldots, x_{(\#J)})$ to be the non-decreasing re-ordering of $\mathbf{x}$ and notice that as vectors, $\widetilde{\mathbf{x}}$ and $\mathbf{x}$ are equal up to a permutation (that depends on $\mathbf{x}$) which we denote by $\widetilde{\pi}$ and write $\widetilde{\pi}(\mathbf{x}) = \widetilde{\mathbf{x}}$. Of course, for random functions on $J$, the re-ordering is done $\omega$ by $\omega$.

### 5.2 Hypothesis $(H')$ for ordered random times

We introduce some notations. For a fixed $n \in \mathbb{N}^+$, we denote the set $\{1, \dots, n\}$ by $(1{:}n)$ and the set of all injections from $(1{:}k)$ into $(1{:}n)$ by $\mathtt{inj}(k,n)$. For aesthetic purposes, we will drop the bracket in $(1{:}n)$ and write $1{:}n$, when it appears in a superscript or in a subscript.

For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and an injective map $\varrho \in \mathtt{inj}(k,n)$, we set $\mathbf{x}_\varrho := (x_{\varrho(j)} : j \in (1{:}k))$ and we also write $\varrho(\mathbf{x}) \equiv \mathbf{x}_\varrho$. For a subset $J \subset (1{:}n)$, we set $\mathbf{x}_J := (x_j : j \in J)$ and denote the projection map $\mathbf{x} \to \mathbf{x}_J$ defined on $\mathbb{R}^n$ by $\mathtt{P}_J$. In particular, for $i = 1, \dots, n$, the projection map $\mathbf{x} \to x_i$ is denoted by $\mathtt{P}_i$. These notations will be applied to random functions on $(1{:}n)$.

In the rest of the paper, we consider the $n$-dimensional vector of random times $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ and its non-decreasing re-ordering $\widetilde{\boldsymbol{\tau}} = (\tau_{(1)}, \dots, \tau_{(n)})$ as random functions on $(1{:}n)$. Using the notation introduced above, for any $k \le n$, we have $\widetilde{\boldsymbol{\tau}}_{1:k} = (\tau_{(1)}, \dots, \tau_{(k)})$, which represents the first $k$ ordered random times. In other words, the first $k$ ordered times $\widetilde{\boldsymbol{\tau}}_{1:k}$ can be obtained by first permuting, $\omega$ by $\omega$, the non-ordered times $\boldsymbol{\tau}$ into ordered times $\widetilde{\boldsymbol{\tau}}$ and then projecting it onto $(1{:}k)$.

The aim of this section is to apply the results established in the first part of the paper to investigate the hypothesis $(H')$ between $\mathbb{F}$ and $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$, for given $1 \le k \le n$. The intuition is the following. As the random times $(\tau_1, \dots, \tau_n)$ are not ordered, to study the hypothesis $(H')$ between $\mathbb{F}$ and $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$, we must consider every permutation or all possible ordering of $k$ random times from $(\tau_1, \dots, \tau_n)$. This is done by introducing for all $\varrho \in \mathtt{inj}(k,n)$, the sets

$$D^*_\varrho := \bigcap_{i=1}^{k} \{\tau_{(i)} = \tau_{\varrho(i)}\} = \{\widetilde{\boldsymbol{\tau}}_{1:k} = \boldsymbol{\tau}_\varrho\}. \tag{5.1}$$

Note that the family of sets $\{D^*_\varrho\}_{\varrho \in \mathtt{inj}(k,n)}$ may not be pairwise disjoint, however one always can find a partition $\{D_\varrho\}_{\varrho \in \mathtt{inj}(k,n)}$ of $\Omega$ such that $D_\varrho \subseteq D^*_\varrho$. By using this partition, the study of non-ordered random times is effectively reduced to the study of ordered random times of size $k$ on each partition $D_\varrho$ (which can be empty), and the result on each partition is then aggregated to obtain the drift operator in $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$. To applied the machinery developed in the first part of the paper, we note the following result.

**Corollary 5.4** *Suppose* $\mathbb{F} \hookrightarrow^{H'} \mathbb{F}^{\tau_\varrho}$ *holds for all* $\varrho \in \mathtt{inj}(k,n)$ *then the triplet* $(\mathbb{F}, (\mathbb{F}^{\boldsymbol{\tau}_\varrho})_{\varrho \in \mathtt{inj}(k,n)}, \mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}})$ *satisfies Assumption 1.1 with respect to the partition* $\{D_\varrho\}_{\varrho \in \mathtt{inj}(k,n)}$.

*Proof* It is sufficient to note that $D_\varrho \subseteq D^*_\varrho$ and apply Lemma 5.3. □

Given the family $(\mathbb{F}^{\boldsymbol{\tau}_\varrho})_{\varrho \in \mathtt{inj}(k,n)}$ and the partition $\{D_\varrho\}_{\varrho \in \mathtt{inj}(k,n)}$, we define the direct sum filtration $\widehat{\mathbb{F}}^{\widetilde{\boldsymbol{\tau}}_{1:k}} = (\widehat{\mathcal{F}}_t^{\widetilde{\boldsymbol{\tau}}_{1:k}})_{t \ge 0}$ by setting for every $t \ge 0$, (see formula (1.2)),

$$\widehat{\mathcal{F}}_t^{\widetilde{\boldsymbol{\tau}}_{1:k}} := \{A \in \mathcal{F} \,|\, \forall \varrho \in \mathtt{inj}(k,n),\, \exists A_t^\varrho \in \mathcal{F}_t^{\boldsymbol{\tau}_\varrho} \text{ such that } A \cap D_\varrho = A_t^\varrho \cap D_\varrho\}.$$

The $\widehat{\mathbb{F}}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$ and $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-semimartingale decompositions of $\mathbb{F}$-martingales are now readily available.

**Lemma 5.5** *Let $M$ be an $\mathbb{F}$-martingale such that for every $\varrho \in \mathtt{inj}(k,n)$, it is an $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-semimartingale with $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-semimartingale decomposition of $M$ is given by $M = M^\varrho + \Gamma^\varrho(M)$, where $M^\varrho$ is an $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-local martingale and $\Gamma^\varrho(M)$ is an $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-predictable process of finite variation. We set*

$$\begin{aligned}
\widetilde{N}^\varrho &:= {}^{o,\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}}(\mathbb{1}_{D_\varrho}) \\
N^\varrho &:= {}^{o,\mathbb{F}^{\boldsymbol{\tau}_\varrho}}(\mathbb{1}_{D_\varrho}) \\
\widehat{V}^\varrho &:= \Gamma^\varrho(M)\mathbb{1}_{D_\varrho} + \frac{\mathbb{1}_{D_\varrho}}{N^\varrho_-} \centerdot \langle N^\varrho, M\rangle^\varrho
\end{aligned}$$

*where the predictable bracket $\langle .,.\rangle^\varrho$ is computed with respect to $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$. Then*
*(i) the $\widehat{\mathbb{F}}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of $M$ is given by*

$$\sum_{\varrho \in \mathtt{inj}(k,n)} \widehat{V}^\varrho,$$

*(ii) the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of $M$ is given by*

$$\sum_{\varrho \in \mathtt{inj}(k,n)} \widetilde{N}^\varrho_- * \psi(\widehat{V}^\varrho)$$

*where the operations $\psi$ and $*$ are defined in Lemma 3.2 and equality (3.5) respectively.*

### 5.3 Computation under the density hypothesis

From Lemma 4.2, we see that in order to compute the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of an $\mathbb{F}$-martingale $M$, one can compute the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-dual predictable projection of the process

$$\widehat{V}^\varrho = \Gamma^\varrho(M)\mathbb{1}_{D_\varrho} + \frac{\mathbb{1}_{D_\varrho}}{N^\varrho_-} \centerdot \langle N^\varrho, M\rangle^\varrho,$$

which requires computing $N^\varrho$ and the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-drift of $N^\varrho M$ (see Remark 4.3). In order to make the computations as explicit as possible, we will work under the *density hypothesis* (Assumption 5.6) and compute fully explicitly the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-conditional expectation, the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-drift, the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-conditional expectation and the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift. We point out that some of the computations are quite involved and therefore we shall postponed them to the appendix.

In the recent works of El-Karoui et al. [11,12], the authors have also considered multiple default times under the density hypothesis (Jacod's criteria). However the goals of [11,12] are very different compared to the current paper as the authors do not consider hypothesis $(H')$ and do not compute the semimartingale decomposition in the enlarged filtration. In [11], for pricing purposes, the authors have focused their computational efforts on the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-conditional expectation, the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-intensity of default and the characterization of $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-martingales in the case ordered default times and have only commented on the link between the density of non-ordered default times and ordered default times (see section 5.1 within [11] or (5.7) below). While in [12], the authors studied similar problems to that of [11] and the form of the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$ and $\mathbb{F}^{\boldsymbol{\tau}_{\varrho}}$-conditional expectations for non-ordered default times was stated without computation in (26) within and much of the computational effort was devoted to the characterization of $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-martingales.

**Assumption 5.6** Density Hypothesis: For $t \in \mathbb{R}_+$, let $\nu_t$ denote the regular conditional law of the vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ with respect to the $\sigma$-algebra $\mathcal{F}_t$. Assume that there exists a non-atomic $\sigma$-finite measure $\mu$ on $\mathbb{R}_+$ and a non negative $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $a_t(\omega, \mathbf{x})$ such that $\nu_t$ has $a_t$ as density function with respect to $\mu^{\otimes n}$, i.e., for any non-negative Borel function $h$ on $\mathbb{R}_+^n$, for $t \in \mathbb{R}_+$, we have $\mathbb{P}$-almost surely,

$$\mathbb{E}_{\mathbb{P}}(h(\boldsymbol{\tau}) \,|\, \mathcal{F}_t) = \int_{\mathbb{R}_+^n} h(\mathbf{x}) a_t(\mathbf{x}) \mu^{\otimes n}(d\mathbf{x}). \tag{5.2}$$

The function $a_t$ will be called the conditional density function at $t$. Under this hypothesis, the probability that $\tau_i = \tau_j$ for $i \neq j$ is zero. In the context of credit risk modeling, this implies that we exclude the possibility of simultaneous defaults.

The choice of the non-atomic product measure $\mu^{\otimes n}(d\mathbf{x})$ is taken to ease the already complex notations and aid certain computations. The fact that the product measure is symmetric is used to simplify integration against permuted coordinates. Also the non-atomic assumption implies that the family of sets $\{D^*_\varrho\}_{\varrho \in \mathtt{inj}(k,n)}$ defined in (5.1) is almost surely pairwise disjoint and therefore $D_\varrho = D^*_\varrho$ up to a set of probability zero. This fact is used in the proof of Lemma 5.18 and Theorem 5.21.

