Weak infinite-energy solutions for the Navier–Stokes equations in $\mathbb{R}^3$

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Abstract:
We prove existence of suitable weak solutions (in the sense of Caffarelli, Kohn and Nirenberg) for the Navier–Stokes equations on $(0, \infty) \times \mathbb{R}^3$ when the initial value is uniformly locally square integrable and vanishes at infinity.

Keywords : Besov spaces, Calderón–Zygmund operators, divergence free vector wavelets, energy inequalities, Littlewood–Paley decomposition, mild solutions, Morrey–Campanato spaces, Navier–Stokes equations, uniqueness, wavelets, weak solutions

AMS Classification :

Introduction.

The main purpose of this paper is to use the Caffarelli–Kohn–Nirenberg regularity criterion [CKN] [LIN] for weak solutions to the Navier–Stokes equations on $\mathbb{R}^3$ to get new results on Leray's weak solutions [LER].

More precisely, let us consider the following Navier–Stokes equations on $\vec{u}(t,x), t \in (0,T), x \in \mathbb{R}^3$:

$$(1) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0. A *weak solution* is a vector field $\vec{u}$ associated to a pressure field p such that $\vec{u} \in L^2_{loc}((0,T) \times \mathbb{R}^3)$, $p \in \mathcal{D}'((0,T) \times \mathbb{R}^3)$ and such that (1) is satisfied when the derivatives are defined in the sense of distributions.

Classical results on existence of weak solutions are the theorems of Leray (1934) and Kato (1984) :

Leray's theorem : [LER]

For all $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$, there exists a weak solution $\vec{u} \in L^\infty((0,\infty),(L^2)^3) \cap L^2((0,\infty),(\dot{H}^1)^3)$ for the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^3$ such that $\lim_{t\to 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0$. Moreover, we may choose this solution $\vec{u}$ fullfilling the energy inequality

$$(2) \qquad \forall t > 0 \quad \|\vec{u}(t,.)\|_2^2 + 2\int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, ds \le \|\vec{u}_0\|_2^2$$

(We write $\dot{H}^1$ for the homogeneous Sobolev space of functions whose gradient is square integrable, equipped with the norm $(\int |\vec{\nabla} f|^2 \, dx)^{1/2}$.)

Kato's theorem : [KAT 1]

(A) Let $p \in (3,\infty)$. Then, for all $\vec{u}_0 \in (L^p(\mathbb{R}^3))^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T and a (unique) weak solution $\vec{u} \in \mathcal{C}([0,T),(L^p)^3)$ for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ such that $\vec{u}(0,.) = \vec{u}_0$. This solution is then smooth on $(0,T) \times \mathbb{R}^3$.

(B) For all $\vec{u}_0 \in (L^3(\mathbb{R}^3))^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T and a weak solution $\vec{u} \in \mathcal{C}([0,T),(L^3)^3)$ for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ such that $\vec{u}(0,.) = \vec{u}_0$. Moreover, this solution may be chosen such that for all $t \in (0,T)$ we have $\sup_{0<s<t} \sqrt{s}\, \|\vec{u}(s,.)\|_\infty < \infty$ and $\lim_{t\to 0} \sqrt{t}\, \|\vec{u}(t,.)\|_\infty = 0$. With this extra condition on the L^∞ norm, such a solution is unique and, moreover, it is smooth on $(0,T) \times \mathbb{R}^3$.

(C) There exists $\epsilon_0 > 0$ such that if $\|\vec{u}_0\|_3 < \epsilon_0$ then the existence time T in point (B) for the smooth solution $\vec{u}$ is equal to $+\infty$ and, moreover, $\lim_{t\to\infty} \sqrt{t}\, \|\vec{u}(t,.)\|_\infty = \lim_{t\to\infty} \|\vec{u}(t,.)\|_3 = 0$.

In this paper, we are going to discuss local versions of these two theorems (more precisely, we shall consider uniformly locally L^2 or L^3 initial values). Kato's theorem is easily adapted to the locally L^3 setting (or to the more general setting of Morrey–Campanato spaces [CAN] [FED] [FLT][KAT 2] [TAY]). Leray's theorem is more difficult : we have to use a local version of energy inequality (2). This refinement of Leray's inequality was introduced by Scheffer [SCH] and is the basis for the regularity criterion of Cafarelli, Kohn and Nirenberg :

Definition 1 : (suitable solutions)

A suitable weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ is a vector field $\vec{u}(t,x)$ such that

a) $\vec{u}$ is locally square integrable on $(0,T) \times \mathbb{R}^3$

b) $\vec{\nabla}.\vec{u} = 0$

c) $\exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^3)$ $\partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p$

d) for all $0 < T_1 < T_2 < T$ and all compact subset $K \subset \mathbb{R}^3$, we have $\sup_{T_1<t<T_2} \int_K |\vec{u}|^2 \, dx < \infty$.

e) $\vec{\nabla} \otimes \vec{u}$ is locally square integrable on $(0,T) \times \mathbb{R}^3$
f) p is locally $L^{3/2}$ on $(0,T) \times \mathbb{R}^3$
g) for all $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ such that $\phi \geq 0$ we have

$$(3) \qquad 2 \int\int |\vec{\nabla} \otimes \vec{u}|^2 \ \phi(t,x) \ dx \ dt \leq \int\int |\vec{u}|^2 \ (\partial_t \phi + \Delta \phi) \ dx \ dt + \int\int (|\vec{u}|^2 + 2p)(\vec{u}.\vec{\nabla})\phi \ \ dx \ dt$$

Caffarelli, Kohn and Nirenberg's regularity criterion : [CKN] [LIN]
There exist two positive constants ϵ_1 and C_1 such that if $\vec{u}$ is a suitable weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$, if $x_0 \in \mathbb{R}^3$ and $0 < t_0$ and if for some r in $(0, \sqrt{t_0})$ we have

$$\int\int_{|x-x_0|<r, t_0-r^2<s<t_0} |\vec{u}|^3 + |p|^{3/2} \ dx \ ds < \epsilon_1 \ r^2$$

then

$$\sup_{|x-x_0|<r/2, t_0-r^2/4<s<t_0} |\vec{u}| < C_1 \ r^{-1}.$$

We may now state our results concerning locally L^p initial values which vanish at infinity :

Definition 2 : (locally L^p functions vanishing at infinity)
For $1 \leq p < \infty$, the space E_p of locally L^p functions vanishing at infinity is defined by: $f \in E_p$ if and only if :
a) f is locally L^p on $\mathbb{R}^3$
b) $\lim_{x \to \infty} \int_{|y-x| \leq 1} |f(y)|^p \ dy \ = 0$
It is normed by :

$$||f||_{E_p} = \sup_{x \in \mathbb{R}^3} (\int_{|y-x|\leq 1} |f(y)|^p \ dy)^{\frac{1}{p}}$$

It is easy to check that E_p is the closure of L^p (or of the test functions) in the Morrey space of uniformly locally L^p functions (defined by $\sup_{x \in \mathbb{R}^3} \ \int_{|y-x|\leq 1} |f(y)|^p \ dy \ < +\infty$).

Definition 3 : (local Leray solutions)
Let $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$. A local Leray solution $\vec{u}$ on $(0,T) \times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec{u}_0$ is a weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ such that :
α) $\vec{u} \in \cap_{t<T} L^\infty((0,t), (E_2)^3)$: $\sup_{x_0 \in \mathbb{R}^3, s<t} \int_{|x-x_0|\leq 1} |\vec{u}(s,x)|^2 \ dx < \infty$
β) For all $t < T$ we have $\sup_{x_0 \in \mathbb{R}^3} \int\int_{0<s<t, |x-x_0|\leq 1} |\vec{\nabla} \otimes \vec{u}|^2 \ dx \ ds < \infty$
γ) $\lim_{t \to 0^+} \|\vec{u} - \vec{u}_0\|_{E_2} = 0$
δ) $\vec{u}$ is suitable in the sense of Caffarelli, Kohn and Nirenberg.

Theorem 1 : (locally square integrable initial value)
For all $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$, there exists a local Leray solution $\vec{u}$ on $(0,\infty) \times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec{u}_0$.

Theorem 2 : (locally L^3 initial value)
(A) Let $\vec{u}_0 \in (E_3(\mathbb{R}^3))^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$. There exists a positive T and a unique solution $\vec{u} \in \mathcal{C}([0,T), (E_3)^3)$ for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ such that $\vec{u}(0,.) = \vec{u}_0$.
(B) If $\vec{u} \in \mathcal{C}([0,T), (E_3)^3)$ is a solution for the Navier–Stokes equations, then for all $t \in (0,T)$ we have $\sup_{0<s<t} \sqrt{s} \ \|\vec{u}(s,.)\|_\infty \ < \ \infty$ and $\lim_{t \to 0} \sqrt{t} \ \|\vec{u}(t,.)\|_\infty \ = \ 0$. If T is maximal and if $T < \infty$, then $\lim_{t \to T} \|\vec{u}(t,.)\|_\infty = \infty$.

(C) If $\vec{u} \in \mathcal{C}([0,\infty),(L^3)^3)$ is a solution for the Navier–Stokes equations, then $\sup_{0<s} \sqrt{s}\ \|\vec{u}(s,.)\|_\infty < \infty$ and $\lim_{t\to\infty} \sqrt{t}\ \|\vec{u}(t,.)\|_\infty = 0$.

(D) If $\vec{u} \in \mathcal{C}([0,T),(E_3)^3)$ is a solution for the Navier–Stokes equations, then $\vec{u}$ and p are smooth on $(0,T)\times\mathbb{R}^3$ and for all $t<T$ we have $\sup_{x_0\in\mathbb{R}^3} \int\int_{0<s<t,|x-x_0|\le 1} |\vec{\nabla}\otimes\vec{u}|^2\ dx\ ds < \infty$. Moreover, if $\vec{v}$ is a local Leray solution $\vec{v}$ on $(0,T)\times\mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec{u}(0)$, then $\vec{v}=\vec{u}$.

Remark : In Theorem 2, point (A) is a result of [FLT] ; points (B), (C) and (D) are new. There is a very simple and direct proof of (B) which was found by May [MAY] while we were finishing the redaction of this paper ; point (D) is a generalization on the Sohr and Von Wahl uniqueness theorem [WAH].

The requirement that $\vec{u}$ vanish at infinity is essential to get rid of the pressure term in the Navier–Stokes equations. Indeed, taking the divergence of equation (1), we obtain $\Delta p + \sum_k \sum_l \partial_k\partial_l(u_l u_k) = 0$. This is enough for getting p from $\vec{u}$ when $\vec{u}$ vanishes at infinity [FLT].

The main idea of the proof of theorem 1 is to approximate $\vec{u}_0$ through divergence-free square integrable vector fields $\vec{u}_{N,0}$. We then begin with constructing suitable solutions $\vec{u}_N$ for the Navier–Stokes equations with $\vec{u}_{N,0}$ as initial values, by using the Leray method [LER]. Then, we use estimates on the size of $\|\vec{u}_N\|_{E_2}$ to show that we may find a subsequence $\vec{u}_{N_k}$ weakly convergent in $\mathcal{D}'((0,T)\times\mathbb{R}^3$ toward a suitable solution $L^\infty((0,T),(E_2)^3)$ with $\vec{u}_0$ as initial value. The regularity criterion of Caffarelli, Kohn and Nirenberg [CKN] then gives us that for almost all $t\in(0,T)$ this solution is uniformly locally L^3. We then deal with the case of a uniformly locally L^3 initial value by using Kato's method of mild solutions [KAT 1] [KAT 2] and we show that this mild solution locally coincides with our previous solution due to our generalization of the Von Wahl uniqueness theorem [WAH]. We may then extend our solution to all positive times.

We finish the paper with a discussion of the case of singular initial values.

Contents :

1. The Leray method for the L^2 initial value problem.

In this section, we shall present the method developed by Leray in 1934 [LER] to provide weak solutions for the Navier–Stokes equations. We shall consider a generalized equation which will be used in many occasions that can be dealt with the same approach. We consider the equation :

$$(4)\qquad \begin{cases} \partial_t\vec{u} = \Delta\vec{u} - \eta\vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla}.(\vec{V}\otimes\vec{u}) - \vec{\nabla}.(\vec{u}\otimes\vec{W}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

where

$$\begin{cases} \eta \in \{0,1\} \\ \vec{\nabla}.\vec{V} = \vec{\nabla}.\vec{W} = 0 \\ \vec{\nabla}.\vec{u}_0 = 0 \qquad \vec{u}_0 \in (L^2(\mathbb{R}^3))^3 \end{cases}$$

We assume the following regularity assumption on $\vec{V}$ and $\vec{W}$:

Definition 4 : (A) For $0 \leq r < 3/2$, we define the space $X_r(\mathbb{R}^3)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly $H^r(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm of X_r is given by the operator norm of pointwise multiplication:

$$\|f\|_{X_r} = \sup\{ \|fg\|_2 \ / \ \|g\|_{H^r} \leq 1\}$$

We write $X_r^{(0)}$ for the closure of $\mathcal{S}(\mathbb{R}^3)$ in X_r.

(B) $\vec{W}$ and $\vec{V}$ will be said to satisfy the X_r regularity requirement on $(0,T)$ if we have that $\vec{W}$ and $\vec{V}$ belong to $L^\sigma((0,T),(X_r^{(0)})^3)$ with $2/\sigma = 1-r$ $(0 \leq r < 1)$ or to $\mathcal{C}([0,T],(X_1^{(0)})^3)$ $(r=1)$.

Remark : The reader may check that $L^p \subset X_r^{(0)}$ for $r = 3/p$ and $3 \leq p < \infty$. Thus, $\vec{V}$ satisfies the X_r regularity requirement whenever $\vec{V}$ satisfies the Serrin uniqueness condition [SER] $\vec{V} \in L^\sigma((L^p)^3)$ with $2/\sigma = 1 - 3/p$.

We may now adapt Leray's method to get the following theorem :

Theorem 3 : Let $T < \infty$, $r \in [0,1]$ and let $\vec{V},\vec{W}$ satisfy the X_r regularity requirement on $(0,T)$ and $\vec{\nabla}.\vec{V} = \vec{\nabla}.\vec{W} = 0$. Then for $\vec{u}_0 \in (L^2)^3$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\eta = 0$ or 1, the problem (4) has a solution $\vec{u} \in L^\infty((0,T),(L^2)^3) \cap L^2((0,T),(H^1)^3)$ and $\lim_{t\to 0} \|\vec{u}-\vec{u}_0\|_2 = 0$.

If $\eta = 0$ (linear case), then we have in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$:

$$\partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2 = \Delta|\vec{u}|^2 - \vec{\nabla}.(|\vec{u}|^2\vec{V}) - 2\vec{\nabla}.((\vec{u}.\vec{W})\ \vec{u}) + 2((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W} - 2\vec{\nabla}.(p\ \vec{u}) \tag{5}$$

If $\eta = 1$ (non-linear case), then one may choose $\vec{u}$ such that there exists a non-negative locally finite measure μ such that we have in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$:

$$\partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2 = \Delta|\vec{u}|^2 - \vec{\nabla}.(|\vec{u}|^2\vec{u}) - \vec{\nabla}.(|\vec{u}|^2\vec{V}) - 2\vec{\nabla}.((\vec{u}.\vec{W})\ \vec{u}) + 2((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W} - 2\vec{\nabla}.(p\ \vec{u}) - \mu \tag{6}$$

so that we have in particular

$$\|\vec{u}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{u}\|_2^2\ ds \leq \|\vec{u}_0\|_2^2 + 2\int\int_0^t ((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}\ dx\ ds \tag{7}$$

Remark : (7) is the Leray energy inequality [LER], while (6) is the local energy inequality of Scheffer [SCH] and Caffarelli, Kohn and Nirenberg [CKN].

Proof :

Fisrt step : elimination of the pressure.

If we take the divergence of (4), we obtain

$$\Delta p = -\eta(\vec{\nabla}\otimes\vec{\nabla}).(\vec{u}\otimes\vec{u}) - (\vec{\nabla}\otimes\vec{\nabla}).(\vec{V}\otimes\vec{u}) - (\vec{\nabla}\otimes\vec{\nabla}).(\vec{u}\otimes\vec{W})$$

For general classes of solutions, this allows to get rid of the pressure by the formula $\vec{\nabla}p = \vec{\nabla}\frac{1}{\Delta}\Delta p$. Thus, we are led to introduce the following singular integral operators $\mathcal{R}$ and $\mathbb{P}$:

Definition 5 : (Singular integral operators associated to the pressure)

We will call in the following $R_l, \mathcal{R}, \mathbb{P}$ the singular integral operators listed below :
a) $R_l = \frac{1}{\sqrt{-\Delta}}\partial_l, 1 \leq l \leq 3$, is the Riesz transform associated to the Fourier multiplier $\frac{i\xi_l}{|\xi|}$.
b) $\mathcal{R}((v_{k,l})_{1\leq k,l\leq 3}) = \sum_k \sum_l R_k R_l(v_{k,l})$. We will write $R(x,y)$ for the distribution kernel of $\mathcal{R}$.
c) $\mathbb{P}(\vec{f}) = \vec{f} + (\sum_l R_k R_l f_l)_{1\leq k\leq 3}$. this is the orthogonal projection operator onto the divergence free vector fields (*Leray's projection*).

The main properties we shall need for these operators is the size estimates on their kernel (they have kernels $K(x,y)$ such that $|K(x,y)| + |x-y|\ |\vec{\nabla}_x K(x,y)| + |x-y|\ |\vec{\nabla}_y K(x,y)| \leq C|x-y|^{-3}$) and their boundedness properties on L^q for $1 < q < \infty$ ($R_l \in \mathcal{L}(L^q, L^q)$).

Now, we may prove in the same way as in [FLT] the following result :

Proposition 1 : (Equivalent formulations for the Navier–Stokes equations)
Let $\vec{u} \in \cap_{T<T^*} L^2((0,T),(E_2)^3)$. Then the following three assertions are equivalent :
(A) $\vec{u}$ is a weak solution of the generalized Navier–Stokes equations on $(0,T^*)\times \mathbb{R}^3$

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \eta\vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla}.(\vec{V}\otimes\vec{u}) - \vec{\nabla}.(\vec{u}\otimes\vec{W}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

where $\eta, \vec{V}$ and $\vec{W}$ satisfy the hypotheses of Theorem 3.
(B) $\vec{u}$ is a solution of $\vec{\nabla}.\vec{u} = 0, \partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\eta\ \vec{u}\otimes\vec{u} + \vec{V}\otimes\vec{u} + \vec{u}\otimes\vec{W})$
(C) there exists $\vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^3))^3$, $\vec{\nabla}.\vec{u}_0 = 0$, such that: $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\eta\ \vec{u}\otimes\vec{u} + \vec{V}\otimes\vec{u} + \vec{u}\otimes\vec{W})\ ds$.

We recall briefly the idea of the proof. We just notice that $\mathbb{P}$ operates on high frequencies in $(\mathcal{S}'(\mathbb{R}^3))^3$; thus, we introduce the Littlewood–Paley decomposition in dyadic blocks $f = S_J(f) + \sum_{j\geq J}\Delta_j(f)$ ([FJW], [TRI],[PEE]) to project the Navier–Stokes equations on high frequencies into :

$$\partial_t(Id - S_J)\vec{u} = \Delta(Id - S_J)\vec{u} - \mathbb{P}(Id - S_J)\vec{\nabla}.(\eta\ \vec{u}\otimes\vec{u} + \vec{V}\otimes\vec{u} + \vec{u}\otimes\vec{W})$$

then, since $\vec{u}$ vanishes at infinity, we obtain $\lim_{J\to-\infty} S_J\vec{u} = 0$ in $\mathcal{S}'$ hence we have

$$\partial_t\vec{u} = \Delta\vec{u} - \lim_{J\to-\infty}\mathbb{P}(Id - S_J)\vec{\nabla}.(\eta\ \vec{u}\otimes\vec{u} + \vec{V}\otimes\vec{u} + \vec{u}\otimes\vec{W})$$

The Bernstein inequalities give, for $j < 0$, $\|\mathbb{P}\Delta_j\vec{\nabla}.(\vec{u}\otimes\vec{u})\|_\infty \leq C2^j\|S_0(\vec{u}\otimes\vec{u})\|_\infty \leq C'2^j\|\vec{u}\|^2_{E_2}$, and thus we see that $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ is well defined. In the same way, the product of a function in E_2 by a function in X_r is uniformly locally in H^{-r} and since a function f in the Schwartz class will satisfy, for any $\phi \in \mathcal{D}(\mathbb{R}^3)$ with $\sum_{k\in\mathbb{Z}^3}\phi(x-k) = 1$, the inequality $\sum_{k\in\mathbb{Z}^3}\|\phi(x-k)f\|_{H^r} < \infty$, we get that $\|\mathbb{P}\Delta_j\vec{\nabla}.(\vec{u}\otimes\vec{V})\|_\infty \leq C2^j\|S_0(\vec{u}\otimes\vec{V})\|_\infty \leq C'2^j\|\vec{u}\|_{E_2}\|\vec{V}\|_{X_r}$ so that $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{V})$ is well defined (and the same for $\mathbb{P}\vec{\nabla}.(\vec{W}\otimes\vec{u})$). On the other hand, high frequencies of p are easily estimated by $(Id-S_J)p = \mathcal{R}(Id-S_J)(\eta\ \vec{u}\otimes\vec{u}+\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W})$ whereas $\vec{\nabla}p$ is given by $\vec{\nabla}p = -(Id-\mathbb{P})\vec{\nabla}.(\eta\ \vec{u}\otimes\vec{u}+\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W})$; p is then given (up to a constant) by

$$p_j = \mathcal{R}\Delta_j(\eta\ \vec{u}\otimes\vec{u}+\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W}) \ \text{ and } \ p = \sum_{j<J} p_j(x) - p_j(0) + \sum_{j\geq J} p_j(x) \tag{8}$$

Now, the proof of Proposition 1 is easy.

Step 2 : the regularized equation.

We begin with a modified equation, choosing a function $\omega \in \mathcal{D}(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3}\omega\ dx = 1$ and a function $\zeta \in \mathcal{D}(\mathbb{R}^4)$ with $\int_{\mathbb{R}^4}\zeta\ dt\ dx = 1$ and solving for $\epsilon > 0$ the equation

$$\begin{cases} \partial_t\vec{u} = \Delta\vec{u} - \eta\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{u}) - \vec{\nabla}.((\vec{V}*\zeta_\epsilon)\otimes\vec{u}) - \vec{\nabla}.(\vec{u}\otimes(\vec{W}*\zeta_\epsilon)) - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases} \tag{9 - \epsilon}$$

where $\omega_\epsilon = \frac{1}{\epsilon^3}\omega(\frac{x}{\epsilon})$ and $\zeta_\epsilon = \frac{1}{\epsilon^4}\zeta(\frac{t}{\epsilon},\frac{x}{\epsilon})$ (where we extend $\vec V$ to the whole $\mathbb{R}\times\mathbb{R}^3$ by taking $\vec V = 0$ for $t<0$ or $t>T$ $(0\le r<1)$ or, for $r=1$ and for a $\gamma\in\mathcal{D}(\mathbb{R}), \gamma(0)=1$, $\vec V=\gamma(t)\vec V(0)$ for $t<0$ and $\vec V = \gamma(t-T)\vec V(T)$ for $t>T$). Following Proposition 1, we are lead to solve the fixed–point problem $\vec u = e^{t\Delta}\vec u_0 - B_\epsilon(\vec u) - A_\epsilon(\vec u)$ where the bilinear operator B_ϵ is given by $B_\epsilon(\vec u,\vec v) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.((\vec u*\omega_\epsilon)\otimes\vec v)\, ds$ and the linear operator A_ϵ is given by $A_\epsilon(\vec u) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.((\vec V*\zeta_\epsilon)\otimes\vec u + \vec u\otimes(\vec W*\zeta_\epsilon))\, ds$. Now, we notice that for $\vec u_0\in(L^2)^3$, we have $e^{t\Delta}\vec u_0\in\mathcal{C}([0,T],(L^2)^3)\cap L^2([0,T],(H^1)^3)$. Moreover $\|B_\epsilon(\vec u,\vec v)(t)\|_2\le C_\epsilon\sqrt{t}\ \sup_{0<s<t}\|\vec u\|_2\ \sup_{0<s<t}\|\vec v\|_2$ and $\|A_\epsilon(\vec u)(t)\|_2\le C_\epsilon\sqrt{t}\ \sup_{0<s<t}\|\vec u\|_2$ where C_ϵ does not depend on t. This proves that one can find a solution of $(9-\epsilon)$ in $\mathcal{C}([0,T_\epsilon],(L^2)^3)$ with a positive T_ϵ which is $O(\|\vec u_0\|_2^{-2})$. Moreover, this solution belongs to $L^2([0,T_\epsilon],(H^1)^3)$ since we have $\|B_\epsilon(\vec u,\vec u)\|_{L^2([0,T_\epsilon],(\dot H^1)^3)}\le C'_\epsilon\sup_{0<s<t}\|\vec u\|_2^2$ and $\|A_\epsilon(\vec u)(t)\|_{L^2([0,T_\epsilon],(\dot H^1)^3)}\le C'_\epsilon\sup_{0<s<t}\|\vec u\|_2$.

This solution can be extended to $[0,T]\times\mathbb{R}^3$: indeed, we may compute $\partial_t|\vec u|^2$ as $2\ \langle\partial_t\vec u|\vec u\rangle$ (since $\vec u\in L^2([0,T_\epsilon],(H^1)^3)$ while $\partial_t\vec u\in L^2([0,T_\epsilon],(H^{-1})^3)$) and get (writing $\vec V_\epsilon$ for $\vec V*\zeta_\epsilon$ and $\vec W_\epsilon$ for $\vec W*\zeta_\epsilon$)

$$\partial_t|\vec u|^2 + 2|\vec\nabla\otimes\vec u|^2 = \Delta|\vec u|^2 - \eta\ \vec\nabla.(|\vec u|^2(\vec u*\omega_\epsilon)) - \vec\nabla.(|\vec u|^2\vec V_\epsilon) - 2\vec\nabla.((\vec u.\vec W_\epsilon)\ \vec u) + 2((\vec u.\vec\nabla)\ \vec u).\vec W_\epsilon - 2\vec\nabla.(p\ \vec u)$$

and finally (integrating in x)

$$\|\vec u(t)\|_2^2 + 2\int\int_0^t|\vec\nabla\otimes\vec u|^2\ dx\ ds = \|\vec u_0\|_2^2 + 2\int\int_0^t((\vec u.\vec\nabla)\ \vec u).\vec W_\epsilon\ dx\ ds$$

Since $\vec W_\epsilon$ is bounded, we find that $2\int\int_0^t((\vec u.\vec\nabla)\ \vec u).\vec W_\epsilon\ dx\ ds\le\int\int_0^t|\vec\nabla\otimes\vec u|^2\ dx\ ds + C_\epsilon\int_0^t\|\vec u\|_2^2\ ds$ where C_ϵ depends only on $\|\vec W_\epsilon\|_\infty$ and not on t nor on T_ϵ. The Gronwall lemma then gives existence of $\vec u$ on the whole $[0,T]\times\mathbb{R}^3$.

Step 3 : the weak limit.

We call $\vec u_\epsilon$ the solution of $(9-\epsilon)$. We have seen that $\|\vec u_\epsilon(t)\|_2^2 + 2\int\int_0^t|\vec\nabla\otimes\vec u_\epsilon|^2\ dx\ ds = \|\vec u_0\|_2^2 + 2\int\int_0^t((\vec u_\epsilon.\vec\nabla)\ \vec u_\epsilon).\vec W_\epsilon\ dx\ ds$. Moreover, we have that the norm of X_r is invariant under translation so that $\|\vec W_\epsilon\|_{L^\sigma((0,T),(X_r)^3)}\le\|\vec W\|_{L^\sigma((0,T),(X_r)^3)}$ and thus that

$$2\int\int_0^t((\vec u_\epsilon.\vec\nabla)\ \vec u_\epsilon).\vec W_\epsilon\ dx\ ds\le C\|\vec u_\epsilon\|_{L^2((0,T),(\dot H^1)^3)}\|\vec u_\epsilon\|_{L^\rho((0,T),(H^r)^3)}\|\vec W\|_{L^\sigma((0,T),(X_r)^3)}$$

where $2/\sigma = 1-r$ and $1/\rho = r/2$. If $r<1$, we write

$$\|\vec u_\epsilon\|_{L^2((H^1)^3)}\|\vec u_\epsilon\|_{L^\rho((H^r)^3)}\le\|\vec u_\epsilon\|_{L^2((H^1)^3)}^{1+r}\|\vec u_\epsilon\|_{L^\infty(,(L^2)^3)}^{1-r}\le\frac{(1+r)\alpha}{2}\|\vec u_\epsilon\|_{L^2((H^1)^3)}^2 + \frac{(1-r)}{2\alpha^{(1+r)/(1-r)}}\|\vec u_\epsilon\|_{L^\infty((L^2)^3)}^2$$

which , for α small enough, gives (again by the Gronwall lemma) that $\sup_{\epsilon>0}\sup_{0<t<T}\|\vec u_\epsilon\|_2<\infty$ and $\sup_{\epsilon>0}\int_0^T\|\vec u_\epsilon\|_{H^1}\ dt<\infty$. For the case $r=1$, then we split $\vec W$ in $\vec X+\vec Y$ where $\sup_{0<t<T}\|\vec X\|_{X_1}\le\alpha$ and $\vec Y\in(L^\infty((0,T)\times\mathbb{R}^3)^3$. We then obtain

$$2\int\int_0^t((\vec u_\epsilon.\vec\nabla)\ \vec u_\epsilon).\vec W_\epsilon\ dx\ ds\le C\|\vec u_\epsilon\|_{L^2((\dot H^1)^3)}\big(\|\vec u_\epsilon\|_{L^2((H^1)^3)}\|\vec X\|_{L^\infty((X_1)^3)} + \sqrt{t}\ \|\vec u_\epsilon\|_{L^\infty((H^1)^3)}\|\vec Y\|_\infty\big)$$

which gives again (for α small enough) the control of $\vec u_\epsilon$ in norms $L^\infty((L^2)^3)$ and $L^2((H^1)^3)$.