*Remark 5.7* According to Lemma 1.8 in [17], the density function $a_t(\omega, \mathbf{x})$ can be chosen everywhere càdlàg in $t \in \mathbb{R}_+$, with however a rougher measurability: $a_t$ being $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^n))$-measurable. Moreover, for $\nu_t$ almost every $\mathbf{x}$, the process $a_t(\mathbf{x}), t \in \mathbb{R}_+$, is an $\mathbb{F}$-martingale under $\mathbb{P}$. In this section, we will take this càdlàg version of the density function. Without loss of the generality, we assume that almost everywhere

$$\int_{\mathbb{R}_+^n} a_t(\mathbf{x}) \mu^{\otimes n}(d\mathbf{x}) = 1, \quad t \in \mathbb{R}_+.$$

The identity (5.2) in Assumption 5.6 remains valid for any bounded $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^n))$-measurable function.

**Lemma 5.8** *For any* $t \in \mathbb{R}_+$ *and any* $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^n))$*-measurable bounded function* $H$ *on* $\Omega \times \mathbb{R}_+^n$*, we have*

$$\mathbb{E}_{\mathbb{P}}(H(\boldsymbol{\tau}) \,|\, \mathcal{F}_t) = \int_{\mathbb{R}_+^n} a_t(\mathbf{x}) H(\mathbf{x}) \mu^{\otimes n}(d\mathbf{x}).$$

*Proof* For any $s > t$, $H$ is $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable. Let $B \in \mathcal{F}_t$, then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_B H(\boldsymbol{\tau})) &= \mathbb{E}_{\mathbb{P}}\big(\int_{\mathbb{R}_+^n} \mathbb{1}_B H(\mathbf{x}) a_s(\mathbf{x}) \mu^{\otimes n}(d\mathbf{x})\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B \int_{\mathbb{R}_+^n} H(\mathbf{x}) a_t(\mathbf{x}) \mu^{\otimes n}(d\mathbf{x})\big)\end{aligned}$$

because, for every $\mathbf{x}$, the process $a_t(\mathbf{x}), t \in \mathbb{R}_+$, is an $\mathbb{F}$-martingale, and $H(\mathbf{x})$ is $\mathcal{F}_t$-measurable, thanks to the right-continuity of $\mathbb{F}$. The lemma follows from the fact that $\int_{\mathbb{R}_+^n} H(\mathbf{x}) a_t(\mathbf{x}) \mu^{\otimes n}(d\mathbf{x})$ also is $\mathcal{F}_t$-measurable. □

We introduce the following system of notations to help us integrate with respect to a subset of coordinates. Let $J \subset (1{:}n)$ with cardinal $\#J = k$. For vectors $\mathbf{z} \in \mathbb{R}_+^k$ and $\mathbf{y} \in \mathbb{R}_+^{n-k}$, we define a vector $\mathbf{z} \triangledown_J \mathbf{y}$ in $\mathbb{R}_+^n$ by:

$$(\mathbf{z} \triangledown_J \mathbf{y})_j = \begin{cases} z_i, & \text{if } j \text{ is the } i\text{-th element of } J \text{ in its natural order} \\ y_i, & \text{if } j \text{ is the } i\text{-th element of } J^c \text{ in its natural order.} \end{cases}$$

In other words, the $n$-dimensional vector $\mathbf{z} \triangledown_J \mathbf{y}$ is created by placing entries of $\mathbf{z}$ in the coordinates described by $J$ and entries of $\mathbf{y}$ in coordinates described by $J^c$, while keeping the order for which they have naturally appeared in $\mathbf{z}$ and $\mathbf{y}$ respectively. To better illustrate this, suppose $n = 4$, $J = (2, 4) \subset (1, 2, 3, 4)$, $\mathbf{z} = (z_1, z_2)$ and $\mathbf{y} = (y_1, y_2)$ then $\mathbf{z} \triangledown_J \mathbf{y} = (y_1, z_1, y_2, z_2)$.

For non-negative Borel function $g$, we denote

$$g_J(\mathbf{z}) = \int_{\mathbb{R}_+^{n-k}} g(\mathbf{z} \triangledown_J \mathbf{y}) \mu^{\otimes(n-k)}(d\mathbf{y}).$$

The function $g_J : \mathbb{R}^k \to \mathbb{R}$ is computed by integrating $g(\mathbf{x}) = g(x_1, \ldots, x_n)$ only with respect to those coordinates $x_j$ for which $j \in J^c$. By taking $g = a_t$ in the above expression, we obtain the family $(a_t)_J, t \in \mathbb{R}_+$ given by

$$(a_t)_J(\mathbf{z}) = \int_{\mathbb{R}_+^{n-k}} a_t(\mathbf{z} \triangledown_J \mathbf{y}) \mu^{\otimes(n-k)}(d\mathbf{y}),$$

which is the conditional density functions of the random times $\boldsymbol{\tau}_J$. Following Lemma 1.8 in Jacod [17], $(a_t)_J$ admits a càdlàg version that we denote by $a_{J,t}, t \in \mathbb{R}_+$. Notice that, we have for all $t \in \mathbb{R}_+$, for almost all $\omega$, $(a_t)_J(\omega, \mathbf{x}) = a_{J,t}(\omega, \mathbf{x})$, $\mu^{\otimes n}$-almost everywhere. Using this notation, the conditional expectation of $g$ under $\nu_t(\omega)$ given $\mathtt{P}_J = \mathbf{z}$ is

$$\mathbb{E}^{\nu_t}(g \,|\, \mathtt{P}_J = \mathbf{z}) = \frac{(g a_t)_J(\mathbf{z})}{a_{J,t}(\mathbf{z})} \mathbb{1}_{\{a_{J,t}(\mathbf{z}) > 0\}}.$$

*5.3.1 Computing the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-conditional expectation*

In this subsection, we fix $\varrho \in \mathtt{inj}(k,n)$ and compute the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-conditional expectation. In the case of progressive enlargement of $\mathbb{F}$ with a single random time $\tau$, the progressive enlargement coincides with $\mathbb{F}$ before $\tau$ and with the initial enlargement after $\tau$. One can then split the computations into before and after $\tau$. The idea here is similar, however, unlike the case of a single random time, the random times $\boldsymbol{\tau}_\varrho$ are not ordered and the computations are not so straightforward, because without a good system of notation they quickly become intractable as one must consider every possible ordering of $\boldsymbol{\tau}_\varrho$.

To overcome the above-mentioned issues, in the following, we will develop systematically the necessary notations and techniques to perform explicit computations. For $a, b \in [0,\infty]$, we set

$$a \nmid b = \begin{cases} a & \text{if } a \le b, \\ \infty & \text{if } a > b. \end{cases}$$

and we write $\boldsymbol{\tau}_\varrho \nmid b$ for the random map $j \in J \to \tau_{\varrho(j)} \nmid b$ and consider $\sigma(\boldsymbol{\tau} \nmid b) := \sigma(\tau_j \nmid b : j \in J)$. We let $T \subset (1{:}k)$ with $j = \#T$. Note that for $t \in \mathbb{R}_+$, the $\sigma$-algebra $\mathcal{F}_t^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}$ is generated by $h(\boldsymbol{\tau}_{\varrho(T)})$ where $h$ runs over the family of all bounded functions on $\Omega \times \mathbb{R}_+^j$ which are $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^j))$-measurable. For $\mathbf{x} \in \mathbb{R}_+^n$, and $t \in \mathbb{R}_+$, we set

$$\begin{aligned} U_{T,\varrho}(\mathbf{x}) &:= \max_{i \in T} x_{\varrho(i)}, \\ L_{T,\varrho}(\mathbf{x}) &:= \min_{i \in 1:k\setminus T} x_{\varrho(i)}, \\ \mathtt{A}_{t,T,\varrho} &:= \{\mathbf{x} \in \mathbb{R}_+^n : U_{T,\varrho}(\mathbf{x}) \le t < L_{T,\varrho}(\mathbf{x})\}. \end{aligned}$$

The random times $U_{T,\varrho}(\boldsymbol{\tau})$ and $L_{T,\varrho}(\boldsymbol{\tau})$ are $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-stopping times. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ be the non-increasing re-ordering of $\boldsymbol{\tau}_\varrho$. We see that for $T \subset (1{:}k)$ and $j = 0, \dots, k$, the intervals $[\![ U_{T,\varrho}(\boldsymbol{\tau}), L_{T,\varrho}(\boldsymbol{\tau}) [\![$ are disjoint and

$$\bigcup_{T:\,\#T=j} \{U_{T,\varrho}(\boldsymbol{\tau}) \le t < L_{T,\varrho}(\boldsymbol{\tau})\} = \{\gamma_j \le t < \gamma_{j+1}\}. \tag{5.3}$$

Note that if $j = 0$, then $T = \emptyset$, $L_{\emptyset,\varrho}(\boldsymbol{\tau}) = \gamma_1$ and as convention, we set $\gamma_0 = 0$, $U_{\emptyset,\varrho}(\boldsymbol{\tau}) := 0$, $L_{1:k,\varrho}(\boldsymbol{\tau}) = \infty$ and $\gamma_{k+1} = \infty$. Recall (see for example [21]) that, for any random time $S$, $\mathcal{F}_S$ (resp. $\mathcal{F}_{S-}$) denotes the $\sigma$-algebra generated by $K_S$, where $K$ describes the set of $\mathbb{F}$-optional (resp. predictable) processes.

**Lemma 5.9** *Under Assumption 5.6, for any bounded $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-stopping time $S$,*

$$\mathcal{F}_S^{\boldsymbol{\tau}_\varrho} = \mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_\varrho \nmid S). \tag{5.4}$$

*For any subset $T$ of $(1{:}k)$, the process $\mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau}), t \in \mathbb{R}_+$, is $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-optional and*

$$\begin{aligned} \mathcal{F}_S^{\boldsymbol{\tau}_\varrho} \cap \{\boldsymbol{\tau} \in \mathtt{A}_{S,T,\varrho}\} &= (\mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_\varrho \nmid S)) \cap \{\boldsymbol{\tau} \in \mathtt{A}_{S,T,\varrho}\} \\ &= \big(\mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_{\varrho(T)})\big) \cap \{\boldsymbol{\tau} \in \mathtt{A}_{S,T,\varrho}\}. \end{aligned} \tag{5.5}$$

*Proof* See appendix. □

Let us point out that the key result of Lemma 5.9 is the equalities in (5.5). From (5.5) we see that the computation of the $\mathcal{F}_S^{\boldsymbol{\tau}_\varrho}$-conditional expectation on the set $\{\boldsymbol{\tau} \in \mathtt{A}_{S,T,\varrho}\}$ is reduced to the computation of the $\mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_{\varrho(T)})$-conditional expectation, which is given in Lemma 5.10. The $\mathcal{F}_S^{\boldsymbol{\tau}_\varrho}$-conditional expectation is obtained in Lemma 5.13 by aggregating the computations over the disjoint sets $\{\boldsymbol{\tau} \in \mathtt{A}_{S,T,\varrho}\}$ where $T$ runs over all possible subsets of $(1{:}k)$.

**Lemma 5.10** *For any bounded $\mathbb{F}^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}$-stopping time $S$ and any non-negative $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^n))$-measurable function $g$, we have*

$$\mathbb{E}_{\mathbb{P}}\big(g(\boldsymbol{\tau}) \,|\, \mathcal{F}_S^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}\big) = \frac{(ga_S)_{\varrho(T)}\big(\boldsymbol{\tau}_{\varrho(T)}\big)}{a_{\varrho(T),S}\big(\boldsymbol{\tau}_{\varrho(T)}\big)} \mathbb{1}_{\{a_{\varrho(T),S}(\boldsymbol{\tau}_{\varrho(T)})>0\}}.$$

*If $S$ is a bounded $\mathbb{F}^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}$-predictable stopping time, we also have*

$$\mathbb{E}_{\mathbb{P}}\big(g(\boldsymbol{\tau}) \,|\, \mathcal{F}_{S-}^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}\big) = \frac{(ga_{U-})_{\varrho(T)}\big(\boldsymbol{\tau}_{\varrho(T)}\big)}{a_{\varrho(T),S-}\big(\boldsymbol{\tau}_{\varrho(T)}\big)} \mathbb{1}_{\{a_{\varrho(T),S-}(\boldsymbol{\tau}_{\varrho(T)})>0\}}.$$

*Proof* See appendix. □

In the above formulae, we can remove the indicator $\mathbb{1}_{\{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})>0\}}$, because by Corollary 1.11 in [17] the process $\mathbb{1}_{\{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})=0\}}$ is evanescent.

**Corollary 5.11** *For any Borel function $g$ on $\mathbb{R}_+^n$ such that $g(\boldsymbol{\tau})$ is integrable, the process $(ga_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})$, $t \in \mathbb{R}_+$ is càdlàg and its left limit is the process $(ga_{t-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)}), t \in \mathbb{R}_+$.*

*Remark 5.12* Corollary 5.11 implies that $(a_t)_{\varrho(T)}, t \in \mathbb{R}_+$ is càdlàg so that the process $(a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)}), t \in \mathbb{R}_+$ coincides with $a_{t,\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)}), t \in \mathbb{R}_+$. It is an important property in practice (for example, numerical implantation) because it gives a concrete way to compute $a_{t,\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})$.

**Lemma 5.13** *For any bounded $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-stopping time $S$ and any non-negative $\mathcal{F}_S \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function $g$, we have*

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau}) \,|\, \mathcal{F}_S^{\boldsymbol{\tau}_\varrho}) = \sum_{T \subset 1:k} \mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau}) \le S < L_{T,\varrho}(\boldsymbol{\tau})\}} \frac{(\mathbb{1}_{\{S<L_{T,\varrho}\}} ga_S)_{\varrho(T)}\big(\boldsymbol{\tau}_{\varrho(T)}\big)}{(\mathbb{1}_{\{S<L_{T,\varrho}\}} a_S)_{\varrho(T)}\big(\boldsymbol{\tau}_{\varrho(T)}\big)}.$$

*Proof* See appendix. □

**Corollary 5.14** *For any bounded $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-predictable stopping time $S$ and any non-negative $\mathcal{F}_{S-} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function $g$, we have*

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau}) | \mathcal{F}_{S-}^{\boldsymbol{\tau}_\varrho}) = \sum_{T \subset 1:k} \mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau}) < S \le L_{T,\varrho}(\boldsymbol{\tau})\}} \frac{(\mathbb{1}_{\{S \le L_{T,\varrho}\}} ga_{S-})_{\varrho(T)}\big(\boldsymbol{\tau}_{\varrho(T)}\big)}{(\mathbb{1}_{\{S \le L_{T,\varrho}\}} a_{S-})_{\varrho(T)}\big(\boldsymbol{\tau}_{\varrho(T)}\big)}.$$

*5.3.2 The $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-semimartingale decomposition*

This subsection is devoted to the computation of the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-drift of $N^\varrho M$. For simplicity, we assume from now that $M$ is a bounded $\mathbb{F}$-martingale (we expect the following results to hold for any locally square integrable $\mathbb{F}$-martingale). For a bounded Borel function $g$ on $\mathbb{R}_+^n$, we set $\forall t \in \mathbb{R}_+$

$$Y_t^g := \mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})|\mathcal{F}_t^{\boldsymbol{\tau}_\varrho}),$$

and in particular to compute $N^\varrho$, we will take $g(\boldsymbol{\tau}) = \mathbb{1}_{D_\varrho}$. Notice that a computational formula for $Y^g$ is already available in Lemma 5.13, however this formula is not adapted to the computation that we will do in this subsection. From Theorem 2.5 in [17], there exists a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable process $u_v^M(\omega, \mathbf{x})$ such that

$$d\langle a(\mathbf{x}), M\rangle_v = u_v^M(\mathbf{x}) d\langle M, M\rangle_v, \tag{5.6}$$

where the predictable bracket $\langle M, M\rangle$ is calculated in the filtration $\mathbb{F}$. The process $u^M(\mathbf{x})$ is known to satisfy

$$\int_0^t \frac{1}{a_{s-}(\boldsymbol{\tau})} |u_s^M(\boldsymbol{\tau})| d\langle M, M\rangle_s < \infty, \ \forall t \in \mathbb{R}_+,$$

(so that we assume that $u_s^M(\mathbf{x}) = 0$ whenever $a_{s-}(\mathbf{x}) = 0$). However, the computations below will require a stronger condition.

**Assumption 5.15** There exists an increasing sequence $(R_n)_{n\in\mathbb{N}^+}$ of bounded $\mathbb{F}$-stopping times converging to infinity such that

$$\mathbb{E}_{\mathbb{P}}\big(\int_0^{R_n} \frac{1}{a_{s-}(\boldsymbol{\tau})} |u_s^M(\boldsymbol{\tau})| d\langle M, M\rangle_s\big) < \infty, \ \forall n \in \mathbb{N}^+.$$

Notice that the above inequality is equivalent to

$$\mathbb{E}_{\mathbb{P}}\big(\int\int_0^{R_n} |u_s^M(\mathbf{x})| d\langle M, M\rangle_s \mu^{\otimes n}(d\mathbf{x})\big) < \infty.$$

We give a sufficient condition for Assumption 5.15 to hold.

**Lemma 5.16** *Suppose that there exists an increasing sequence $(R_n)_{n\in\mathbb{N}^+}$ of bounded $\mathbb{F}$-stopping times converging to infinity such that*

$$\int \mathbb{E}_{\mathbb{P}}\big(\sqrt{[a(\mathbf{x}), a(\mathbf{x})]_{R_n}}\,\big)\mu^{\otimes n}(d\mathbf{x}) < \infty.$$

*Then, for any bounded $\mathbb{F}$-martingale $M$, we have for all $n \in \mathbb{N}^+$*

$$\mathbb{E}_{\mathbb{P}}\big(\int\int_0^{R_n} |u_s^M(\mathbf{x})| d\langle M, M\rangle_s \mu^{\otimes n}(d\mathbf{x})\big) < \infty.$$

*Proof* See appendix. □

**Theorem 5.17** *Let $M$ be a bounded $\mathbb{F}$-martingale, then under Assumption 5.15, for any bounded Borel function $g$ on $\mathbb{R}_+^n$, the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-drift of $Y^g M$ is*

$$\sum_{T\subset 1:k}\int_0^t \mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})<v\le L_{T,\varrho}(\boldsymbol{\tau})\}}\frac{(\mathbb{1}_{\{v\le L_{T,\varrho}\}}gu_v^M)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\{v\le L_{T,\varrho}\}}a_{v-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}d\langle M,M\rangle_v,\ t\in\mathbb{R}_+,$$

*where $U_{T,\varrho}(\mathbf{x}) := \max_{i\in T} x_{\varrho(i)}$ and $L_{T,\varrho}(\mathbf{x}) := \min_{i\in 1:k\setminus T} x_{\varrho(i)}$.*

*Proof* The result follows from direct computation and Lemma 5.13. See appendix for details. □

*5.3.3 Computing the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-conditional expectation*

In this subsection, we compute the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-conditional expectation, which is used to compute the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-dual predictable projection.

Let us recall that for $\mathbf{x}\in\mathbb{R}_+^n$, the vector $\widetilde{\mathbf{x}}$ is the non-decreasing re-ordering of $\mathbf{x}$ and the map $\widetilde{\pi}:\mathbf{x}\to\widetilde{\mathbf{x}}$ maps $\mathbf{x}$ into this non-decreasing ordering. Let $\mathfrak{S}$ be the symmetric group on the set $(1{:}n)$. For any Borel function $g$, the symmetrization of $g$ is denoted by

$$\overline{g}(\mathbf{x}) := \sum_{\pi\in\mathfrak{S}} g(\pi(\mathbf{x})),$$

and the relationships $\pi^{-1}\pi(\mathbf{x})=\mathbf{x}$, $\pi^{-1}(\mu^{\otimes n})=\mu^{\otimes n}$ and $\overline{g}(\pi(\mathbf{x}))=\overline{g}(\mathbf{x})$ hold. In particular, we have $\overline{g}(\mathbf{x})=\overline{g}(\widetilde{\pi}(\mathbf{x}))$.