Moreover, the control on $\vec u$ gives a control on the associated pressure p_ϵ and on $\partial_t\vec u_\epsilon$: first, we have $\sup_\epsilon\int_0^T\|\vec V_\epsilon\otimes\vec u_\epsilon + \vec u_\epsilon\otimes\vec W_\epsilon\|_2^2\ ds<\infty$ and $\sup_\epsilon\int_0^T\|\vec u_\epsilon\otimes\vec u_\epsilon\|_{3/2}^2\ ds<\infty$; we get that for any $\varphi\in\mathcal{D}(\mathbb{R}^3)$ we have $\sup_\epsilon\int_0^T\|\varphi\ p_\epsilon\|_{3/2}^2\ ds<\infty$ and $\sup_\epsilon\int_0^T\|\varphi\partial_t\vec u_\epsilon\|_{H^{-3/2}}^2\ ds<\infty$. Now, we use the following lemma :

Lemma 1 : a) If $(f_\epsilon)_{0<\epsilon<1}$ is a family of functions on $(0,T)\times\mathbb{R}^3$ such that for all $\phi\in\mathcal{D}(]0,T[\times\mathbb{R}^3)$ we have : $(\phi f_\epsilon)_{0<\epsilon<1}$ is bounded in $L^2((0,T),H^1)$ and $(\phi\ \partial_t f_\epsilon)_{0<\epsilon<1}$ is bounded in $L^2((0,T),H^{-3/2})$, then there exists $f_\infty \in L^2_{loc}((0,T)\times \mathbb{R}^3)$ and a sequence ϵ_n converging to 0 such that for all $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ the sequence ϕf_{ϵ_n} converges to ϕf_∞ strongly in $L^2((0,T)\times\mathbb{R}^3)$ and weakly in $L^2((0,T),H^1)$.

b) If moreover for all $\phi \in \mathcal{D}(]0,T[\times\mathbb{R}^3)$ we have that $(\phi f_\epsilon)_{0<\epsilon<1}$ is bounded in $L^\infty((0,T),L^2)$, then the sequence ϕf_{ϵ_n} converges to ϕf_∞ strongly in $L^p((0,T),L^2)$ for all $p\in[1,\infty)$ and strongly in $L^3((0,T)\times\mathbb{R}^3)$.

Proof : It is enough to check that $L^2(\mathbb{R},H^1)\cap H^1(\mathbb{R},H^{-3/2})\subset H^{2/5}(\mathbb{R}^4)$ and $L^6(\mathbb{R},L^2)\cap L^2(\mathbb{R},L^6)\subset L^3(\mathbb{R}^4)$. For a given ϕ (with support K), the family $(\phi f_\epsilon)_{0<\epsilon<1}$ is bounded in $H^{2/5}(K)$, hence by the compcatness of the imbedding $H^{2/5}(K)\subset L^2(K)$, we find that we may choose a subsequence f_{ϵ_n} such that we have weak convergence of ϕf_{ϵ_n} in $L^2(\mathbb{R},H^1)$ and strong convergence in $L^2(\mathbb{R}^4)$. In order to finish the proof, we notice that it is enough to prove the convergence results for a sequence ϕ_N of functions such that $\phi_N\equiv 1$ on $(1/N,T-1/N)\times\{x\ /\ |x|\le N\}$; since the convergence of $\phi_N f_{\epsilon_n}$ in strong $L^2(L^2)$ and weak $L^2(H^1)$ for a sequence ϵ_n and for all N can be obtained through the diagonal extraction process, the lemma is proved.

We apply the lemma to $\vec{u}_\epsilon$. We thus find a sequence ϵ_n and a vector field $\vec{u}$ such that for all $\phi\in\mathcal{D}((0,T)\times\mathbb{R}^3)$ we have convergence of $\phi\ \vec{u}_{\epsilon_n}$ to $\phi\ \vec{u}$ strongly in $L^2((0,T),(L^2)^3)$ and weakly in $L^2((0,T),(H^1)^3)$. Moreover, we may impose in the same way weak convergence of ϕp_{ϵ_n} in $L^2((0,T),L^{3/2})$ to ϕp. Then, we have that each term of equation $(9-\epsilon)$ applied to $\vec{u}_\epsilon$ converges in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ (we write ϵ instead of ϵ_n): $\vec{u}_\epsilon$ converges (in $(L^2_{loc}((0,T)\times\mathbb{R}^3))^3$) to $\vec{u}$, hence $\partial_t\vec{u}_\epsilon$ converges to $\partial_t\vec{u}$ and $\Delta\vec{u}_\epsilon$ converges to $\Delta\vec{u}$, p_ϵ converges (weakly in $L^{3/2}_{loc}((0,T)\times\mathbb{R}^3)$) to p hence $\vec{\nabla}p_\epsilon$ converges to $\vec{\nabla}p$; moreover $\vec{V}_\epsilon$ converges strongly to $\vec{V}$ in $L^\sigma((0,T),(X_r)^3)$ (with $2/\sigma=1-r$) while $\vec{u}_\epsilon$ converges strongly in $(L^2_{loc}((0,T)\times\mathbb{R}^3)^3)$ and this gives strong convergence of $\vec{V}_\epsilon\otimes\vec{u}_\epsilon$ in $L^{2/r}((0,T),(H^{-r})^9)$; thus, $\vec{\nabla}.(\vec{V}_\epsilon\otimes\vec{u}_\epsilon+\vec{u}_\epsilon\otimes\vec{W}_\epsilon)$ converges to $\vec{\nabla}.(\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W})$; finally, $\vec{u}_\epsilon\otimes\vec{u}_\epsilon$ converges strongly in $(L^{3/2}_{loc}((0,T)\times\mathbb{R}^3))^3$ to $\vec{u}\otimes\vec{u}$, hence $\vec{\nabla}.((\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon)$ converges to $\vec{\nabla}.(\vec{u}\otimes\vec{u})$.

Moreover, for all $\varphi\in\mathcal{D}(\mathbb{R}^3)$, $\varphi\ \partial_t\vec{u}_\epsilon$ is bounded in $L^2((0,T),(H^{-3/2})^3)$ and thus $\|\phi(\vec{u}_\epsilon(t)-\vec{u}_0)\|_{H^{-3/2}}\le C(\varphi)\sqrt{t}$ where $C(\varphi)$ does not depend on t nor on ϵ ; this proves that $\vec{u}(t)$ converges to $\vec{u}_0$ in $H^{-3/2}$. Thus, $\vec{u}$ is a solution of (4).

Step 4 : energy estimates.

If $\eta=0$, we find that $\vec{u}\in L^2((0,T),(H^1)^3)$ and $\partial_t\vec{u}\in L^2((0,T),(H^{-1})^3)$, so that $\partial_t|\vec{u}|^2=2\langle\partial_t\vec{u}|\vec{u}\rangle$ and equality (5) is obvious. If $\eta=1$, we start from the equality

$$\partial_t|\vec{u}_\epsilon|^2+2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2=\Delta|\vec{u}_\epsilon|^2-\vec{\nabla}.(|\vec{u}_\epsilon|^2(\vec{u_\epsilon}*\omega_\epsilon))-\vec{\nabla}.(|\vec{u}_\epsilon|^2\vec{V}_\epsilon)-2\vec{\nabla}.((\vec{u}_\epsilon.\vec{W}_\epsilon)\ \vec{u}_\epsilon)+2((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon-2\vec{\nabla}.(p_\epsilon\ \vec{u}_\epsilon)$$

The strong convergence of $\vec{u}_\epsilon$ in $(L^3_{loc}((0,T)\times\mathbb{R}^3)^3$ and the weak convergence of p_ϵ in $L^{3/2}_{loc}((0,T)\times\mathbb{R}^3)$ ensure the convergence (in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$) of $\partial_t|\vec{u}_\epsilon|^2$ to $\partial_t|\vec{u}|^2$, of $\Delta|\vec{u}_\epsilon|^2$ to $\Delta|\vec{u}|^2$, of $\vec{\nabla}.(|\vec{u}_\epsilon|^2(\vec{u_\epsilon}*\omega_\epsilon))$ to $\vec{\nabla}.(|\vec{u}|^2\ \vec{u})$ and of $\vec{\nabla}.(p_\epsilon\ \vec{u}_\epsilon)$ to $\vec{\nabla}.(p\ \vec{u})$. The strong convergence of $\vec{V}_\epsilon$ in $L^\sigma((0,T),(X_r)^3)$ (with $2/\sigma=1-r$) and of $\vec{u}_\epsilon$ in $L^2((0,T),(L^2)^3)$ and the weak convergence of $\vec{u}_\epsilon$ in $L^{2/r}((0,T),(H^r)^3)$ gives the convergence of $\vec{\nabla}.(|\vec{u}_\epsilon|^2\vec{V}_\epsilon)$ in $\mathcal{D}'$ to $\vec{\nabla}.(|\vec{u}|^2\vec{V})$, and we get in the same way the convergence of $\vec{\nabla}.((\vec{u}_\epsilon.\vec{W}_\epsilon)\ \vec{u}_\epsilon)$ to $\vec{\nabla}.((\vec{u}.\vec{W})\ \vec{u})$.

The convergence of $((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon$ to $((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}$ is more delicate to get. We should use the convergence of $\vec{\nabla}\otimes\vec{u}_\epsilon$ to $\vec{\nabla}\otimes\vec{u}$ in $L^2((0,T),(L^2)^9)$ and of $\vec{u}_\epsilon$ to $\vec{u}$ in $L^{2/r}((0,T),(H^r)^3)$, but both are weak convergences, and, thus,we cannot deal directly with the bilinear term. However, the solution is quite easy. For any positive α, we may split $\vec{W}$ in $\vec{X}+\vec{Y}$ where $\|\vec{X}\|_{L^\sigma((0,T),(X_r)^3}<\alpha$ and $\vec{Y}\in(\mathcal{D}(\mathbb{R}^4))^3$; now, we have that $((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{Y}_\epsilon$ converges to $((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{Y}$ while $\sup_\epsilon\int_0^T|((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{X}_\epsilon|\ dx\ ds<C\alpha$; letting α go to 0, we get the desired convergence result.

Finally, the weak convergence of $\vec{\nabla}\otimes\vec{u}_\epsilon$ in $(L^2_{loc}((0,T)\otimes\mathbb{R}^3)^9)$ gives that for a non-negative measure μ (and a subsequence ϵ_n) $2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2$ converges to $2|\vec{\nabla}\otimes\vec{u}|^2+\mu$. Thus the local energy inequality (6) is proved.

The Leray energy inequality is obtained in a similar way : we start from

$$\|\vec{u}_\epsilon(t)\|_2^2+2\int\int_0^t|\vec{\nabla}\otimes\vec{u}_\epsilon|^2\ dx\ ds=\|\vec{u}_0\|_2^2+2\int\int_0^t((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon\ dx\ ds$$

When ϵ_n goes to 0, the right-hand side of the equality converges to $\|\vec{u}_0\|_2^2+2\int\int_0^t((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}\ dx\ ds$. The weak convergence of $\vec{\nabla}\otimes\vec{u}_\epsilon$ gives that $\int\int_0^t|\vec{\nabla}\otimes\vec{u}|^2\ dx\ ds\le\liminf\int\int_0^t|\vec{\nabla}\otimes\vec{u}_\epsilon|^2\ dx\ ds$. Moreover, the strong convergence of $\vec{u}_\epsilon$ to $\vec{u}$ in $(L^2_{loc}((0,T)\times\mathbb{R}^3)^3$ gives that there exists a subsequence ϵ_n such that for almost all t we have the convergence of $\vec{u}_\epsilon(t)$ to $\vec{u}(t)$ almost everywhere in $\mathbb{R}^3$; thus for almost every t, we have $\|\vec{u}(t)\|_2 \leq \liminf \|\vec{u}_{\epsilon_n}(t)\|_2$, hence inequality (7) is valid for almost every t ; we then conclude that it is valid for every t since the continuity of $\vec{u}$ in $(H^{-3/2}_{loc}(\mathbb{R}^3))^3$ and the boundedness of $\vec{u}$ in $(L^2)^3$ imply the weak convergence of $\vec{u}(t')$ to $\vec{u}(t)$ in $(L^2)^3$ when t' goes to t.

Thus, Theorem 3 is proved.

Remarks : A) The restriction $T<\infty$ is only a tool to alleviate the notations. In particular, we have proved that the Navier–Stokes equations have for a given square integrable divergence free initial value $\vec{u}_0$ have a suitable Leray solution on $(0,\infty)\times\mathbb{R}^3$.

B) The restriction to $X_r^{(0)}$ gives us strong convergence of $\vec{V}_\epsilon$ to $\vec{V}$ and of $\vec{W}_\epsilon$ to $\vec{W}$. If we consider the X_r spaces, we shall have weak convergence. This is no problem, except at the very end of the proof, when considering the term $((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon$ in the energy inequality.