**Lemma 5.18** *For any $t\in\mathbb{R}_+$ and any non-negative $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^n)$-measurable function $g$, we have*

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})\,|\,\mathcal{F}_t^{\sigma(\widetilde{\boldsymbol{\tau}})}) = \frac{\overline{ga_t}(\widetilde{\boldsymbol{\tau}})}{\overline{a_t}(\widetilde{\boldsymbol{\tau}})}\mathbb{1}_{\{\overline{a_t}(\widetilde{\boldsymbol{\tau}})>0\}}.$$

*Proof* See appendix. □

*Remark 5.19* The result of Lemma 5.18 is equivalent to

$$\mathbb{E}^{\nu_t}(g\,|\,\widetilde{\pi}=\mathbf{x}) = \frac{\overline{ga_t}(\mathbf{x})}{\overline{a_t}(\mathbf{x})}\mathbb{1}_{\{\overline{a_t}(\mathbf{x})>0\}},$$

and an useful consequence of the computations in Lemma 5.18 is that, under Assumption 5.6, the ordered times $\widetilde{\boldsymbol{\tau}}$ satisfy also the density hypothesis with respect to $\mu^{\otimes n}$, and the density function for $t\in\mathbb{R}_+$ is given by

$$\widetilde{a}_t(\mathbf{x}) = \mathbb{1}_{\{x_1<x_2<\cdots<x_n\}}\overline{a_t}(\mathbf{x}). \tag{5.7}$$

In the following, we apply the results of subsection 5.3.1 to the first $k$ smallest times $\widetilde{\boldsymbol{\tau}}_{1:k}=(\tau_{(1)},\ldots,\tau_{(k)})$. For notational simplicity, we set

$$\Pi(g)(\mathbf{x}) := \frac{\overline{ga_t}(\mathbf{x})}{\overline{a_t}(\mathbf{x})}\mathbb{1}_{\{\overline{a_t}(\mathbf{x})>0\}}$$

and recall that $\mathtt{P}_i$ is the projection map on $\mathbb{R}^n$, such that $\mathtt{P}_i(\mathbf{x})=x_i$.

**Lemma 5.20** *For any bounded $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-stopping time $S$ and any non-negative $\mathcal{F}_S \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function $g$, we have*

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau}) \,|\, \mathcal{F}_S^{\widetilde{\boldsymbol{\tau}}_{1:k}}) = \sum_{j=0}^{k} \mathbb{1}_{\{\tau_{(j)} \le S < \tau_{(j+1)}\}} \frac{(\mathbb{1}_{\{S < \mathtt{P}_{j+1}\}} \Pi(g) \widetilde{a}_S)_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})}{(\mathbb{1}_{\{S < \mathtt{P}_{j+1}\}} \widetilde{a}_S)_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})},$$

*where $\tau_{(0)} = 0$ and $\tau_{(k+1)} = \infty$. For any bounded $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-predictable stopping time $S$ and any non-negative $\mathcal{F}_{S-} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function $g$, we have*

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau}) \,|\, \mathcal{F}_{S-}^{\widetilde{\boldsymbol{\tau}}_{1:k}}) = \sum_{j=0}^{k} \mathbb{1}_{\{\tau_{(j)} < S \le \tau_{(j+1)}\}} \frac{(\mathbb{1}_{\{S \le \mathtt{P}_{j+1}\}} \Pi(g) \widetilde{a}_{S-})_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})}{(\mathbb{1}_{\{S \le \mathtt{P}_{j+1}\}} \widetilde{a}_{S-})_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})}, \quad (5.8)$$

*where $\tau_{(0)} = 0$ and $\tau_{(k+1)} = \infty$.*

*Proof* By conditioning with respect to $\mathbb{F}^{\sigma(\widetilde{\boldsymbol{\tau}})}$ and using the equality in Lemma 5.18, we have

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau}) \,|\, \mathcal{F}_S^{\widetilde{\boldsymbol{\tau}}_{1:k}}) = \mathbb{E}_{\mathbb{P}}(\Pi(g)(\widetilde{\boldsymbol{\tau}}) \,|\, \mathcal{F}_S^{\widetilde{\boldsymbol{\tau}}_{1:k}}).$$

One can now apply Lemma 5.13 to the ordered times $\widetilde{\boldsymbol{\tau}}$ with density $\widetilde{a}_t$ given in (5.7) and $\varrho \in \mathtt{inj}(k,n)$ taken to be the map $i_{1:k} : j \mapsto j$. To further simplify the result, we observe that $\mathbb{1}_{\mathtt{A}_{t,T,i_{1:k}}}(\widetilde{\boldsymbol{\tau}}) \neq 0$, only if $T$ is of the form $T = (1{:}j)$ for $0 \le j \le k$ ($T = \emptyset$ if $j = 0$). The claim of the lemma then follows as

$$\mathbb{1}_{\mathtt{A}_{t,1:j,i_{1:k}}}(\widetilde{\boldsymbol{\tau}}) = \mathbb{1}_{\{\tau_{(j)} \le t < \tau_{(j+1)}\}},$$

and on $\{x_1 < x_2 < \cdots < x_n\}$, we have $L_{1:j,i_{1:k}}(\mathbf{x}) = x_{j+1} = \mathtt{P}_{j+1}(\mathbf{x})$. □

*5.3.4 The $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-semimartingale decomposition*

Before presenting our final result, we first give some discussions on the change of measure approach to computing the drift operator in the enlarged filtration. It is well known in the enlargement of filtration literature, e.g., in Jeulin and Yor [23], Yœurp [31] and Jacod [17], that there is a close relationship between the semimartingale decomposition in the enlarged filtration and the Girsanov transform. This relationship was systematically studied in Song [27] and the current paper can also be treated using the change of measure approach.

If the density $a_t$ of $\boldsymbol{\tau}$ is strictly positive, then it is straightforward to apply the Girsanov transform to prove hypothesis $(H')$. To see this, we suppose that $\mathcal{F}_0$ is the trivial $\sigma$-algebra and we note that the conditional density of $\widetilde{\boldsymbol{\tau}}_{1:k}$ is given by $\widetilde{a}_{1:k,t}$. For fixed $l > 0$, because $\widetilde{a}_{1:k,l}(\widetilde{\boldsymbol{\tau}}_{1:k})$ is strictly positive, one can define an equivalent probability measure $\mathbb{Q}_l$ on $\mathcal{F}_l \vee \sigma(\widetilde{\boldsymbol{\tau}}_{1:k})$ through

$$\frac{d\mathbb{Q}_l}{d\mathbb{P}} := \frac{\widetilde{a}_{1:k,0}(\widetilde{\boldsymbol{\tau}}_{1:k})}{\widetilde{a}_{1:k,l}(\widetilde{\boldsymbol{\tau}}_{1:k})}.$$

This implies that for $t \le l$, $\frac{d\mathbb{Q}_l}{d\mathbb{P}}|_{\mathcal{F}_t} = 1$ and under $\mathbb{Q}_l$, the random times $\widetilde{\boldsymbol{\tau}}_{1:k}$ are independent of $\mathcal{F}_l$. It follows that under $\mathbb{Q}_l$, every $\mathbb{F}$-martingale is again an $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-martingale (up to time $l$). Therefore an $(\mathbb{P}, \mathbb{F})$-martingale $M$ is an

$(\mathbb{Q}_l, \mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}})$-martingale and to retrieve the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of $M$, it is sufficient to set $Z_t := \frac{d\mathbb{P}}{d\mathbb{Q}_S}\big|_{\mathcal{F}_t^{\widetilde{\boldsymbol{\tau}}_{1:k}}}$ and apply the Girsanov transform to show that

$$M_t - \int_0^t \frac{1}{Z_s}\, d[M,Z]_s, \quad 0 \le t \le l$$

is $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-local martingale under $\mathbb{P}$.

However, the change of measure approach is only helpful on a conceptual level and does not substantially simplify the computation of the drift. It is evident from the above that in order to obtain an explicit expression for the drift, one must first compute the dynamic of $(Z_t)_{t\ge 0}$, which is very involved and requires computational results such as those of Lemma 5.20 and Lemma 5.13, and then compute the quadratic variation $[M,Z]$. Interested readers can compare our results in Lemma 5.13 and Lemma 5.13 with those stated in (24) and (26) in section 4.1 of [12], where the authors are working under strict positivity of the density, for ordered and non-ordered multi-defaults.

We present now the final result of the current paper in Theorem 5.21, where we compute fully explicitly the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of the bounded $\mathbb{F}$-martingale $M$. We first prepare the necessarily notations. In the following, we set

$$\mathtt{D}_\varrho := \{\mathbf{x} \in \mathbb{R}_+^n : x_{\varrho(1)} < \cdots < x_{\varrho(k)} < \min_{i \in 1:n \setminus \varrho(1:k)} x_i\,\}$$

and under Assumption 5.6, the set equality $\{\tau \in \mathtt{D}_\varrho\} = D_\varrho$ holds almost surely. We recall that $\mathtt{P}_i$ is the projection map $\mathbf{x} \to x_i$ on $\mathbb{R}_+^n$, and for simplicity, let

$$d\Gamma_v^{j,\varrho}(\boldsymbol{\tau}) := \frac{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_\varrho} u_v^M)_{\varrho(1:j)}(\widetilde{\boldsymbol{\tau}}_{1:j})}{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_\varrho} a_{v-})_{\varrho(1:j)}(\widetilde{\boldsymbol{\tau}}_{1:j})} d\langle M, M\rangle_v, \tag{5.9}$$

$$\Omega_v^{j,\varrho}(\boldsymbol{\tau}) := \frac{(\mathbb{1}_{\{v \le \mathtt{P}_{j+1}\}} \widetilde{a}^{\varrho}_{v-})_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})}{(\mathbb{1}_{\{v \le \mathtt{P}_{j+1}\}} \widetilde{a}_{v-})_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})} \tag{5.10}$$

where for $\mathbf{x} \in \mathbb{R}_+^n$

$$\widetilde{a}^{\varrho}_{v-}(\mathbf{x}) := \mathbb{1}_{\{x_1 < x_2 < \ldots < x_n\}} \sum_{\substack{\pi \in \mathfrak{S}:\, \forall i \in 1:k, \\ \pi \circ \varrho(i) = i}} a_{v-}(\pi(\mathbf{x})).$$

We remark that in the formula for the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of the bounded $\mathbb{F}$-martingale $M$ given below in Theorem 5.21, the driving process $\Gamma_v^{j,\varrho}(\boldsymbol{\tau})$ is coming from the $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-semimartingale decomposition of $M$ and it appears in the formula with weights given by $\Omega_v^{j,\varrho}(\boldsymbol{\tau})$.

**Theorem 5.21** *Let $M$ be a bounded $\mathbb{F}$-martingale, then under Assumption 5.15, the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of $M$ is given by*

$$\sum_{\varrho \in \mathtt{inj}(k,n)} \sum_{j=0}^{k} \int_0^t \mathbb{1}_{\{\tau_{(j)} < v \le \tau_{(j+1)}\}} \Omega_v^{j,\varrho}(\boldsymbol{\tau})\, d\Gamma_v^{j,\varrho}(\boldsymbol{\tau})$$

*where $\tau_{(0)} = 0$, $\tau_{(k+1)} = \infty$, and $\Gamma_v^{j,\varrho}(\boldsymbol{\tau})$ and $\Omega_v^{j,\varrho}(\boldsymbol{\tau})$ are given in* (5.9) *and* (5.10) *respectively.*

*Proof* According to Theorem 4.1, to obtain the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-drift of $M$, we need only to calculate the process $\widetilde{N}^{\varrho}_{-} * \psi(\widehat{V}^{\varrho})$ for $\varrho \in \mathtt{inj}(k,n)$ (see also Lemma 5.5 for notations), which by Lemma 4.2 is the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-dual predictable projection of $\widehat{V}^{\varrho}$ for $\varrho \in \mathtt{inj}(k,n)$. Therefore, we first compute $\widehat{V}^{\varrho}$ and then its $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-dual predictable projection. To do this, we first apply Lemma 4.2 to write

$$\widehat{V}^{\varrho} = \frac{\mathbb{1}_{D_{\varrho}}}{N^{\varrho}_{-}} \centerdot (N^{\varrho}_{-} \centerdot \widehat{V}^{\varrho}) = \frac{\mathbb{1}_{D_{\varrho}}}{N^{\varrho}_{-}} \centerdot (\mathbb{1}_{D_{\varrho}} A)$$

where $A$ denotes the $\mathbb{F}^{\boldsymbol{\tau}_{\varrho}}$-drift of $N^{\varrho}M$, which can be computed using Theorem 5.17 with $g = \mathbb{1}_{D_{\varrho}}$. To compute explicit the process $A$, we notice that in Theorem 5.17, the stopping times $U_{T,\varrho}(\boldsymbol{\tau})$ and $L_{T,\varrho}(\boldsymbol{\tau})$ can be simplified. To see this, we note that on the set $\mathtt{D}_{\varrho}$, the set

$$\mathtt{A}_{v,T,\varrho} = \{\mathbf{x} \in \mathbb{R}^n_+ : U_{T,\varrho}(\mathbf{x}_{\varrho}) < v \le L_{T,\varrho}(\mathbf{x}_{\varrho})\}$$

is non-empty only if $T = (1{:}j)$ for $j = 0, 1 \ldots, k$ (for $j = 0$, we set $T = \emptyset$ and $U_{\emptyset,\varrho} = x_{\varrho(0)} = x_{(0)} = 0$). Also on the set $\mathtt{D}_{\varrho}$, we have

$$\begin{aligned} U_{1:j,\varrho}(\mathbf{x}) &= x_{\varrho(j)} = x_{(j)} \\ L_{1:j,\varrho}(\mathbf{x}) &= x_{\varrho(j+1)} = x_{(j+1)}, \end{aligned}$$

where by convention $L_{1:k,\varrho} = \infty$.

This shows that $U_{1:j,\varrho}$ and $L_{1:j,\varrho}$ are projection maps onto the $\varrho(j)$-th and $\varrho(j+1)$-th coordinates respectively. Consequently, for $t \in \mathbb{R}_+$, $\widehat{V}^{\varrho}_t$ is given by

$$\sum_{j=0}^{k} \int_0^t \frac{\mathbb{1}_{\mathtt{D}_{\varrho}}(\boldsymbol{\tau})}{N^{\varrho}_{v-}} \mathbb{1}_{\{\tau_{\varrho(j)} < v \le \tau_{\varrho(j+1)}\}} \frac{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_{\varrho}} u^M_v)_{\varrho(1:j)}(\boldsymbol{\tau}_{\varrho(1:j)})}{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} a_{v-})_{\varrho(1:j)}(\boldsymbol{\tau}_{\varrho(1:j)})} d\langle M, M\rangle_v.$$

The process $N^{\varrho}_{-}$ can then computed using Corollary 5.14. That is on $D_{\varrho}$,

$$\begin{aligned} N^{\varrho}_{v-} &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\mathtt{D}_{\varrho}}(\boldsymbol{\tau}) | \mathcal{F}^{\boldsymbol{\tau}_{\varrho}}_{v-}) \\ &= \sum_{T \subset 1:k} \mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau}) < v \le L_{T,\varrho}(\boldsymbol{\tau})\}} \frac{(\mathbb{1}_{\{v \le L_{T,\varrho}\}} \mathbb{1}_{\mathtt{D}_{\varrho}} a_{v-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\{v \le L_{T,\varrho}\}} a_{v-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})} \\ &= \sum_{j=0}^{k} \mathbb{1}_{\{\tau_{\varrho(j)} < v \le \tau_{\varrho(j+1)}\}} \frac{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_{\varrho}} a_{v-})_{\varrho(1:j)}(\boldsymbol{\tau}_{\varrho(1:j)})}{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} a_{v-})_{\varrho(1:j)}(\boldsymbol{\tau}_{\varrho(1:j)})}. \end{aligned}$$

This yields for $t \in \mathbb{R}_+$, the following expression for $\widehat{V}^{\varrho}_t$,

$$\begin{aligned} &\sum_{j=0}^{k} \int_0^t \mathbb{1}_{\{\tau_{\varrho(j)} < v \le \tau_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_{\varrho}}(\boldsymbol{\tau}) \frac{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_{\varrho}} u^M_v)_{\varrho(1:j)}(\boldsymbol{\tau}_{\varrho(1:j)})}{(\mathbb{1}_{\{v \le \mathtt{P}_{\varrho(j+1)}\}} \mathbb{1}_{\mathtt{D}_{\varrho}} a_{v-})_{\varrho(1:j)}(\boldsymbol{\tau}_{\varrho(1:j)})} d\langle M, M\rangle_v \\ &=: \sum_{j=0}^{k} \int_0^t \mathbb{1}_{\{\tau_{(j)} < v \le \tau_{(j+1)}\}} \phi_{v,j,\varrho}(\boldsymbol{\tau}) d\langle M, M\rangle_v. \end{aligned}$$

where $\tau_{(0)} = 0$, $\tau_{(k+1)} = \infty$. The next step is to compute the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-dual predictable projection of $\widehat{V}^{\varrho}$. Since the process $\langle M, M\rangle$ is $\mathbb{F}$-predictable, it is enough to compute the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-predictable projection of the integrand $\phi_{v,j,\varrho}(\boldsymbol{\tau})$ (see [15, Theorem 5.25]), by taking $g = \phi_{v,j,\varrho}$ in Lemma 5.20. The numerator terms in (5.8) can be computed as follows,

$$
\begin{aligned}
&\Pi(\phi_{v,j,\varrho})\,\widetilde{a}_{v-}(\mathbf{x})\\
&= \frac{\overline{\phi_{v,j,\varrho}a_{v-}}(\mathbf{x})}{\overline{a_{v-}}(\mathbf{x})}\mathbb{1}_{\{\overline{a_{v-}}(\mathbf{x})>0\}}\mathbb{1}_{\{x_1<x_2<\ldots<x_n\}}\overline{a_{v-}}(\mathbf{x})\\
&= \overline{\phi_{v,j,\varrho}a_{v-}}(\mathbf{x})\mathbb{1}_{\{x_1<x_2<\ldots<x_n\}} =\\
&\sum_{\pi\in\mathfrak{S}}\mathbb{1}_{\mathtt{D}_\varrho}(\pi(\mathbf{x}))\frac{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}u_v^M)_{\varrho(1:j)}(\pi(\mathbf{x})_{\varrho(1:j)})}{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}a_{v-})_{\varrho(1:j)}(\pi(\mathbf{x})_{\varrho(1:j)})}a_{v-}(\pi(\mathbf{x}))\mathbb{1}_{\{x_1<x_2<\ldots<x_n\}}.
\end{aligned}
$$

We note that the intersection of the set $\{\mathbf{x}\in\mathbb{R}_+^k : \pi(\mathbf{x})\in\mathtt{D}_\varrho\}$ and the set $\{\mathbf{x}\in\mathbb{R}_+^k : x_1 < x_2 < \cdots < x_n\}$ is non-empty only if $\pi\in\mathfrak{S}$ and is such that $\forall i\in(1\!:\!k), \pi\circ\varrho(i) = i$. Using this observation, we can continue with the above equalities and write

$$
\begin{aligned}
&= \sum_{\substack{\pi\in\mathfrak{S}:\,\forall i\in 1:k,\\ \pi\circ\varrho(i)=i}}\frac{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}u_v^M)_{\varrho(1:j)}(\mathbf{x}_{1:j})}{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}a_{v-})_{\varrho(1:j)}(\mathbf{x}_{1:j})}a_{v-}(\pi(\mathbf{x}))\mathbb{1}_{\{x_1<x_2<\ldots<x_n\}}\\
&= \frac{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}u_v^M)_{\varrho(1:j)}(\mathbf{x}_{1:j})}{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}a_{v-})_{\varrho(1:j)}(\mathbf{x}_{1:j})}\mathbb{1}_{\{x_1<x_2<\ldots<x_n\}}\sum_{\substack{\pi\in\mathfrak{S}:\,\forall i\in 1:k,\\ \pi\circ\varrho(i)=i}}a_{v-}(\pi(\mathbf{x}))\\
&= \frac{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}u_v^M)_{\varrho(1:j)}(\mathbf{x}_{1:j})}{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}a_{v-})_{\varrho(1:j)}(\mathbf{x}_{1:j})}\,\widetilde{a}^{\varrho}_{v-}(\mathbf{x}).
\end{aligned}
$$

Finally, by inserting the above into (5.8), we obtain the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-predictable projection of the integrand $\phi_{v,j,\varrho}$ and consequently the $\mathbb{F}^{\widetilde{\boldsymbol{\tau}}_{1:k}}$-dual predictable projection of $\widehat{V}^{\varrho}$ is given by

$$
\begin{aligned}
&\sum_{j=0}^{k}\int_0^t \mathbb{1}_{\{\tau_{(j)}<v\le\tau_{(j+1)}\}}\frac{(\mathbb{1}_{\{v\le \mathtt{P}_{j+1}\}}\widetilde{a}^{\varrho}_{v-})_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})}{(\mathbb{1}_{\{v\le \mathtt{P}_{j+1}\}}\widetilde{a}_{v-})_{1:j}(\widetilde{\boldsymbol{\tau}}_{1:j})}\\
&\qquad\times\frac{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}u_v^M)_{\varrho(1:j)}(\widetilde{\boldsymbol{\tau}}_{1:j}))}{(\mathbb{1}_{\{v\le \mathtt{P}_{\varrho(j+1)}\}}\mathbb{1}_{\mathtt{D}_\varrho}a_{v-})_{\varrho(1:j)}(\widetilde{\boldsymbol{\tau}}_{1:j}))}d\langle M, M\rangle_v, \qquad \forall t\in\mathbb{R}_+.
\end{aligned}
$$

□

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# A Appendix

We present here the proofs of Lemma 5.9, Lemma 5.10, Lemma 5.13, Lemma 5.16, Theorem 5.17 and Lemma 5.18.