2. The approximation lemma.

In Section 1, we discussed a class of initial value problems with square integrable divergence–free initial values. But we shall mostly consider in this paper problems with initial values which are locally square integrable (or even worse...) but not globally square integrable. In order to be able to use the Leray method, we shall be able to approximate locally square integrable divergence free vector fiels by square integrable vector fields. This is the purpose of this section :

Proposition 2 : (The approximation lemma)

Let $p\in]1,\infty[$ and let E_p be the space of locally L^p functions which vanish at infinity (defined by the conditions $\|f\|_{E_p} = \sup_{x\in\mathbb{R}^3}(\int_{|x-y|<1}\ |f(y)|^p\ dy)^{1/p} < \infty$ and $\lim_{x\to\infty}\int_{|x-y|<1}\ |f(y)|^p\ dy = 0$). Let $\vec{u}\in(L^2+E_p)^3$ such that $\vec{\nabla}.\vec{u}=0$. Then for all $\epsilon>0$, there exists $\vec{v}_\epsilon\in(L^2)^3, \vec{w}_\epsilon\in(E_p)^3, \vec{\nabla}.\vec{v}_\epsilon = \vec{\nabla}.\vec{w}_\epsilon = 0$ such that $\vec{u}=\vec{v}_\epsilon+\vec{w}_\epsilon$ et $\|\vec{w}_\epsilon\|_{E_p}\leq\epsilon$.

Proof : Since $\mathcal{D}$ is dense in E_p, we may decompose each component u_i of $\vec{u}$ in V_i+W_i where the E_p norm of W_i is small. The difficulty lies in the request that $\vec{v}$ and $\vec{w}$ be divergence-free. It is not enough projecting $\vec{V}$ and $\vec{W}$ on divergence free vector fields with help of the projection operator $\mathbb{P}$, because $\mathbb{P}$ is not bounded on $(E_p)^3$. Instead of $\mathbb{P}$, we may use projection operators onto a divergence free vector basis of compactly supported wavelets [LEM 1]. Indeed, let φ,φ^* be compactly supported univariate bi-orthogonal scaling functions such that, for instance, φ and φ^* are $\mathcal{C}^2$. Let $\tilde\varphi$ and $\tilde\varphi^*$ be the scaling functions defined by :

$$\frac{d}{dx}\varphi(x) = \tilde\varphi(x) - \tilde\varphi(x-1)$$

$$\tilde\varphi^*(x) = \int_x^{x+1} \varphi^*(t)\ dt$$

If $V_j, V_j^*, \tilde V_j$ and $\tilde V_j^*$ are the multi-resolution analyses generated by $\varphi,\varphi^*,\tilde\varphi$ and $\tilde\varphi^*$, if P_j is the projection operator onto V_j in the direction of $(V_j^*)^\perp$ and if $\tilde P_j$ is the projection operator onto $\tilde V_j$ in direction of $(\tilde V_j^*)^\perp$, then we have :

$$\frac{d}{dx}(P_j f) = \tilde P_j(\frac{d}{dx}f)$$

where the formula is valid for all locally integrable f .

We then define the vector projection operator $\vec{P_j} = (P_j\otimes\tilde P_j\otimes\tilde P_j, \tilde P_j\otimes P_j\otimes\tilde P_j, \tilde P_j\otimes\tilde P_j\otimes P_j)$ so that for all $f\in(L^1_{loc})^3$ we have :

$$\vec{\nabla}.\vec{P_j}(\vec{f}) = \tilde P_j\otimes\tilde P_j\otimes\tilde P_j(\vec{\nabla}.\vec{f})$$

Thus, a divergence free vector field is projected onto a vector field which is still divergence free. The main property of this approximation process for vector fields is the fact that the term $(\vec{P}_{j+1}-\vec{P}_j)\vec{f}$ may be locally reorganized into divergence free "atoms" when $\vec{f}$ is itself a divergence free vector field. Indeed, in the "classical" wavelet theory, we would decompose $(\vec{P}_{j+1}-\vec{P}_j)\vec{f}$ into

$$(\vec{P}_{j+1}-\vec{P}_j)\vec{f}=\sum_{1\leq l\leq 3}\ \sum_{1\leq\alpha\leq 7}\ \sum_{k\in\mathbb{Z}^3}\langle f_l|\psi^*_{\alpha,l,j,k}\rangle\ \vec{\psi}_{\alpha,l,j,k}$$

where $\psi^*_{\alpha,l,j,k}=2^{3j/2}\ \psi^*_{\alpha,l}(2^jx-k)$ and $\vec{\psi}_{\alpha,l,j,k}=2^{3j/2}\ \vec{\psi}_{\alpha,l}(2^jx-k)$ and where the 21 wavelets $\vec{\psi}_{\alpha,l}$ and their 21 dual wavelets $\psi^*_{\alpha,l}$ are derived from the dual wavelets ψ,ψ^* (associated to φ,φ^*) and $\tilde{\psi},\tilde{\psi}^*$ (associated to $\tilde{\varphi},\tilde{\varphi}^*$) through the following formulas where we write $(\vec{e}_l)_{1\leq l\leq 3}$ for the canonical basis of $\mathbb{R}^3$ (so that $\vec{f}=\sum_l\ f_l\ \vec{e}_l$) :

$$\varphi^{\beta}_{[l,n]}=\begin{cases}=\varphi\text{ for }l=n\text{ and }\beta=0,\ =\tilde{\varphi}\text{ for }l\neq n\text{ and }\beta=0\\ =\psi\text{ for }l=n\text{ and }\beta=1,\ =\tilde{\psi}\text{ for }l\neq n\text{ and }\beta=1\end{cases}$$

$$\varphi^{*\beta}_{[l,n]}=\begin{cases}=\varphi^*\text{ for }l=n\text{ and }\beta=0,\ =\tilde{\varphi}^*\text{ for }l\neq n\text{ and }\beta=0\\ =\psi^*\text{ for }l=n\text{ and }\beta=1,\ =\tilde{\psi}^*\text{ for }l\neq n\text{ and }\beta=1\end{cases}$$

$$\vec{\psi}_{\alpha,l}=\varphi^{\beta_1}_{[l,1]}\otimes\varphi^{\beta_2}_{[l,2]}\otimes\varphi^{\beta_3}_{[l,3]}\quad\vec{e}_l\qquad\text{for }\alpha=\sum_{n=1}^{3}\ \beta_n\ 2^n$$

$$\psi^*_{\alpha,l}=\varphi^{*\beta_1}_{[l,1]}\otimes\varphi^{*\beta_2}_{[l,2]}\otimes\varphi^{*\beta_3}_{[l,3]}\qquad\text{for }\alpha=\sum_{n=1}^{3}\ \beta_n\ 2^n$$

In the theory of divergent free vector wavelets, we decompose $(\vec{P}_{j+1}-\vec{P}_j)\vec{f}$ in seven groups (following the values of α) :

$$\vec{Q}_{\alpha,j}\vec{f}=\sum_{1\leq l\leq 3}\ \sum_{k\in\mathbb{Z}^3}\langle f_l|\psi^*_{\alpha,l,j,k}\rangle\ \vec{\psi}_{\alpha,l,j,k}$$

which satisfy a formula of the kind $\vec{\nabla}.\vec{Q}_{\alpha,j}\vec{f}=\tilde{Q}_{\alpha,j}(\vec{\nabla}.\vec{f})$. Finally, we just notice that, since $\alpha\neq 0$, at least one of the associated β_n's is equal to 1; we call n the smallest index such that $\beta_n=1$ and we use the fact that $\tilde{\psi}=\frac{d}{dx}\psi$ to get that for $l\neq n$ we have $\vec{\psi}_{\alpha,l}=\partial_n\Psi_{\alpha,l}\ \vec{e}_l$ where $\Psi_{\alpha,l}$ is defined in the same way as $\psi_{\alpha,l}$ (replacing the n-th component of the tensor product $\tilde{\psi}$ by ψ). We then define $\vec{\gamma}_{\alpha,l}=\vec{\nabla}\wedge(\Psi_{\alpha,l}\ (\vec{e}_l\wedge\vec{e}_n))$ et $\vec{\gamma}_{\alpha,l,j,k}=2^{3j/2}\ \vec{\gamma}_{\alpha,l}(2^jx-k)$. We may write :

$$\vec{Q}_{\alpha,j}\vec{f}=\sum_{l\neq n}\ \sum_{k\in\mathbb{Z}^3}\langle f_l|\psi^*_{\alpha,l,j,k}\rangle\ \vec{\gamma}_{\alpha,l,j,k}+R_n(\vec{f})\ \vec{e}_n$$

When $\vec{f}\in(E_1)^3$ and $\vec{\nabla}.\vec{f}=0$, we get that $\partial_nR_n(\vec{f})=0$ with $R_n(\vec{f})$ vanishing at infinity, and thus $R_n(\vec{f})=0$.

The next step is to notice that the operator $g\to\vec{A}_{j_O,\alpha,l}(g)=\sum_{j\geq j_O}\sum_{k\in\mathbb{Z}^3}\langle g|\psi^*_{\alpha,l,j,k}\rangle\ \vec{\gamma}_{\alpha,l,j,k}$ is a Calderón-Zygmund operator [MEY 2] which is bounded from L^p to $(L^p)^3$ for $1<p<\infty$ and thus from E_p to $(E_p)^3$ for $1<p<\infty$ (since $1_{|x-y|\leq 1}\vec{A}_{j_O,\alpha,l}(g)(y)=1_{|x-y|\leq 1}\vec{A}_{j_O,\alpha,l}(1_{|x-.|\leq 1+C2^{-j_0}}g(.))(y))$. Thus, we get the following lemma :

Lemma 2 : The operators $\vec{P}_{j_0}$ and $\vec{A}_{j_0}=\sum_{1\leq\alpha\leq 7}\sum_{l\neq n(\alpha)}\vec{A}_{j_0,\alpha,l}$ satisfy the following boundedness properties :

(i) $\vec{P}_{j_0}$ is bounded from $(E_p+L^2)^3$ to $(C_0)^3$; moreover, when $\vec{\nabla}.\vec{f}=0$, we have $\vec{\nabla}.\vec{P}_{j_0}\vec{f}=0$.

(ii) For $\vec{f}\in(E_p+L^2)^3$ we have $\lim_{j_0\to-\infty}\|\vec{P}_{j_0}\vec{f}\|_\infty=0$.

(iii) $\vec{A}_{j_0}$ is bounded from $(L^2)^3$ to $\{\vec{f}\in(L^2)^3\ /\ \vec{\nabla}.\vec{f}=0\}$ and from $(E_p)^3$ to $\{\vec{f}\in(E_p)^3\ /\ \vec{\nabla}.\vec{f}=0\}$.

(iv) For all $\vec{f}\in(E_p+L^2)^3$ such that $\vec{\nabla}.\vec{f}=0$ and for all j_0, $\vec{f}=\vec{P}_{j_0}\vec{f}+\vec{A}_{j_0}\vec{f}$.

Proposition 2 is then straightforward. Indeed, we first choose j_0 such that $\|\vec{P}_{j_0}(\vec{u})\|_{E_p}<\epsilon/2$. Let us then decompose $\vec{u}$ into $\vec{V}_\epsilon+\vec{W}_\epsilon$ with $\|\vec{W}_\epsilon\|_{E_p}$ small enough to ensure that $\|\vec{A}_{j_0}\vec{W}_\epsilon\|_{E_p}$ is less than $\epsilon/2$; then, we may conclude by defining $\vec{v}_\epsilon=\vec{A}_{j_0}\vec{V}_\epsilon$ and $\vec{w}_\epsilon=\vec{P}_{j_0}(\vec{u})+\vec{A}_{j_0}\vec{W}_\epsilon$.

3. Local size estimates for a suitable Leray solution.

In this section, we consider a local Leray solution of the Navier–Stokes equations and we shall prove that, at least for small times, we may find an estimate for the E_2 norm of the solution which depends only of the E_2 norm of the initial value :

Proposition 3 : Let $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$ and let $\vec{u}$ be a local Leray solution on $(0,T) \times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec{u}_0$. Define $\alpha_0 = \|\vec{u}_0\|_{E_2}$ and $\alpha_1 = \min(1, \alpha_0)$. Then, there exist a positive constant C_0 (which does not depend on $\vec{u}$) such that for all $0 < t < \min(1, \frac{\alpha_1^2}{\alpha_0^2 C_0^4}, T)$ we have

$$\|\vec{u}(t,.)\|_{E_2} \leq \sqrt{C_0}\|\vec{u}_0\|_{E_2}\big(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2} t\big)^{-1/4} \tag{10}$$

Proof : A convenient tool to estimate E_2 norms will be the use of translates of one given smooth function $\varphi_0 \in \mathcal{D}(\mathbb{R}^3)$ to localize L^2 norms :

Definition 6 : Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^3)$ such that $\varphi_0 \geq 0$ and $\sum_{k\in\mathbb{Z}^3} \varphi_0(x-k) = 1$.

(A) We define $\mathcal{B} = \{\varphi_0(x - x_0) \ / \ x_0 \in \mathbb{R}^3\}$; the norm $\|f\|_{E_2}$ is then equivalent to $\sup_{\varphi\in\mathcal{B}} \|f\varphi\|_2$.

(B) We fix ω_0 and $\psi_0 \in \mathcal{D}(\mathbb{R}^3)$ such that ω_0 is identically equal to 1 in the neighbourhood of the support of φ_0 and similarly ψ_0 is identically equal to 1 in the neighbourhood of the support of ω_0. Then, for $\varphi \in \mathcal{B}, \varphi = \varphi_0(x - x_\varphi)$, we define $\psi = \psi_0(x - x_\varphi)$.

Now, we define $\alpha(t) = \sup_{\varphi\in\mathcal{B}} \|\vec{u}(t,.)\varphi(x)\|_2^2$ and $\beta(t) = \sup_{\varphi\in\mathcal{B}} \int_0^t \|(\vec{\nabla} \otimes \vec{u}(s,.))\varphi(x)\|_2^2 \, ds$. We estimate α and β with help of a series of technical lemmas. We call p the pressure associated to $\vec{u}$ and we begin with a lemma to estimate the size of p.