### Proof of Lemma 5.9

*Proof* Notice that we can write

$$\{\boldsymbol{\tau} \in \mathtt{A}_{t,T,\varrho}\} = \{\forall i \in T, \tau_{\varrho(i)} \nmid t < \infty,\ \forall i \in (1\!:\!k) \setminus T,\ \tau_{\varrho(i)} \nmid t = \infty\},$$

which is a $\sigma(\boldsymbol{\tau}_\varrho \nmid t)$-measurable set. Hence, the process $\mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})$ for $t \in \mathbb{R}_+$ is $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-adapted. It is also càdlàg and therefore $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-optional.

The density hypothesis with respect to $\mathbb{F}$ holds for $\boldsymbol{\tau}_\varrho$, since it holds for $\boldsymbol{\tau}$. It is proved in [29, Theorem 6.9] that the optional splitting formula holds with respect to $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$ (see Definition 6.3 in [29]). As a consequence, for $0 \le j \le k$,

$$\begin{aligned}\mathcal{F}_S^{\boldsymbol{\tau}_\varrho} \cap \{\gamma_j \le S < \gamma_{j+1}\} &= \Big(\mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_\varrho \nmid \gamma_j)\Big) \cap \{\gamma_j \le S < \gamma_{j+1}\} \\ &= \Big(\mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_\varrho \nmid S)\Big) \cap \{\gamma_j \le S < \gamma_{j+1}\}.\end{aligned}$$

From (5.3), we see that

$$\mathbb{1}_{\{\gamma_j \le S < \gamma_{j+1}\}} = \sum_{\substack{T \subset 1:k, \\ \#T = j}} \mathbb{1}_{\mathtt{A}_{S,T,\varrho}}(\boldsymbol{\tau})$$

is $\mathcal{F}_S^{\boldsymbol{\tau}_\varrho}$ as well as $\mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_\varrho \nmid S)$-measurable. Hence, we can take the union of the above identities to conclude

$$\mathcal{F}_S^{\boldsymbol{\tau}_\varrho} = \mathcal{F}_S \vee \sigma(\boldsymbol{\tau}_\varrho \nmid S).$$

If the cardinal of $T$ is equal to $j$, the last claim of the lemma follows from the above identities together with the fact that

$$\sigma(\boldsymbol{\tau}_\varrho \nmid S) \cap \mathtt{A}_{S,T,\varrho} = \sigma(\boldsymbol{\tau}_{\varrho(T)}) \cap \mathtt{A}_{S,T,\varrho}.$$

□

### Proof of Lemma 5.10

*Proof* By monotone convergence theorem we only need to prove the lemma for $0 \le g \le 1$. Let us show firstly the lemma for a constant stopping time $S = t \in \mathbb{R}_+$. For a bounded function $h$ on $\Omega \times \mathbb{R}_+^j$ which is $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^j))$-measurable, according to Lemma 5.8,

$$\begin{aligned}\mathbb{E}_\mathbb{P}(g(\boldsymbol{\tau})h(\boldsymbol{\tau}_{\varrho(T)})) &= \mathbb{E}_\mathbb{P}\big(\int_{\mathbb{R}_+^n} g(\mathbf{x})h(\mathbf{x}_{\varrho(T)})a_t(\mathbf{x})\mu^{\otimes n}(d\mathbf{x})\big) \\ &= \mathbb{E}_\mathbb{P}\Big(\frac{(ga_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})}\mathbb{1}_{\{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})>0\}}h(\boldsymbol{\tau}_{\varrho(T)})\Big),\end{aligned}$$

where the last equality comes from Lemma 5.8 applied with respect to $\boldsymbol{\tau}_{\varrho(T)}$.

The formula is proved for $S = t$, because $\frac{(ga_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})}\mathbb{1}_{\{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})>0\}}$ is $\mathcal{F}_t^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}$-measurable. Next, for any $n \in \mathbb{N}^+$ and $(\omega, \mathbf{x}) \in \Omega \times \mathbb{R}_+^n$, we let

$$R^n(\omega, \mathbf{x}) = \inf\{s \in \mathbb{Q}_+ : a_s(\omega, \mathbf{x}) > n\}.$$

Then, for $b > 0$, we have

$$\{R^n \geq b\} = \{(\omega, \mathbf{x}) \in \Omega \times \mathbb{R}_+^n : \forall s \in \mathbb{Q}_+ \cap [0, b), a_s(\omega, \mathbf{x}) \leq n\} \in \mathcal{F}_{b-} \otimes \mathcal{B}(\mathbb{R}_+^n).$$

By applying the above formula to $g(\boldsymbol{\tau})\mathbb{1}_{\{t<R^n(\boldsymbol{\tau})\}}$ at constant time $t > 0$, we can write

$$\mathbb{E}_{\mathbb{P}}\big(g(\boldsymbol{\tau})\mathbb{1}_{\{t<R^n(\boldsymbol{\tau})\}}|\mathcal{F}_t^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}\big) = \frac{(g\mathbb{1}_{\{t<R^n\}}a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})}\mathbb{1}_{\{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})>0\}}.$$

Note that $g\mathbb{1}_{\{t<R^n\}}a_t \leq n$. By the dominated convergence theorem, for almost all $\omega$, the map

$$t \to (g\mathbb{1}_{\{t<R^n\}}a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)}),$$

is right-continuous. By Lemma 1.8 in [17], the process $\mathbb{1}_{\{a_{\varrho(T),t}(\boldsymbol{\tau}_{\varrho(T)})>0\}}$ is also right-continuous. Hence, the above formula can be extended to any bounded $\mathbb{F}^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}$-stopping time $S$, that is

$$\mathbb{E}_{\mathbb{P}}\big(g(\boldsymbol{\tau})\mathbb{1}_{\{S<R^n(\boldsymbol{\tau})\}}|\mathcal{F}_S^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}\big) = \frac{(g\mathbb{1}_{\{S<R^n\}}a_S)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{a_{\varrho(T),S}(\boldsymbol{\tau}_{\varrho(T)})}\mathbb{1}_{\{a_{\varrho(T),S}(\boldsymbol{\tau}_{\varrho(T)})>0\}}.$$

Note that, by dominated convergence theorem, $(g\mathbb{1}_{\{t<R^n\}}a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})$ has left limit given by $(g\mathbb{1}_{\{t\leq R^n\}}a_{t-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})$. If $S$ is a bounded $\mathbb{F}^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}$-predictable stopping time, we also have

$$\mathbb{E}_{\mathbb{P}}\big(g(\boldsymbol{\tau})\mathbb{1}_{\{S\leq R^n(\boldsymbol{\tau})\}}|\mathcal{F}_{S-}^{\sigma(\boldsymbol{\tau}_{\varrho(T)})}\big) = \frac{(g\mathbb{1}_{\{S\leq R^n\}}a_{S-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{a_{\varrho(T),S-}(\boldsymbol{\tau}_{\varrho(T)})} \times \mathbb{1}_{\{a_{\varrho(T),S-}(\boldsymbol{\tau}_{\varrho(T)})>0\}}.$$

We can now conclude by letting $n \uparrow \infty$. □

### Proof of Lemma 5.13

*Proof* By monotone class theorem, we only need to prove the lemma for bounded Borel functions $g$ on $\mathbb{R}_+^n$. Let us firstly consider $S = t \in \mathbb{R}_+$ and we introduce the set $\mathtt{F}_{t,T,\varrho} = \{\mathbf{x} \in \mathbb{R}_+^n : L_{T,\varrho}(\mathbf{x}) > t\}$.

For any bounded $\mathcal{B}(\mathbb{R}^k)$-measurable function $h$, there exists a $\mathcal{B}(\mathbb{R}^{\#T})$-measurable function $h'$ such that $h(\boldsymbol{\tau}_\varrho \nmid t) = h'(\boldsymbol{\tau}_{\varrho(T)})$ on $\mathtt{A}_{t,T,\varrho}$. Let $B \in \mathcal{F}_t$, we compute

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_B h(\boldsymbol{\tau}_\varrho \nmid t)\mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})\mathbb{E}\big(g(\boldsymbol{\tau})\,|\,\mathcal{F}_t^{\boldsymbol{\tau}_\varrho}\big)\right)\\
&\quad = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_B h'(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})\leq t\}}\mathbb{1}_{\{L_{T,\varrho}(\boldsymbol{\tau})>t\}}g(\boldsymbol{\tau})\right)\\
&\quad = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_B h'(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})\leq t\}}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{L_{T,\varrho}>t\}}g(\boldsymbol{\tau})|\mathcal{F}_t^{\sigma(\boldsymbol{\tau}_{\varrho(T)})})\right)
\end{aligned}$$

and by Lemma 5.10 and Remark 5.12, we have

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_B h'(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})\le t\}}\frac{(\mathbb{1}_{\mathtt{F}} g a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}\\
&\qquad \times \mathbb{1}_{\{(a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})>0\}}\mathbb{1}_{\{(\mathbb{1}_{\mathtt{F}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})>0\}}\Big)
\end{aligned}
$$

and again by Lemma 5.10,

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_B h'(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})\le t\}}\mathbb{1}_{\{L_{T,\varrho}(\boldsymbol{\tau})>t\}}\frac{(a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\mathtt{F}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}\\
&\qquad \times \frac{(\mathbb{1}_{\mathtt{F}} g a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}\mathbb{1}_{\{(a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})>0\}}\mathbb{1}_{\{(\mathbb{1}_{\mathtt{F}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})>0\}}\Big)\\
&= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_B h(\boldsymbol{\tau}_{\varrho} \nmid t)\mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})\frac{(\mathbb{1}_{\mathtt{F}} g a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\mathtt{F}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}\Big).
\end{aligned}
$$

Note that the random variables

$$
\mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau}) \text{ and } \mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})\frac{(\mathbb{1}_{\mathtt{F}} g a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\mathtt{F}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}
$$

are $\mathcal{F}_t^{\boldsymbol{\tau}_\varrho} = \mathcal{F}_t \vee \sigma(\boldsymbol{\tau}_\varrho \nmid t)$-measurable. By Lemma 5.9, the above computation implies that

$$
\mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})\,|\,\mathcal{F}_t^{\boldsymbol{\tau}_\varrho}) = \mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})\frac{(\mathbb{1}_{\mathtt{F}} g a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\mathtt{F}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}.
$$

Recall that $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$ is the increasing re-ordering of $\boldsymbol{\tau}_\varrho$ and $\gamma_0 = 0, \gamma_{k+1} = \infty$. Notice also that, under Assumption 5.6, $\mathbb{P}(\tau_i = \tau_j) = 0$ for any pair of $i, j$ such that $i \neq j$. It results that

$$
\sum_{\substack{T\subset 1:k,\\ \#T=j}} \mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau}) = \mathbb{1}_{\{\gamma_j \le t < \gamma_{j+1}\}}, \tag{A.1}
$$

and by using the above,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})\,|\,\mathcal{F}_t^{\boldsymbol{\tau}_\varrho}) &= \sum_{j=0}^{k} \mathbb{1}_{\{\gamma_j \le t < \gamma_{j+1}\}}\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})\,|\,\mathcal{F}_t^{\boldsymbol{\tau}_\varrho})\\
&= \sum_{j=0}^{k}\sum_{\substack{T\subset 1:k,\\ \#T=j}} \mathbb{1}_{\mathtt{A}_{t,T,\varrho}}(\boldsymbol{\tau})\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})\,|\,\mathcal{F}_t^{\boldsymbol{\tau}_\varrho})\\
&= \sum_{T\subset 1:k} \mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})\le t < L_{T,\varrho}(\boldsymbol{\tau})\}}\frac{(\mathbb{1}_{\{t<L_{T,\varrho}\}} g a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\{t<L_{T,\varrho}\}} a_t)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}.
\end{aligned}
$$

Applying Corollary 5.11, we can extend this formula to any bounded $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-stopping time $S$. □

**Proof of Lemma 5.16**

*Proof* Let $H_s(\mathbf{x}) = \mathtt{sign}(u_s(\mathbf{x}))$. We have

$$\begin{aligned}&\int \mathbb{E}_{\mathbb{P}}\Big(\int_0^{R_n} |d[a(\mathbf{x}), H(\mathbf{x}) \centerdot M]_s|\Big)\mu^{\otimes n}(d\mathbf{x})\\&\leq \int \sqrt{2}\mathbb{E}_{\mathbb{P}}\big([a(\mathbf{x}), a(\mathbf{x})]_{R_n}^{1/2}\big)\|H(\mathbf{x}) \centerdot M\|_{\mathtt{BMO}}\mu^{\otimes n}(d\mathbf{x}).\end{aligned}$$

We note that $\|H(\mathbf{x}) \centerdot M\|_{\mathtt{BMO}}$ is computed using its bracket (see [15, Theorem 10.9]) so that it is uniformly bounded by a multiple of $\|M_\infty\|_\infty$. This boundedness together with the assumption of the lemma enables us to write

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\Big(\int\int_0^{R_n} |u_s^M(\mathbf{x})| d\langle M, M\rangle_s \mu^{\otimes n}(d\mathbf{x})\Big)\\&= \int \mathbb{E}_{\mathbb{P}}\Big(\int_0^{R_n} H_s(\mathbf{x})u_s^M(\mathbf{x}) d\langle M, M\rangle_s\Big)\mu^{\otimes n}(d\mathbf{x})\\&= \int \mathbb{E}_{\mathbb{P}}\Big(\int_0^{R_n} d\langle a(\mathbf{x}), H(\mathbf{x}) \centerdot M\rangle_s\Big)\mu^{\otimes n}(d\mathbf{x})\\&= \int \mathbb{E}_{\mathbb{P}}\big([a(\mathbf{x}), H(\mathbf{x}) \centerdot M]_{R_n}\big)\,\mu^{\otimes n}(d\mathbf{x}) < \infty.\end{aligned}$$

□

**Proof of Theorem 5.17**

*Proof* Let $T \subset (1{:}k)$ and $j = \#T$. Let $R$ be one of $R_n$ in Assumption 5.15. We compute the following for $s, t \in \mathbb{R}_+$, $s \leq t$, $B \in \mathcal{F}_s$ and $h$ a bounded Borel function on $\mathbb{R}_+^j$. Note that Fubini's theorem can be applied because of the boundedness of $M$ and Assumption 5.15. Since $\varrho$, $T$, and $t \in \mathbb{R}_+$ are fixed, we simply write $U(\mathbf{x}) = U_{T,\varrho}(\mathbf{x})$ and $L(\mathbf{x}) = L_{T,\varrho}(\mathbf{x})$.

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\Big(\int_s^t \mathbb{1}_B h(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{U(\boldsymbol{\tau})<v\leq L(\boldsymbol{\tau})\}}\mathbb{1}_{\{v\leq R\}} d(Y^g M)_v\Big)\\&\quad= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_B h(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{s\vee U(\boldsymbol{\tau})<t\wedge L(\boldsymbol{\tau})\wedge R\}}\\&\qquad\qquad \times((Y^g M)_{t\wedge L(\boldsymbol{\tau})\wedge R} - (Y^g M)_{(s\vee U(\boldsymbol{\tau}))\wedge(t\wedge L(\boldsymbol{\tau})\wedge R)}))\end{aligned}$$

and by using the fact that $U_{T,\varrho}(\boldsymbol{\tau})$ and $L_{T,\varrho}(\boldsymbol{\tau})$ are $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-stopping times,

$$\begin{aligned}&\quad= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_B h(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{s\vee U(\boldsymbol{\tau})<t\wedge L(\boldsymbol{\tau})\wedge R\}} g(\boldsymbol{\tau})\\&\qquad\qquad \times(M_{t\wedge L(\boldsymbol{\tau}_\varrho)\wedge R} - M_{(s\vee U(\boldsymbol{\tau}))\wedge(t\wedge L(\boldsymbol{\tau})\wedge R)}))\\&\quad= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_B h(\boldsymbol{\tau}_{\varrho(T)}) g(\boldsymbol{\tau})(M_{t\wedge L(\boldsymbol{\tau})\wedge R} - M_{(s\vee U(\boldsymbol{\tau}))\wedge(t\wedge L(\boldsymbol{\tau})\wedge R)})).\end{aligned}$$

Using the the density hypothesis for $\boldsymbol{\tau}$, integration by parts formula and (5.6), the above is equal to

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}(\mathbb{1}_B \int h(\mathbf{x}_{\varrho(T)})g(\mathbf{x})(M_{t\wedge L(\mathbf{x})\wedge R} \\
&\qquad - M_{(s\vee U(\mathbf{x}))\wedge(t\wedge L(\mathbf{x})\wedge R)})a_t(\mathbf{x})\mu^{\otimes n}(d\mathbf{x})) \\
&= \int h(\mathbf{x}_{\varrho(T)})g(\mathbf{x})\mathbb{E}_{\mathbb{P}}(\mathbb{1}_B(M_{t\wedge L(\mathbf{x})\wedge R} \\
&\qquad - M_{(s\vee U(\mathbf{x}))\wedge(t\wedge L(\mathbf{x})\wedge R)})a_t(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}) \\
&= \int h(\mathbf{x}_{\varrho(T)})g(\mathbf{x})\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B \int_s^t \mathbb{1}_{\{U(\mathbf{x})<v\le L(\mathbf{x})\wedge R\}}u_v(\mathbf{x})d\langle M,M\rangle_v\big)\mu^{\otimes n}(d\mathbf{x})
\end{aligned}$$

Next, we apply Fubini's theorem to $\mu^{\otimes n}(d\mathbf{x}) \times d\langle M,M\rangle$ to obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B \int_s^t \Big(\int h(\mathbf{x}_{\varrho(T)})\mathbb{1}_{\{U(\mathbf{x})<v\le L(\mathbf{x})\wedge R\}}g(\mathbf{x})u_v(\mathbf{x})\mu^{\otimes n}(d\mathbf{x})\Big)d\langle M,M\rangle_v\big) \\
&= \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B \int_s^t \Big(\int h(\mathbf{x}_{\varrho(T)})\mathbb{1}_{\{U(\mathbf{x})<v\le L(\mathbf{x})\wedge R\}}\frac{(\mathbb{1}_{\{v\le L\}}gu_v^M)_{\varrho(T)}(\mathbf{x}_{\varrho(T)})}{(\mathbb{1}_{\{v\le L\}}a_{v-})_{\varrho(T)}(\mathbf{x}_{\varrho(T)})} \\
&\qquad \times a_{v-}(\mathbf{x})\mu^{\otimes n}(d\mathbf{x})\Big)d\langle M,M\rangle_v\big) \\
&= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_B \int_s^t h(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{U(\boldsymbol{\tau}_\varrho)<v\le L(\boldsymbol{\tau}_\varrho)\wedge R\}} \\
&\qquad \times \frac{(\mathbb{1}_{\{v\le L\}}gu_v^M)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\{v\le L\}}a_{v-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}d\langle M,M\rangle_v)
\end{aligned}$$

where the last equality follows from Lemma 1.10 in [17]. We notice that by letting $R = R_n$ tend to infinity, $\mathbb{1}_B h(\boldsymbol{\tau}_{\varrho(T)})\mathbb{1}_{\{s<v\le t\}}$ generate all bounded $\mathbb{F}^{\boldsymbol{\tau}_\varrho}$-predictable processes on the interval $]\!]\, U_{T,\varrho}(\boldsymbol{\tau}), L_{T,\varrho}(\boldsymbol{\tau})\,]\!]$. The above computations shows that the drift of the process $\mathbb{1}_{]\!] U_{T,\varrho}(\boldsymbol{\tau}),L_{T,\varrho}(\boldsymbol{\tau})]\!]} \centerdot (Y^g M)$ is

$$\int_0^t \mathbb{1}_{\{U_{T,\varrho}(\boldsymbol{\tau})<v\le L_{T,\varrho}(\boldsymbol{\tau})\}}\frac{(\mathbb{1}_{\{v\le L\}}gu_v^M)_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}{(\mathbb{1}_{\{v\le L\}}a_{v-})_{\varrho(T)}(\boldsymbol{\tau}_{\varrho(T)})}d\langle M,M\rangle_v,\ t\in\mathbb{R}_+.$$