Lemma 3 : Let $\vec{u}$ be a weak E_2 solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$. Assume moreover that, for all $0 < T_0 < T_1 < T$, $\vec{u} \in L^\infty((T_0,T_1),(E_2)^3)$ and that $\vec{\nabla} \otimes \vec{u}$ is locally square integrable on $(0,T)\times\mathbb{R}^3$. Then, the pressure p is locally $L^{3/2}$ on $(0,T)\times\mathbb{R}^3$. More precisely, for all $\varphi \in \mathcal{B}$ ($\varphi = \varphi_0(x-x_0)$ for some $x_0 \in \mathbb{R}^3$), there exists a function $p_\varphi(t)$ such that for all $0 < T_0 < T_1 < T$:

$$\Big(\iint_{(T_0,T_1)\times\mathbb{R}^3} |p(t,x) - p_\varphi(t)|^{3/2} \ \varphi(x) \ dx \ dt\Big)^{2/3} \leq C\big(\|\vec{u}\|^2_{L^3([T_0,T_1],E_2)} + \|\psi\vec{u}\|_{L^6([T_0,T_1],L^2)} \ \|\psi\vec{u}\|_{L^2([T_0,T_1],H^1)}\big)$$

where C does not depend neither on φ nor on T_0, T_1.

Proof : In order to prove this estimate, we compute $p - p_\varphi(t)$ through the formula:

$$p(t,x) - p_\varphi(t) = \mathcal{R}\big(\psi_0^2(y-x_0)\vec{u}(t,y) \otimes \vec{u}(t,y)\big)(x) + \int (R(x,y) - R(x_0,y))(1 - \psi_0^2(y-x_0))\vec{u}(t,y) \otimes \vec{u}(t,y) \ dy$$

$$= p_1(t,x) + p_2(t,x)$$

It is easy to check that, on the support of $\varphi(x) = \varphi_0(x - x_0)$, $p_2(t,x)$ is bounded by $C\|\vec{u}(t,.)\|^2_{E_2}$. On the other hand, $\|p_1(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3)} \leq C\|\psi^2 \ \vec{u} \otimes \vec{u}\|_{L^{3/2}}$ (R being a Calderón–Zygmund operator). But $H^1(\mathbb{R}^3) \subset L^6$ and thus $L^6(L^2) \cap L^2(H^1) \subset L^3(L^3)$. p_1 is finally controlled by : $\|p_1(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3} \leq C\|\psi\vec{u}\|_{L^6([T_0,T_1],L^2)} \ \|\psi\vec{u}\|_{L^2([T_0,T_1],H^1)}$. Thus, we have proved the lemma.

We now prove the size estimates for a suitable Leray solution.

Lemma 4 : For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality :

$$\|\vec{u}(t,.)\varphi(x)\|_2^2 + 2\int_0^t \|(\vec{\nabla} \otimes \vec{u}(s,.))\varphi(x)\|_2^2 \ ds \leq \tag{11}$$

$$\leq \|\vec{u}_0\varphi(x)\|_2^2 + \int\int_0^t |\vec{u}|^2\ \Delta(\varphi^2(x))\ dx\ ds + \int\int_0^t (|\vec{u}|^2 + 2(p-p_\varphi))(\vec{u}.\vec{\nabla})\varphi^2(x)\ \ dx\ ds$$

Proof : We start from the local energy inequality (3) applied to $\phi_{\tau,\epsilon}(t,x) = \varphi(x)\ g_{\tau,\epsilon}(t)$ with $g_{\tau,\epsilon}(t) = G(t/\epsilon) - G((t-\tau)/\epsilon)$ where G is a fixed smooth function equal to 0 on $(-\infty, 1]$ and to 1 on $[2, +\infty)$. We may replace p by $p - p_\varphi$ since we have $\int \vec{u}(t,x).\vec{\nabla}\varphi(x)\ dx = \int \vec{\nabla}.(\varphi(x)\ \vec{u}(t,x))\ dx = 0$. We fix $\tau \in (0,T)$ and let ϵ go to 0. We obtain that

$$\limsup_{\epsilon\to 0} \int_0^T \|\vec{u}(s,.)\varphi(x)\|_2^2 \frac{1}{\epsilon}\big(G'((s-\tau)/\epsilon) - G'(s/\epsilon)\big)\ ds + 2\int_0^\tau \|(\vec{\nabla}\otimes\vec{u}(s,.))\varphi(x)\|_2^2\ ds \leq$$

$$\leq \int\int_0^\tau |\vec{u}|^2\ \Delta(\varphi^2(x))\ dx\ ds + \int\int_0^\tau (|\vec{u}|^2 + 2(p-p_\varphi))(\vec{u}.\vec{\nabla})\varphi^2(x)\ \ dx\ ds$$

Now, if τ is a Lebesgue point of $t \to \|\vec{u}(t,x)\varphi(x)\|_2^2$, we obtain that

$$\lim_{\epsilon\to 0} \int_0^T \|\vec{u}(s,.)\varphi(x)\|_2^2 \frac{1}{\epsilon}\big(G'((s-\tau)/\epsilon) - G'(s/\epsilon)\big)\ ds = \|\vec{u}(\tau,x)\varphi(x)\|_2^2 - \|\vec{u}_0\varphi(x)\|_2^2$$

Thus, when φ is fixed, inequality (11) is true for almost every t in $(0,T)$. But $t \to \vec{u}(t,.)$ is weakly continuous and thus if τ_n is a sequence of times converging to t, we have $\|\vec{u}(t,x)\varphi(x)\|_2^2 \leq \liminf \|\vec{u}(\tau_n,x)\varphi(x)\|_2^2$. Finally, we may conclude that inequality (11) is valid for all t.

Lemma 5 : For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality :

$$\int\int_0^t |\vec{u}|^2\ \Delta(\varphi^2(x))\ dx\ ds \leq C_1 \int_0^t \alpha(s)\ ds$$

Proof : This is obvious.

Lemma 6 : For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality :

$$\int\int_0^t |\vec{u}|^2(\vec{u}.\vec{\nabla})\varphi^2(x)\ \ dx\ ds \leq C_2(\int_0^t \alpha(s)^3\ ds)^{1/4}\ \big(\beta(t) + \int_0^t \alpha(s)\ ds\big)^{3/4}$$

Proof : Let ψ_0 be equal to 1 on the support of φ_0 and let us write, for $\varphi(x) = \varphi_0(x - x_0)$,

$$\int\int_0^t |\vec{u}|^2(\vec{u}.\vec{\nabla})\varphi^2(x)\ \ dx\ ds \leq C\int\int_0^t |\psi_0(x-x_0)\ \vec{u}|^3\ dx\ ds$$

We use the inclusions $L^2(L^6) \cap L^6(L^2) \subset L^3(L^3)$ and $H^1(\mathbb{R}^3) \subset L^6$ to get

$$\int\int_0^t |\psi_0(x-x_0)\ \vec{u}|^3\ dx\ ds \leq C\big(\int_0^t \|\psi_0(x-x_0)\ \vec{u}\|_2^6\ ds\big)^{1/4}\ \big(\int_0^t \|\psi_0(x-x_0)\ \vec{u}\|_{H^1}^2\ ds\big)^{3/4}$$

and we have just to check the (easy) inequalities $\int(|\psi_0(x-x_0)|^2 + |\vec{\nabla}\psi_0(x-x_0)|^2)\ |\vec{u}(t,x)|^2\ dx \leq \alpha(t)$ and $\int\int_0^t |\psi_0(x-x_0)|^2\ |\vec{\nabla}\otimes\vec{u}(s,x)|^2\ dx\ ds \leq \beta(t)$.

Lemma 7 : For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality :

$$\int\int_0^t (p-p_\varphi)(\vec{u}.\vec{\nabla})\varphi^2(x)\ \ dx\ ds \leq C_3(\int_0^t \alpha(s)^3\ ds)^{1/4}(\beta(t) + \int_0^t \alpha(s)\ ds)^{3/4}$$

Proof : We use the results of Lemma 3 to get that

$$\int\int_0^t (p-p_\varphi)(\vec{u}.\vec{\nabla})\varphi_0^2(x-x_0)\ \ dx\ ds \leq$$

$$\leq C\ \big(\int\int_0^t |p(t,x)-p_\varphi(s)|^{3/2}\ \omega_0(x-x_0)\ dx\ dt\big)^{2/3}\ \big(\int\int_0^t |\vec{u}|^3\ \omega_0(x-x_0)\ dx\ dt\big)^{1/3}$$
$$\leq C'\big(\|\vec{u}\|^2_{L^3([0,t],E_2)}+\|\psi_0(x-x_0)\vec{u}\|^2_{L^3([0,t]\times\mathbb{R}^3)}\big)\ \|\psi_0(x-x_0)\vec{u}\|_{L^3([0,t]\times\mathbb{R}^3)}$$

and we conclude since we know that $\|\psi_0(x-x_0)\vec{u}\|_{L^3([0,t]\times\mathbb{R}^3)} \leq C''(\int_0^t \alpha(s)^3\ ds)^{1/12}(\beta(t)+\int_0^t \alpha(s)\ ds)^{1/4}$ and $\|\vec{u}\|_{L^3([0,t],E_2)} \leq \|\vec{u}\|^{1/2}_{L^2([0,t],E_2)}\ \|\vec{u}\|^{1/2}_{L^6([0,t],E_2)} \leq C'''(\int_0^t \alpha(s)\ ds)^{1/4}\ (\int_0^t \alpha(s)^3\ ds)^{1/12}$.

End of the proof : Now, clearly, these lemmas give us the following inequalities (just by looking at the extrema of $\|\vec{u}(t,.)\varphi(x)\|_2^2$ and of $2\int_0^t \|(\vec{\nabla}\otimes\vec{u}(s,.))\varphi(x)\|_2^2\ ds)$:

$$\alpha(t) \leq \alpha(0) + C_1\int_0^t \alpha(s)\ ds + (C_2+2C_3)(\int_0^t \alpha(s)^3\ ds)^{1/4}(\beta(t)+\int_0^t \alpha(s)\ ds)^{3/4}$$
$$2\beta(t) \leq \alpha(0) + C_1\int_0^t \alpha(s)\ ds + (C_2+2C_3)(\int_0^t \alpha(s)^3\ ds)^{1/4}(\beta(t)+\int_0^t \alpha(s)\ ds)^{3/4}$$

We then use the Young inequality

$$(\int_0^t \alpha(s)^3\ ds)^{1/4}(\beta(t)+\int_0^t \alpha(s)\ ds)^{3/4} \leq \frac{1}{4\epsilon^4}\int_0^t \alpha(s)^3\ ds\ + \frac{3\epsilon^{4/3}}{4}(\beta(t)+\int_0^t \alpha(s)\ ds)$$

with $3/4\ (C_2+2C_3)\ \epsilon^{4/3} < 1$ to get rid of $\beta(t)$ and we obtain

$$\alpha(t) \leq C_4\ (\alpha(0)+\int_0^t \alpha(s)\ ds + \int_0^t \alpha(s)^3\ ds)$$

We then write $\alpha(s) \leq \alpha_0 + \frac{\alpha(s)^3}{\alpha_0^2}$ and get for $t<1$:

$$\alpha(t) \leq C_0\ (\alpha(0)+\int_0^t \frac{\alpha(s)^3}{\alpha_1^2}\ ds)$$

Thus, the function $\gamma(t) = \alpha(0)+\int_0^t \frac{\alpha(s)^3}{\alpha_1^2}\ ds$ is a solution of the differential inequality $\gamma' \leq C_0^4/\alpha_1^2\ \gamma^3$. Thus, we obtain $1/\alpha_0^2 - 1/\gamma(t)^2 \leq C_0^4/\alpha_1^2\ t$, so that $\alpha(t) \leq C_4(1/\alpha_0^2 - C_0^4/\alpha_1^2\ t)^{-1/2}$ and this finishes the proof of Proposition 3.

4. Spatial decay estimates at infinity for a suitable Leray solution.

In this section, we still consider a local Leray solution and we shall prove that we may find an estimate for the decay at infinity of the solution which depends only of the decay rate of the initial value and on its E_2 norm :

Proposition 4 : Let $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$ and let $\vec{u}$ be be a local Leray solution on $(0,T^*)\times\mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec{u}_0$. Let $\gamma \in \mathcal{D}(\mathbb{R}^3)$ be equal to 1 in a neighbourhood of 0 and define $\chi_R(x) = 1-\gamma(x/R)$. Then, for all $T<T^*$, there exist a positive constant C_T such that for all $0<t<T$ and all $R>1$ we have

$$\|\vec{u}(t,.)\chi_R\|_{E_2} \leq C_T\ (\|\vec{u}_0\chi_R\|_{E_2} + \sqrt{\frac{1+\ln R}{R}}) \tag{12}$$

Moreover C_T depends only on T, $\sup_{\varphi\in\mathcal{B}}\int_0^T \|(\vec\nabla\otimes\vec u(s,.))\varphi(x)\|_2^2\ ds$ and $\sup_{t<T,\varphi\in\mathcal{B}}\|\vec u(t,.)\varphi(x)\|_2^2$.

Proof : We mimick the proof of Proposition 3. We define again $\alpha(t) = \sup_{\varphi\in\mathcal{B}}\|\vec u(t,.)\varphi(x)\|_2^2$ and $\beta(t) = \sup_{\varphi\in\mathcal{B}}\int_0^t \|(\vec\nabla\otimes\vec u(s,.))\varphi(x)\|_2^2\ ds$. We define moreover $\alpha_R(t) = \sup_{\varphi\in\mathcal{B}}\|\vec u(t,.)\varphi(x)\chi_R(x)\|_2^2$ and $\beta_R(t) = \sup_{\varphi\in\mathcal{B}}\int_0^t \|(\vec\nabla\otimes\vec u(s,.))\varphi(x)\chi_R(x)\|_2^2\ ds$. Lemmas 3 to 7 then are changed in the following ones.

Lemma 8 : We have the following estimate valid for all $R>1$ and all $\varphi\in\mathcal{B}$ ($\varphi(x)=\varphi(x-x_0)$ for some $x_0\in\mathbb{R}^3$) : $(p-p_\varphi)\chi_R^2$ can be decomposed into $(p(t,x)-p_\varphi(t))\ \chi_R^2(x) = p_{\varphi,R}(t,x)+q_{\varphi,R}(t,x)$ such that

$$\Big(\iint_{(T_0,T_1)\times\mathbb{R}^3}|p_{\varphi,R}(t,x)|^{3/2}\ \varphi(x)\ dx\ dt\Big)^{2/3}+\Big(\iint_{(T_0,T_1)\times\mathbb{R}^3}|\vec\nabla q_{\varphi,R}(t,x)|^{3/2}\ \varphi(x)\ dx\ dt\Big)^{2/3}\leq$$

$$\leq C\big(\|\chi_R\vec u\|^2_{L^3(I,E_2)}+\|\psi\chi_R\vec u\|_{L^6(I,L^2)}\ \|\psi\chi_R\vec u\|_{L^2(I,H^1)}+\frac{\ln R}{R}\|\vec u\|^2_{L^3(I,E_2)}+\frac{1}{R}\|\psi\vec u\|_{L^6(I,L^2)}\ \|\psi\vec u\|_{L^2([T_0,T_1],H^1)}\big)$$

where we write $I=[T_0,T_1]$ and where C is a constant which depend neither on x_0, nor on R, nor on T_0,T_1.

Proof : Remember that in the proof of Lemma 3, we split $p-p_\varphi$ into p_1+p_2. Now, we just write :

$$p_1(x,t)\chi_R^2(x)=\mathcal{R}\big(\psi_0^2(y-x_0)\chi_R^2(y)\vec u(t,y)\otimes\vec u(t,y)\big)(x)-[\mathcal{R},\chi_R^2]\big(\psi_0^2(y-x_0)\vec u(t,y)\otimes\vec u(t,y)\big)(x)=q_1+q_2$$

and similarly $p_2=q_3+q_4$ where $q_3(t,x)=\int(R(x,y)-R(x_0,y))(1-\psi_0^2(y-x_0))\chi_R^2(y)\vec u(t,y)\otimes\vec u(t,y)\ dy$ and $q_4(t,x)=\int(R(x,y)-R(x_0,y))(1-\psi_0^2(y-x_0))(\chi_R^2(x)-\chi_R^2(y))\ \vec u(t,y)\otimes\vec u(t,y)\ dy$. On the support of $\varphi_0(x-x_0)$, $q_3(t,x)$ is bounded by $C\|\vec u(t,.)\chi_R\|^2_{E_2}$ and q_4 by $C\frac{\ln R}{R}\|\vec u(t,.)\|^2_{E_2}$. On the other hand, we control q_1 by $\|q_1(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3}\leq C\|\psi\ \chi_R\ \vec u\|_{L^6([T_0,T_1],L^2)}\ \|\psi\ \chi_R\ \vec u\|_{L^2([T_0,T_1],H^1)}$. The term q_2 is the most difficult. We compute $\vec\nabla q_2$ as

$$\vec\nabla q_2=(\vec\nabla\chi_R^2)(x)\mathcal{R}\big(\psi_0^2(y-x_0)\vec u(t,y)\otimes\vec u(t,y)\big)(x)-[\vec\nabla\mathcal{R},\chi_R^2]\big(\psi_0^2(y-x_0)\vec u(t,y)\otimes\vec u(t,y)\big)(x)=q_5+q_6$$

q_5 is then controlled by $\|q_5(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3}\leq C\frac{1}{R}\|\psi\vec u\|_{L^6([T_0,T_1],L^2)}\ \|\psi\vec u\|_{L^2([T_0,T_1],H^1)}$. For q_6, we just notice that the commutator $[\vec\nabla\mathcal{R},\chi_R^2]$ is a Calderón commutator between an operator of order 1 and a Lipschitz function, hence that it is a Calderón-Zygmund operator (see [MEY 2] for instance) whose operator norm is controlled by the L^∞ norm of $\vec\nabla\chi_R^2$; we thus obtain a control on q_5, namely that $\|q_5(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3}\leq C\frac{1}{R}\|\psi\vec u\|_{L^6([T_0,T_1],L^2)}\ \|\psi\vec u\|_{L^2([T_0,T_1],H^1)}$. Thus, we proved the lemma.

Lemma 9 : For all $t>0$ and all $\varphi\in\mathcal{B}$, we have the following inequality :

$$\|\vec u(t,.)\chi_R(x)\varphi(x)\|_2^2+2\int_0^t\|(\vec\nabla\otimes\vec u(s,.))\chi_R(x)\varphi(x)\|_2^2\ ds\leq$$

$$\leq\|\vec u_0\chi_R(x)\varphi(x)\|_2^2+\iint_0^t|\vec u|^2\ \Delta(\chi_R^2(x)\varphi^2(x))\ dx\ ds+\iint_0^t(|\vec u|^2+2(p-p_\phi))(\vec u.\vec\nabla)(\chi_R^2(x)\varphi^2(x))\ \ dx\ ds$$

Proof : Same proof as for Lemma 4.

Lemma 10 : For all $t>0$, all $\varphi\in\mathcal{B}$ and all $R>1$, we have the following inequality :

$$\iint_0^t|\vec u|^2\ \Delta(\chi_R^2(x)\varphi^2(x))\ dx\ ds\leq C_1\ \Big(\int_0^t\alpha_R(s)\ ds+\frac{1}{R}\int_0^t\alpha(s)\ ds\Big)$$

Proof : This is obvious. We just have to consider whether we differentiate χ_R (getting the $O(1/R)$ term) or not (getting the term involving α_R).

Lemma 11 : For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality :

$$\int\int_0^t |\vec{u}|^2(\vec{u}.\vec{\nabla})(\chi_R^2(x)\varphi^2(x)) \ \ dx\ ds \leq C_2 M(t)\{(\int_0^t \alpha_R(s)^3\ ds)^{1/6}\ \left(\beta_R(t) + \int_0^t \alpha_R(s)\ ds\right)^{1/2} + \frac{1}{R}M^2(t)\}$$

with $M(t) = (\int_0^t \alpha(s)^3\ ds)^{1/12}\ \left(\beta(t) + \int_0^t \alpha(s)\ ds\right)^{1/4}$.

Proof : We consider again whether we differentiate χ_R (getting the $O(1/R)$ term) or not. When χ_R is not differentiated, we have to estimate $\int\int_0^t |\vec{u}\chi_R(x)|^2(\vec{u}.\vec{\nabla})\varphi^2(x) \ \ dx\ ds$. We proceed as in Lemma 6 to control the L^3 norm of $\psi(x)\ \vec{u}$ and of $\psi(x)\chi_R(x)\vec{u}$.

Lemma 12 : For all $t > 0$, all $\varphi \in \mathcal{B}$ and all $R > 1$, we have the following inequality :

$$\int\int_0^t (p-p_\varphi)(\vec{u}.\vec{\nabla})(\chi_R^2(x)\varphi^2(x))dx\ ds \leq C_3 M(t)\{(\int_0^t \alpha_R(s)^3\ ds)^{1/6}\left(\beta_R(t)+\int_0^t \alpha_R(s)\ ds\right)^{1/2}+\frac{1+\ln R}{R}M(t)^2\}$$

Proof : We consider one more time whether we differentiate χ_R (getting an $O(1/R)$ term) or not. When χ_R is not differentiated, we have to estimate $\int\int_0^t (p-p_\varphi)\chi_R^2(x)(\vec{u}.\vec{\nabla})\varphi^2(x) \ \ dx\ ds$. We then decompose $(p-p_\varphi)\chi_R^2(x)$ into $p_{x_0,R}(t,x) + q_{x_0,R}(t,x)$ accordingly to Lemma 8 and we write

$$\int\int_0^t (p-p_\varphi)\chi_R^2(x)(\vec{u}.\vec{\nabla})\varphi^2 \ \ dx\ ds = \int\int_0^t p_{x_0,R}(s,x)(\vec{u}.\vec{\nabla})\varphi^2(x) \ \ dx\ ds - \int\int_0^t \varphi^2(x)\ (\vec{u}.\vec{\nabla})q_{x_0,R}(s,x)\ dx\ ds$$

Putting together estimates of Lemma 8 and Lemma 7 gives the required estimate.

End of the proof : Now, clearly, these lemmas give us the following inequalities (just by looking at the extrema of $\|\vec{u}(t,.)\chi_R(x)\varphi(x)\|_2^2$ and of $2\int_0^t \|(\vec{\nabla}\otimes\vec{u}(s,.))\chi_R(x)\varphi(x)\|_2^2\ ds$ and by controlling the size of $M(t)$ on $(0,T)$) :

$$\alpha_R(t) \leq \alpha_R(0) + C_1\int_0^t \alpha_R(s)\ ds + A_T(\int_0^t \alpha_R(s)^3\ ds)^{1/6}(\beta_R(t) + \int_0^t \alpha_R(s)\ ds)^{1/2} + B_T\frac{1+\ln R}{R}$$

$$2\beta_R(t) \leq \alpha_R(0) + C_1\int_0^t \alpha_R(s)\ ds + A_T(\int_0^t \alpha_R(s)^3\ ds)^{1/6}(\beta_R(t) + \int_0^t \alpha_R(s)\ ds)^{1/2} + B_T\frac{1+\ln R}{R}$$

We then use the Young inequality

$$(\int_0^t \alpha_R(s)^3\ ds)^{1/6}(\beta_R(t) + \int_0^t \alpha_R(s)\ ds)^{1/2} \leq \frac{1}{2\epsilon_T^2}(\int_0^t \alpha_R(s)^3\ ds)^{1/3}\ + \frac{\epsilon_T^2}{2}(\beta_R(t) + \int_0^t \alpha_R(s)\ ds)$$

with $A_T\epsilon_T^2/2 < 1$ to get rid of $\beta(t)$ and we obtain

$$\alpha_R(t) \leq D_T\ \left(\alpha_R(0) + \int_0^t \alpha_R(s)\ ds + (\int_0^t \alpha_R(s)^3\ ds)^{1/3} + \frac{1+\ln R}{R}\right)$$

We now use the Hölder inequality : $\int_0^t \alpha_R(s)\ ds \leq T^{2/3}(\int_0^t \alpha_R(s)^3)^{1/3}$ and obtain finally :

$$\alpha_R^3(t) \leq E_T\ \left(\alpha_R^3(0) + \int_0^t \alpha_R(s)^3\ ds + (\frac{1+\ln R}{R})^3\right)$$

Thus, the function $\gamma(t) = \alpha_R^3(0) + \int_0^t \alpha_R(s)^3 \, ds$ is a solution of the differential inequality $\gamma' \leq E_t \, (\gamma + (\frac{1+\ln R}{R})^3)$. This gives Proposition 4.

5. Local existence of a weak solution for the E_2 initial value problem.

We may now begin the proof of Theorem 1. In this section, we prove the existence of a suitable weak solution in $(0,T) \times \mathbb{R}^3$ for some positive T. The extension to $T = \infty$ will be studied in Section 7.

Thus, we consider an initial value $\vec{u}_0 \in (E_2)^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. We know (Proposition 2) that we may approximate $\vec{u}_0$ by $\vec{v}_\epsilon$ such that $\vec{v}_\epsilon \in (L^2)^3$, $\|\vec{u}_0 - \vec{v}_\epsilon\|_{E_2} < \epsilon$ and $\vec{\nabla}.\vec{v}_\epsilon = 0$. Moreover, we know that we may find a suitable solution $\vec{u}_\epsilon$ for the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^3$ with $\vec{v}_\epsilon$ as initial value (Theorem 3). Now, we use the results of section 4 (Proposition 3) to get that for some positive T

$$\sup_\epsilon \sup_{x_0 \in \mathbb{R}^3, t<T} \int_{|x-x_0|\leq 1} |\vec{u}_\epsilon(t,x)|^2 \, dx + \int\int_{0<s<t, |x-x_0|\leq 1} |\vec{\nabla} \otimes \vec{u}_\epsilon|^2 \, dx \, ds < \infty$$

We may thus apply Lemma 1 to get that for some sequence ϵ_n converging to 0 we have convergence of $\vec{u}_\epsilon$ (for $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$, we have strong convergence of $\phi \vec{u}_\epsilon$ in $L^p((0,T),(L^2)^3)$ for all $p < \infty$ and weak convergence in $L^2((0,T),(H^1)^3)$). Moreover, if we use the formula (8) :

$$p = \sum_{j<0} p_j(x) - p_j(0) + \sum_{j\geq 0} p_j(x)$$

we see that for all $t < T$ the pression p_ϵ is bounded in $L^\infty((0,t), B_\infty^{-3,\infty})$ which is a dual space of a separable Banach space, hence we may have weak convergence of p_{ϵ_n}. The weak limits $(\vec{u}, p)$ are then a solution of the Navier–Stokes equations in $(0,T) \times \mathbb{R}^3$.

We have to show that this solution is suitable, that $\vec{u} \in \cap_{t<T} L^\infty((0,t),(E_2)^3)$ and $\lim_{t\to 0^+} \|\vec{u} - \vec{u}_0\|_{E_2} = 0$. Since the space $M^{2,2}$ of uniformly locally square integrable functions is a dual space, the control on the E_2 norm of $\vec{u}_\epsilon$ gives $\vec{u} \in \cap_{t<T} L^\infty((0,t),(M^{2,2})^3)$. Moreover, we know that we may control the decay rate of $\vec{u}_\epsilon$ (as we have seen it in the preceding section) : indeed we have

$$\|\vec{u}_\epsilon(t,.)\chi_R\|_{E_2} \leq C \, (\|\vec{u}_0\chi_R\|_{E_2} + \|\vec{u}_0 - \vec{v}_\epsilon\|_{E_2} + \sqrt{\frac{1+\ln R}{R}})$$

This proves that $\vec{u}$ belongs to $(E_2)^3$ for all t. Moreover,this gives that $\|(\vec{u} - \vec{u}_0)\chi_R\|_{E_2}$ is controlled by $\|\vec{u}_0\chi_R\|_{E_2} + \sqrt{\frac{1+\ln R}{R}}$. Moreover the strong convergence in $(L^2_{loc}((0,T) \times \mathbb{R}^3))^3$ gives that for a subsequence ϵ_n we have for all $N > 0$ and for almost all $t < T$ $\lim_{\epsilon_n \to 0} \int_{|x|<N} |\vec{u}_{\epsilon_n} - \vec{u}|^2 \, dx = 0$.