The result then follows from (A.1) and $\sum_{T\subset 1:k}\mathbb{1}_{]\!] U_{T,\varrho}(\boldsymbol{\tau}),L_{T,\varrho}(\boldsymbol{\tau})]\!]} = \mathbb{1}_{(0,\infty]}$. □

**Proof of Lemma 5.18**

*Proof* For a non-negative $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $h$, we compute

$$\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})h(\widetilde{\pi}(\boldsymbol{\tau}))\,|\mathcal{F}_t\,) = \int_{\mathbb{R}_+^n}(ga_t)(\mathbf{x})h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}).$$

By using the fact that the measure $\mu$ is non-atomic and $\mu^{\otimes n}$ is symmetric, we proceed as follows

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} (ga_t)(\mathbf{x})h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}) \\
&= \sum_{\pi\in\mathfrak{S}} \int_{\mathbb{R}_+^n} (ga_t)(\mathbf{x})h(\pi(\mathbf{x}))\mathbb{1}_{\{x_{\pi(1)}<\ldots<x_{\pi(n)}\}}\mu^{\otimes n}(d\mathbf{x}) \\
&= \sum_{\pi\in\mathfrak{S}} \int_{\mathbb{R}_+^n} (ga_t)(\pi^{-1}(\mathbf{x}))h(\mathbf{x})\mathbb{1}_{\{x_1<\ldots<x_n\}}\mu^{\otimes n}(d\mathbf{x}) \\
&= \int_{\mathbb{R}_+^n} \overline{ga_t}(\mathbf{x})h(\mathbf{x})\mathbb{1}_{\{x_1<\ldots<x_n\}}\mu^{\otimes n}(d\mathbf{x}) \\
&= \frac{1}{n!}\int_{\mathbb{R}_+^n} \overline{ga_t}(\mathbf{x})h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}).
\end{aligned}$$

In particular, if $g = 1$,

$$\int_{\mathbb{R}_+^n} a_t(\mathbf{x})h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}) = \frac{1}{n!}\int_{\mathbb{R}_+^n} \overline{a_t}(\mathbf{x})h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}).$$

Continuing the computation using this property, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(g(\boldsymbol{\tau})h(\widetilde{\pi}(\boldsymbol{\tau}))\,|\,\mathcal{F}_t) &= \frac{1}{n!}\int_{\mathbb{R}_+^n} \overline{a_t}(\mathbf{x})\frac{\overline{ga_t}(\widetilde{\pi}(\mathbf{x}))}{\overline{a_t}(\widetilde{\pi}(\mathbf{x}))}\mathbb{1}_{\{\overline{a_t}(\widetilde{\pi}(\mathbf{x}))>0\}}h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}) \\
&= \int_{\mathbb{R}_+^n} a_t(\mathbf{x})\frac{\overline{ga_t}(\widetilde{\pi}(\mathbf{x}))}{\overline{a_t}(\widetilde{\pi}(\mathbf{x}))}\mathbb{1}_{\{\overline{a_t}(\widetilde{\pi}(\mathbf{x}))>0\}}h(\widetilde{\pi}(\mathbf{x}))\mu^{\otimes n}(d\mathbf{x}) \\
&= \mathbb{E}_{\mathbb{P}}\left(\frac{\overline{ga_t}(\widetilde{\pi}(\boldsymbol{\tau}))}{\overline{a_t}(\widetilde{\pi}(\boldsymbol{\tau}))}\mathbb{1}_{\{\overline{a_t}(\widetilde{\pi}(\boldsymbol{\tau}))>0\}}h(\widetilde{\pi}(\boldsymbol{\tau}))\,\middle|\,\mathcal{F}_t\right).
\end{aligned}$$

□

## References


1. Aksamit, A., Choulli, T. and Jeanblanc, M.: *Classification of random times and applications.* Working paper, arXiv:1605.03905v1, 2016.
2. Ankirchner, S. and Imkeller, P.: Financial markets with asymmetric information: information drift, additional utility and entropy. In: *Stochastic Processes and Applications to Mathematical Finance: Proceedings of the 6th Ritsumeikan International Symposium,* J. Akahori, S. Ogawa and S. Watanabe (eds), World Scientific. (2007)
3. Bielecki, T., Jeanblanc, M. and Rutkowski, M.: Pricing and trading credit default swaps in a hazard process model. *Ann. Appl. Probab.* 18, 6 (2008), pp 2495-2529.
4. Brigo, D, Pallavicini, A and Papatheodorou, V.: Arbitrage-Free Valuation of Bilateral Counterparty Risk for Interest-Rate Products: Impact of Volatilities and Correlations. *Int. J. Theor. Appl. Finance* 14, 773 (2011), pp 773–802.
5. Brigo, D and Capponi, A and Pallavicini, A.: Arbitrage-free Bilateral counterpart risk valuation under collateralization and application to credit default swaps. *Math. Finance* 24, 1 (2014), pp 125–146.

6. Callegaro, G. and Jeanblanc, M. and Zargari, B.: Carthaginian enlargement of filtrations. *ESAIM Probab. Stat.* 17, (2013), pp 550–566.
7. Choukroun, S. Goutte, S and Ngoupeyou, A.: Mean variance hedging under multiple defaults risk. *Stoch. Anal. Appl.* 33, 5 (2015), pp 757–791.
8. Crépey, S.: Bilateral counterparty risk under funding contraints-part I: Pricing. *Math. Finance* 25, 1 (2015), pp 1–22.
9. Crépey, S.: Bilateral counterparty risk under funding contraints-part II: CVA. *Math. Finance* 25, 1 (2015), pp 23–50.
10. Dellacherie, C. and Meyer, P.A.: *Probabilités et potentiel*, Chapitres V-VIII: Théorie des Martingales, Hermann (1992).
11. El-Karoui, N., Jeanblanc, M., and Jiao, Y. : Density approach in modeling successive defaults. *SIAM J. FinancialMath.* 6, 1, (2015), pp 1–21.
12. El-Karoui, N., Jeanblanc, M., and Jiao, Y. : Dynamics of multivariate default system in random environment. Accepted for publication in Stochastic Processes and their Applications 2017.
13. Föllmer, H and Protter, P.: Local martingales and filtration shrinkage. *ESAIM Probab. Stat.* 15, (2011), pp S25–S38.
14. Hillairet,C. and Jiao, Y.: Information asymmetry in pricing of credit derivatives, *Int. J. Theor. Appl. Finance* 14, 5 (2011), pp 611–633.
15. He, S.W., Wang, J.G. and Yan. J.A.: *Semimartingale Theory and Stochastic Calculus.* Chinese edition, Science Press, Beijing, 1995.
16. Jacod, J.: *Calcul stochastique et problèmes de martingales. Séminaire de Probabilités XII, Lecture Notes in Mathematics 714.* Springer, Berlin Heidelberg New York, 1979.
17. Jacod, J.: Grossissement initial, hypothèse $(H')$ et théorème de Girsanov. In: *Grossissements de Filtrations: Exemples et Applications. Lecture Notes in Mathematics 1118*, Th. Jeulin and M. Yor, eds. Springer, Berlin Heidelberg New York, 1985, pp 15–35.
18. Jeanblanc, M. and Le Cam, Y.: Progressive enlargement of filtration with initial times. *Stochastic Process. Appl.* 119, 8 (2009), pp 2523–2543.
19. Jeanblanc, M and Song, S.: Martingale representation property in progressively enlarged filtrations. *Stochastic Process. Appl.* 125, 11 (2015), pp 4242–4271.
20. Laurent, J.P. and Gregory, J.: Basket Default Swaps, CDO's and Factor Copulas. *J. risk.* 7, 4 (2005), pp 1–20.
21. Jeulin, T.: Grossissement d'une filtration et applications. In: *Séminaire de Probabilités XIII, Lecture Notes in Mathematics 721*, Springer, Berlin Heidelberg New York, 1979, pp. 574–609.
22. Jeulin, T.: *Semi-martingales et grossissement d'une filtration. Lecture Notes in Mathematics 833*, Springer, Berlin Heidelberg New York, 1980.
23. Jeulin, T. and Yor, M.: Nouveaux résultats sur le grossissement des tribus. *Ann. Sci. Éc. Norm. Supér. (4)* 11, 3, 1978, pp.429–443.
24. Jiao, Y, Kharroubi, I and Pham, H.: Optimal investment under multiple defaults risk: A BSDE-decomposition approach. *Ann. Appl. Probab.* 23, 2 (2013), pp 455–491.
25. Kchia, Y., Larsson, M., and Protter, P.: Linking progressive and initial filtration expansions. In: *Malliavin Calculus and Stochastic Analysis* A Festschrift in Honor of David Nualart Series: Springer Proceedings in Mathematics & Statistics, Vol. 34 Viens, F., Feng, J., Hu, Y. and Nualart, E. eds. 2013.
26. Meyer, P.A.: Sur un théorème de J. Jacod. In: *Séminaire de Probabilités VII, Lecture Notes in Mathematics 649.* Springer-Verlag, Berlin Heidelberg New York, 1978, pp 57–60.
27. Song, S.: *Grossissement de filtrations et problèmes connexes.* PhD thesis, L'université Paris VI, 1987.
28. Song, S.: Drift operator in a viable expansion of information flow. *Stochastic Process. Appl.* 126, 8 (2016), pp 2297–2322.
29. Song, S.: Optional splitting formula in progressively enlarged filtration. *ESAIM Probab. Stat.* 18 (2014) pp 829–853.

30. Stricker, C.: Quasimartingales, martingales locales, et filtrations naturelles. *Zeitschrift für Wahrscheinlichkeitstheorie and Verwandte Gebiete* 39 (1977), pp 55–63.
31. Yoeurp, C.: Théorème de Girsanov généralisé et grossissement d'une filtration. In: *Grossissement de filtrations: exemples et applications. Lecture Notes 1118*. Th. Jeulin and M. Yor, eds. Springer-Verlag, Berlin Heidelberg New York, 1985, pp 172–196.
32. Yor, M.: Entropie d'une partition, et grossissement initial d'une filtration. In: *Grossissement de filtrations: exemples et applications. Lecture Notes 1118*. Th. Jeulin and M. Yor, eds. Springer-Verlag, Berlin Heidelberg New York, 1980, pp 45–58.
33. Yor, M.: Grossissement de filtrations et absolue continuité de noyaux. In: *Grossissement de filtrations: exemples et applications. Lecture Notes 1118*. Th. Jeulin and M. Yor, eds. Springer-Verlag, Berlin Heidelberg New York, 1980, pp 6–14.