Now, we consider the energy inequality :

$$\|\vec{u}_\epsilon(t,.)\varphi(x)\|_2^2 + 2\int_0^t \|(\vec{\nabla} \otimes \vec{u}_\epsilon(s,.))\varphi(x)\|_2^2 \, ds \leq$$

$$\leq \|\vec{v}_\epsilon\varphi(x)\|_2^2 + \int\int_0^t |\vec{u}_\epsilon|^2 \, \Delta(\varphi^2(x)) \, dx \, ds + \int\int_0^t (|\vec{u}_\epsilon|^2 + 2(p_\epsilon - p_{\epsilon,\varphi}))(\vec{u}_\epsilon.\vec{\nabla})\varphi \, ds \, dx$$

We have $\|\vec{u}(t,.)\varphi(x)\|_2^2 + 2\int_0^t \|(\vec{\nabla} \otimes \vec{u}(s,.))\varphi(x)\|_2^2 \, ds \leq \liminf \|\vec{u}_\epsilon(t,.)\varphi(x)\|_2^2 + 2\int_0^t \|(\vec{\nabla} \otimes \vec{u}_\epsilon(s,.))\varphi(x)\|_2^2 \, ds$. On the other hand, we have $\|\vec{u}_0\varphi(x)\|_2^2 + \int\int_0^t |\vec{u}|^2 \, \Delta(\varphi^2(x)) \, dx \, ds = \lim \|\vec{v}_\epsilon\varphi(x)\|_2^2 + \int\int_0^t |\vec{u}_\epsilon|^2 \, \Delta(\varphi^2(x)) \, dx \, ds$. We check now that we have as well $\int\int_0^t (|\vec{u}|^2 + 2(p - p_\varphi))(\vec{u}.\vec{\nabla})\varphi \, ds \, dx = \lim \int\int_0^t (|\vec{u}_\epsilon|^2 + 2(p_\epsilon - p_{\epsilon,\varphi}))(\vec{u}_\epsilon.\vec{\nabla})\varphi \, ds \, dx$. Indeed, we know that $\vec{u}_\epsilon\psi$ is uniformly bounded in $L^4((0,t),(L^3)^3)$ and that $(p_\epsilon - p_{\epsilon,\varphi}(s))\psi$ is uniformly bounded in $L^2((0,t), L^{3/2})$ so that $\lim_{\delta\to 0} \sup_\epsilon \int\int_0^\delta (|\vec{u}_\epsilon|^2 + 2(p_\epsilon - p_{\epsilon,\varphi}))(\vec{u}_\epsilon.\vec{\nabla})\varphi \, ds \, dx = 0$. Now, we consider

$\int\int_\delta^t (|\vec u_\epsilon|^2 + 2(p_\epsilon - p_{\epsilon,\varphi}))(\vec u_\epsilon.\vec\nabla)\varphi\, ds\, dx$: we know that $\vec u_\epsilon$ converges strongly in $(L^3_{loc}((0,T)\times \mathbb{R}^3)^3$ and thus we have no problem with $\int\int_\delta^t |\vec u_\epsilon|^2(\vec u_\epsilon.\vec\nabla)\varphi\, ds\, dx$; the pressure can be dealt with as in section 3: we write

$$p_\epsilon(s,x) - p_{\epsilon,\phi,R}(s) = \mathcal{R}\big(\gamma_R(y-x_0)\vec u_\epsilon(s,y)\otimes\vec u_\epsilon(s,y)\big)(x) + \int (R(x,y)-R(x_0,y))\,\chi_R(y-x_0)\vec u_\epsilon(s,y)\otimes\vec u_\epsilon(s,y)\, dy\ ;$$

the first term in the right hand side of this equality converges strongly, as R is fixed, in $L^{3/2}((\delta,t)\times\mathbb{R}^3)$ while the second term is uniformly (in ϵ) bounded on the support of ψ by a constant $C(R)$ which goes to O as R goes to ∞ (since we control the decay of $\vec u_\epsilon$). Moreover, we may change p_ϵ by adding a constant $q_\epsilon(s)$ without changing the value $\int\int_\delta^t (p_\epsilon - p_{\epsilon,\varphi})(\vec u_\epsilon.\vec\nabla)\varphi\, ds\, dx$ since $\vec u_\epsilon$ is divergence free ; thus we get the desired convergence.

We have thus obtained the energy inequality

$$\|\vec u(t,.)\varphi(x)\|_2^2 + 2\int_0^t \|(\vec\nabla\otimes\vec u(s,.))\varphi(x)\|_2^2\, ds \leq$$

$$\leq \|\vec u_0\varphi(x)\|_2^2 + \int\int_0^t |\vec u|^2\ \Delta(\varphi^2(x))\ dx\ ds + \int\int_0^t (|\vec u|^2 + 2(p-p_\varphi))(\vec u.\vec\nabla)\varphi\ ds\ dx$$

Moreover, the equicontinuity of $\vec u_\epsilon$ in $(B_\infty^{-4,\infty})^3$, the convergence of $\vec v_\epsilon$ to $\vec u_0$ and the convergence of $\vec u_\epsilon(t)$ to $\vec u(t)$ for almost all t gives that $\vec u_0$ is the weak limit of $\vec u(t)$ as t goes to 0 ; but we have, due to the energy inequality, $\limsup_{t\to 0}\|\vec u_\epsilon\varphi\|_2 \leq \|\vec u_0\varphi\|_2$ which gives that for all $\phi\in\mathcal{B}$ we have $\lim_{t\to 0}\|\phi(\vec u - \vec u_0)\|_2 = 0$. Using this together with te decay estimate, we may conclude that $\lim_{t\to 0}\|\vec u - \vec u_0\|_{E_2} = 0$.

Now, we just conclude that $\vec u$ is suitable, in proving the strong energy inequality in the same way as for the Leray inequality. This far, we have proved the following statement :

Local existence : For all $\vec u_0 \in (E_2(\mathbb{R}^3))^3$ such that $\vec\nabla.\vec u_0 = 0$, there exists a positive T and a local Leray solution $\vec u$ on $(0,T)\times\mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec u_0$.

In order to get a solution beyond T, we need to have a better understanding of the local Leray solutions:

Proposition 5 : If $\vec u$ is a local Leray solution on $(0,T)\times\mathbb{R}^3$ for the Navier–Stokes initial value problem associated to $\vec u_0$ (for some $\vec u_0\in (E_2)^3$, then we have the following regularity results :
(A) for almost all $t\in(0,T)$, $\lim_{s>t, s\to t}\|\vec u(s) - \vec u(t)\|_{E_2} = 0$.
(B) for all $[\delta,t]\subset(0,T)$, $\vec u\in L^4((\delta,t),(E_3)^3)$.

Proof : Point (A) is easy. We use the energy equality on suitable solutions to get that for all $\varphi\in\mathcal{B}$ and for all Lebesgue points $t<\tau$ of $\|\vec u\varphi\|_2$ we have the inequality

$$\|\vec u(\tau,.)\varphi(x)\|_2^2 + 2\int_t^\tau \|(\vec\nabla\otimes\vec u(s,.))\varphi(x)\|_2^2\, ds \leq$$

$$\leq \|\vec u(t,.)\varphi(x)\|_2^2 + \int\int_t^\tau |\vec u|^2\ \Delta(\varphi^2(x))\ dx\ ds + \int\int_t^\tau (|\vec u|^2 + 2(p-p_\varphi))(\vec u.\vec\nabla)\varphi\ ds\ dx$$

Then, we use the weak continuity of $t\to\vec u(t,.)$ to conclude that this equality is valid for almost all t and for all $\tau>t$. Thus, for all $\varphi\in\mathcal{B}$ and for all Lebesgue points t of $\|\vec u\varphi\|_2$ we have $\lim_{\tau>t,\tau\to t}\|(\vec u(\tau)-\vec u(t))\varphi\|_2 = 0$. Moreover, Proposition 4 implies a good uniform control of the decay of $\|\vec u(\tau)\varphi_0(x-k)\|_2$ when $k\in\mathbb{Z}^3$ goes to ∞ whereas we have a good control of $\|(\vec u(\tau)-\vec u(t))\varphi_0(x-k)\|_2$ on the points t which are Lebesgue points for all $\|\vec u\varphi_0(x-k)\|_2, k\in\mathbb{Z}^3$, hence for almost all t. Since $\sum_k\ \varphi_0(x-k) = 1$, this gives (A). In order to prove point (B), we use again Proposition 4and we see from the estimates on $\vec u$ and on p that for all $t<T$ we have

$$\lim_{R\to\infty}\sup_{|x_0|>R}\int\int_{|x-x_0|<\min(\sqrt{t},1),\sup(0,t-1)<s<t} |\vec u|^3 + |p(s,x) - p_{x_0}(s)|^{3/2}\ dx\ ds = 0$$

and thus (due to the Caffarelli, Kohn and Nirenberg regularity criterion)

$$\sup_{|x_0|>R(t),|x-x_0|<\min(1,\sqrt{t})/2,\sup(3t/4,t-1/4)<s<t} |\vec{u}| < C_1 \frac{1}{\min(1,\sqrt{t})}.$$

Since we know that for every $R>0$ and for every $[\delta,t]\subset(0,T)$ $\vec{u}\in L^4((\delta,t),(L^3(B(0,R))^3)$, we get the desired estimate. Proposition 5 is proved.

6. Mild E_3 solutions and boundedness properties.

In this section, we prove Theorem 2.

Existence of a mild solution :

The proof of existence is by now quite classical. For E_3, as for other Morrey–Campanato spaces, it can be found in [CAN], [FED], [FLT], [KAT 2] or [TAY]. Following the lines of [WEI] and [KAT 1], we notice that $E_3\subset B_\infty^{1,\infty}$ or equivalently that for all $T>0$ we have $\sup_{0<t<T}\sqrt{t}\|e^{t\Delta}f\|_\infty \leq C_T\|f\|_{E_3}$; moreover, since test functions are dense in E_3, $\lim_{t\to 0}\sqrt{t}\|e^{t\Delta}f\|_\infty = 0$ for all $f\in E_3$. Thus, we look for a solution $\vec{u}$ in the space $E_T=\{\vec{u}\in\mathcal{C}([0,T],(E_3)^3)\ /\ \vec{\nabla}.\vec{u}=0, \sup_{0<t<T}\sqrt{t}\|e^{t\Delta}\vec{u}\|_\infty$ and $\lim_{t\to 0}\sqrt{t}\|e^{t\Delta}\vec{u}\|_\infty = 0\ \}$. We want $\vec{u}$ to be a fixed point of $\vec{v}\to e^{t\Delta}\vec{u}_0 - B(\vec{v},\vec{v})$ where $B(\vec{f},\vec{g})=\int_0^t e^{(t-s)\Delta}\ \mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\ ds$.

We start from the estimates, for $\sup_{t<T}\|\vec{v}_i\|_{E_3}\leq M_3$ and $\sup_{t<T}\sqrt{t}\|\vec{v}_i\|_\infty\leq M_\infty$,

$$\|e^{(t-s)\Delta}\ \mathbb{P}\vec{\nabla}.(\vec{v}_1\otimes\vec{v}_1-\vec{v}_2\otimes\vec{v}_2)\|_{E_3}\leq C\frac{1}{\sqrt{t-s}\ \sqrt{s}}\ M_\infty\ \|\vec{v}_1-\vec{v}_2\|_{E_3}$$

$$\|e^{(t-s)\Delta}\ \mathbb{P}\vec{\nabla}.(\vec{v}_1\otimes\vec{v}_1-\vec{v}_2\otimes\vec{v}_2)\|_\infty\leq C\frac{1}{\sqrt{t-s}\ \sqrt{s}}\ M_\infty\ \min(\frac{C_T\|\vec{v}_1-\vec{v}_2\|_{E_3}}{\sqrt{t-s}},\frac{\|\vec{v}_1-\vec{v}_2\|_\infty}{\sqrt{s}})$$

hence, defining $\|\vec{u}\|_{E_T}=\sup_{t<T}\|\vec{u}\|_{E_3}+\sup_{t<T}\sqrt{t}\|\vec{u}\|_\infty$, we obtain

$$\begin{cases} \|B(\vec{v}_1,\vec{v}_1)-B(\vec{v}_2,\vec{v}_2)\|_{E_T}\leq C_1(1+C_T)M_\infty\|\vec{v}_1-\vec{v}_2\|_{E_T} \\ \sup_{t<T}\|e^{t\Delta}\vec{u}_0-B(\vec{v},\vec{v})\|_{E_3}\leq M_3(\frac{\|\vec{u}_0\|_{E_3}}{M_3}+C_2M_\infty) \\ \sup_{t<T}\|e^{t\Delta}\vec{u}_0-B(\vec{v},\vec{v})\|_\infty\leq M_\infty(\frac{\sup_{t<T}\|e^{t\Delta}\vec{u}_0\|_\infty}{M_\infty}+C_3\sqrt{C_TM_3M_\infty}) \end{cases}$$

so that we shall get a fixed point in $\{\vec{v}\in E_T\ /\ \sup_{t<T}\|\vec{v}\|_{E_3}\leq M_3$ and $\sup_{t<T}\sqrt{t}\|\vec{v}\|_\infty\leq M_\infty\ \}$ for some $T<T_0$ if we choose $M_3=2\|\vec{u}_0\|_{E_3}$, $C_1(1+C_{T_0})M_\infty<1$, $C_2M_\infty<1/2$, $C_3^2C_{T_0}M_3M_\infty<1/4$ and finally T such that $\sup_{t<T}\|e^{t\Delta}\vec{u}_0\|_\infty<M_\infty/2$.

Remark : We have proved that, given any positive T, there exists $\epsilon>0$ such that a solution exists in $\mathcal{C}([0,T],(E_3)^3)$ as soon as $\|\vec{u}_0\|_{E_3}<\epsilon$.

Uniqueness :

Uniqueness is proved in [FLT]. We give here an alternate proof, based on the proof of Y. Meyer [MEY 3] using Lorentz spaces.

Indeed, if $\vec{u}_1$ and $\vec{u}_2$ are two solutions in $\mathcal{C}([0,T],(E_3)^3)$, the difference $\vec{w}=\vec{u}_1-\vec{u}_2$ is a solution of the integral equation :

$$\vec{w}=-\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}_1\otimes\vec{w}+\vec{w}\otimes\vec{u}_2)\ ds$$

Now, for $T_2<T$, we split $\vec{u}_i$ in $\vec{X}_i+\vec{Y}_i$ where $\sup_{0<s<T}\|\vec{X}_i\|_3<\alpha$ with α small enough and $\sup_{0<s<T}\|\vec{Y}_i\|_\infty<\infty$. We define the space $ML^{p,q}$ based on the Lorents space $L^{p,q}$ by $f\in ML^{p,q}(1<p<\infty,1\leq q\leq\infty)$ if and only if f is locally integrable and $\sup_{x_0\in\mathbb{R}^3}\|f1_{[0,1]^3}(x-x_0)\|_{L^{p,q}}<\infty$. (The space $ML^{p,p}$ is then equal to the Morrey–Campanato space $M^{p,p}$). Since $\vec{w} \in \mathcal{C}([0,T_2],(E_3)^3)$, we have that $\vec{w} \in L^\infty((0,T_2),(ML^{3,\infty})^3)$. We shall prove that, if $\vec{w}$ is equal to 0 on $(0,t_0)$ for some $t_0 \in [0,T_2]$, then for all $t \geq t_0$ we have

$$\|\vec{w}(t)\|_{ML^{3,\infty}} \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{ML^{3,\infty}} \big(\sup_{0<s<t} (\|\vec{X}_1\|_3 + \|\vec{X}_2\|_3) + \sqrt{t-t_0} \sup_{0<s<t} (\|\vec{Y}_1\|_\infty + \|\vec{Y}_2\|_\infty)\big)$$

which is enough (if $2C\alpha < 1$) to grant that $\vec{w} = 0$. The estimate of $\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{Y}_1 \otimes \vec{w})\, ds$ is obvious: since pointwise multiplication by a L^∞ function operates on $ML^{3,\infty}$ and since the norm of $ML^{3,\infty}$ is shift-invariant, we have $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{Y}_1 \otimes \vec{w})\|_{ML^{3,\infty}} \leq C\frac{1}{\sqrt{t-s}}\|\vec{Y}_1(s)\|_\infty \, \|\vec{w}(s)\|_{ML^{3,\infty}}$. For the estimate on $\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{X}_1 \otimes \vec{w})\, ds$, we write that $\vec{X}_1 \otimes \vec{w} \in L^\infty((0,T_2),(ML^{3/2,\infty})^9)$ so that it is enough to check that $\vec{f} \to \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\, ds$ is bounded from $L^\infty((0,T_2),(ML^{3/2,\infty})^9)$ to $L^\infty((0,T_2),(ML^{3,\infty})^3)$. We are going to prove in an equivalent way that $\vec{f} \to A(\vec{f}) = \int_0^{T_2} e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\, ds$ is bounded from $L^\infty((0,T_2),(ML^{3/2,\infty})^9)$ to $(ML^{3,\infty})^3$. We write

$$A(\vec{f}) = A(\vec{f}1_{[-2,2]^3}(y-x_0)) + A(\vec{f}(1 - 1_{[-2,2]^3}(y-x_0))) = \vec{f}_1 + \vec{f}_2$$

Now, we estimate $\vec{f}_1$ by checking that A operates boundedly from $L^\infty((0,T_2),(L^{3/2,\infty})^9)$ to $(L^{3,\infty})^3$ (see [MEY 3] for related results) and $\vec{f}_2$ by checking that $f \to A(\vec{f}(1-1_{[-2,2]^3}(y-x_0)))$ operates boundedly from $L^\infty((0,T_2),(M^{1,1})^9)$ to $(L^\infty(x_0+[0,1]^3)^3)$. More precisely, we have estimates that for $1 < q < 3$ $\|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}\|_{L^{3,1}} \leq C_q\|\vec{g}\|_q \, s^{-\frac{3}{2q}}$ which ensure that $\int_0^\infty \|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}\|_{L^{3,1}}\, ds \leq C\|\vec{g}(x)\|_{L^{3/2,1}}$ (because each $g \in L^{3/2,1}$ can be decomposed into $g = \sum_{n\in\mathbb{N}} g_n$ with $\sum_{n\in\mathbb{N}} \|g_n\|_1^{2/3}\|g_n\|_\infty^{1/3} \leq C\|g\|_{L^{3/2,1}}$) and by duality we see that A operates boundedly from $L^\infty((0,T_2),(L^{3/2,\infty})^9)$ to $(L^{3,\infty})^3$; on the other hand we have that $|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}| \leq C\int_{\mathbb{R}^3} \frac{\sqrt{s}}{|x-y|^4}|\vec{g}(y)|\, dy$ and this gives the control on $\vec{f}_2$.

Boundedness of the E_3 solutions :

There is a direct proof by May [MAY] that a solution in $\mathcal{C}([0,T],(E_3)^3)$ is bounded : $\sup_{0<t<T} \sqrt{t}\|\vec{u}\|_\infty < \infty$.

We give here an indirect proof. We shall check that a solution in $\mathcal{C}([0,T],(E_3)^3)$ is suitable and we shall use the Caffarelli, Kohn and Nirenberg regularity criterion.

In this subsection, we check that a *suitable* solution which belongs to $\mathcal{C}([0,T],(E_3)^3)$ is indeed bounded in $1/\sqrt{t}$. The set $\{\vec{u}(t) \ / \ 0 \leq t \leq T\}$ is compact in $(E_3)^3$ and thus $\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^3, t\leq T}\int_{|x-x_0|<\delta} |\vec{u}|^3\, dx = 0$. Moreover, we may easily control the pressure : we choose $\omega \in \mathcal{D}(\mathbb{R}^3)$ equal to 1 in the neighbourhood of 0 and we compute $p(t,x)$ in the neighbourhood of x_0 up to a constant p_{t,x_0} by

$$p(t,x) - p_{t,x_0} = \mathcal{R}(\omega(y-x_0)(\vec{u}\otimes\vec{u})(t,y)) + \int(\mathcal{R}(x,y) - \mathcal{R}(x_0,y))(1-\omega(y-x_0))(\vec{u}\otimes\vec{u})(t,y)\, dy$$

We find that $q_{t,x_0} = \mathcal{R}(\omega(y-x_0)(\vec{u}\otimes\vec{u})(t,y))$ belongs to a compact set of $L^{3/2}$ when x_0 varies in $\mathbb{R}^3$ and t in $(0,T]$ while $r_{t,x_0} = \int(\mathcal{R}(x,y) - \mathcal{R}(x_0,y))(1-\omega(y-x_0))(\vec{u}\otimes\vec{u})(t,y)\, dy$ remains bounded on $|x-x_0| \leq \gamma_0$ (where γ_0 is a positive constant which does not depend on t nor on x_0) uniformly in t and x_0. This gives that $\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^3, t\leq T}\int_{|x-x_0|<\delta} |p - p_{x_0,t}|^{3/2}\, dx = 0$

We then fix δ_0 such that $\omega(\delta,t) = \sup_{x_0\in\mathbb{R}^3}\int_{|x-x_0|<\delta} |\vec{u}|^3 + |p-p_{x_0,t}|^{3/2}\, dx$ is smaller than ϵ_1 for $\delta \leq \delta_0$ and $t \leq T$. For $t_0 \leq T$, we choose $\delta(t_0) = \min(\sqrt{t_0},\delta_0)$ and the inequality $\int_{t_0-\delta(t_0)^2}^{t_0} \omega(\delta(t_0),t)\, dt \leq \epsilon_1\, \delta(t_0)^2$ gives us (due to the Caffarelli, Kohn and Nirenberg regularity criterion) that : $\|\vec{u}(t_0)\|_\infty \leq C_1\frac{1}{\delta(t_0)}$. This proves that $\|\vec{u}(t)\|_\infty$ is $O(1/\sqrt{t})$. This proves even that it is $o(1/\sqrt{t})$ (the Caffarelli, Kohn and Nirenberg criterion [CKN] gives that the local L^∞ norm of $\vec{u}$ goes to 0 if the local L^3 norm of $\vec{u}$ and the local $L^{3/2}$ of p go to 0), but this is also a consequence of the uniqueness of weak solutions in $\mathcal{C}([0,T),(E_3)^3)$ and the fact that we may find for a small time T_0 a Kato solution in in $\mathcal{C}([0,T_0),(E_3)^3)$ whose L^∞ norm is $o(1/\sqrt{t})$.

Suitability of the E_3 solutions.

Now, we prove that a solution $\vec{u} \in \mathcal{C}([0,T),(E_3)^3)$ is suitable.. We shall first prove that the problem

$$(13) \qquad \begin{cases} \partial_t \vec{z} = \Delta \vec{z} - \vec{\nabla}.(\vec{u} \otimes \vec{z}) - \vec{\nabla} q \\ \vec{\nabla}.\vec{z} = 0 \\ \vec{z}(0,.) = \vec{u}_0 \end{cases}$$

(which has $(\vec{u}, p)$ as solution) has a solution $(\vec{z}, q)$ such that, for all $t < T$, :

$$(14) \qquad \sup_{x_0 \in \mathbb{R}^3, s<t} \int_{|x-x_0|\leq 1} |\vec{z}(s,x)|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int\int_{0<s<t, |x-x_0|\leq 1} |\vec{\nabla} \otimes \vec{z}|^2 \, dx \, ds < \infty$$

Indeed, let us approximate $\vec{u}_0$ by $\vec{u}_0 = \vec{v}_\epsilon + \vec{w}_\epsilon$ with $\vec{v}_\epsilon \in (L^2)^3$ and $\|\vec{w}_\epsilon\|_{E_3} < \epsilon$. Equation (13) with $\vec{v}_\epsilon$ as initial value instead of $\vec{u}_0$ may be solved by the Leray method, which provides a solution $\vec{z}_\epsilon$. We know that this solution is in $\cap_{t<T} L^\infty((0,t),(L^2)^3) \cap L^2((0,t),(H^1)^3)$. We define $\alpha_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|\leq 1} |\vec{z}_\epsilon(t,x)|^2 \, dx$ and $\beta_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \int\int_{0<s<t, |x-x_0|\leq 1} |\vec{\nabla} \otimes \vec{z}_\epsilon|^2 \, dx \, ds$; we shall show now that for all $t < T$ we have $\sup_\epsilon (\sup_{s<t} \alpha_\epsilon(s) + \beta_\epsilon(t)) < \infty$. Indeed, we start from the identities

$$\begin{cases} \partial_t |\vec{z}_\epsilon|^2 + 2|\vec{\nabla} \otimes \vec{z}_\epsilon|^2 = \Delta |\vec{z}_\epsilon|^2 - \vec{\nabla}.(|\vec{z}_\epsilon|^2 \vec{u}) - 2\vec{\nabla}.(q_\epsilon \, \vec{z}_\epsilon) \\ q_\epsilon = \mathcal{R}(\vec{u} \otimes \vec{z}_\epsilon) \end{cases}$$

Now, we notice that $\int_{|x-x_0|<2} |q_\epsilon(s,x) - C_{s,x_0,\epsilon}|^2 \, dx \leq C \, (\{\int_{|x-x_0|<3} |\vec{z}(s,x)|^6 \, dx\}^{1/3} + \alpha_\epsilon(s)) \, \|\vec{u}(s)\|_3^2$ where C does not depend on x_0 nor on s nor on ϵ and where the constant $C_{s,x_0,\epsilon}$ is formally given by $C_{s,x_0,\epsilon} = \int_{|y-x_0|>3} \mathcal{R}(x_0,y) \vec{u} \otimes \vec{z}_\epsilon(s,y) \, dy$. Thus, if we integrate the energy equality against $1_{(0,t)}(s)\varphi(x-x_0)$ where $\varphi \in \mathcal{D}(\mathbb{R}^3)$ is equal to 1 on the unit ball $B(0,1)$ and to 0 outside from the ball $B(0,2)$, we get

$$\int |\vec{z}_\epsilon(s,x)|^2 \varphi(x-x_0) \, dx + 2 \int\int_0^t |\vec{\nabla} \otimes \vec{z}_\epsilon(s,x)|^2 \varphi(x-x_0) \, dx \leq$$

$$\leq \int |\vec{z}_\epsilon(0,x)|^2 \varphi(x-x_0) \, dx + C_1 \int_0^t \alpha_\epsilon(s) \, ds + C_2 \sup_{0<s<t} \|\vec{u}(s)\|_3 (\int_0^t \alpha_\epsilon(s) \, ds)^{1/2} (\int_0^t \alpha_\epsilon(s) \, ds + \beta_\epsilon(t))^{1/2}$$

$$\leq C_3 \alpha_\epsilon(0) + C_4 (1 + \sup_{0<s<t} \|\vec{u}(s)\|_3^2) \int_0^t \alpha_\epsilon(s) \, ds + \beta_\epsilon(t)$$

For one choice of x_0 (which depends on t and ϵ), we have $\alpha_\epsilon(t) \leq \int |\vec{z}_\epsilon(s,x)|^2 \varphi(x-x_0) \, dx$ while for another good choice of x_0 we have $\beta_\epsilon(t) \leq \int\int_0^t |\vec{\nabla} \otimes \vec{z}_\epsilon(s,x)|^2 \varphi(x-x_0) \, dx$. Thus, we have

$$\alpha_\epsilon(t) \leq 2 \, (C_3 \alpha_\epsilon(0) + C_4 (1 + \sup_{0<s<t} \|\vec{u}(s)\|_3^2) \int_0^t \alpha_\epsilon(s) \, ds)$$

and thus $\alpha_\epsilon(t) \leq 2 \, C_3 \alpha_\epsilon(0) e^{2 \, C_4 (1 + \sup_{0<s<t} \|\vec{u}(s)\|_3^2) \, t}$. This gives the desired inequalities on α_ϵ and β_ϵ.

Now, we use again the lemma 1 to get that for a sequence ϵ_n going to 0 $\vec{z}_{\epsilon_n}$ converges strongly in $(L^2_{loc}((0,T) \times \mathbb{R}^3)^3$ to a solution $\vec{z}$ of problem (13) with energy estimate (14). For such a solution, we may compute $\partial_t |\vec{z}|^2$ as $2 \, \langle \partial_t \vec{z} | \vec{z} \rangle$ and get that $\partial_t |\vec{z}|^2 + 2|\vec{\nabla} \otimes \vec{z}|^2 = \Delta |\vec{z}|^2 - \vec{\nabla}.(|\vec{z}|^2 \vec{u}) - 2\vec{\nabla}.(q \, \vec{z})$. In order to prove the suitability of $\vec{u}$, we only need to show that $\vec{z} = \vec{u}$. Thus, we will show that Problem (13) has a unique solution in $\cap_{t<T} L^\infty((0,t),(E_2)^3)$. (It is not difficult to see that $\vec{z}$, which we proved to belong to the space $\cap_{t<T} L^\infty((0,t),(M^{2,2})^3)$ is indeed in $\cap_{t<T} L^\infty((0,t),(E_2)^3)$; we proved a similar statement in Section 5 in a more difficult case). The proof runs along the same lines as the proof of uniqueness of E_3 solutions : if $\vec{z}_1$ and $\vec{z}_2$ are two such solutions, the difference $\vec{w} = \vec{z}_1 - \vec{z}_2$ is a solution of the integral equation :

$$\vec{w} = -\int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{w}) \, ds$$

For $T_2 < T$, we split $\vec{u}$ in $\vec{X}+\vec{Y}$ where $\sup_{0<s<T}\|\vec{X}\|_3 < \alpha$ with α small enough and $\sup_{0<s<T}\|\vec{Y}\|_\infty < \infty$. Since $\vec{w} \in L^\infty((0,T_2),(E_2)^3)$, we have that $\vec{w} \in L^\infty((0,T_2),(ML^{2,\infty})^3)$. We shall prove that, if $\vec{w}$ is equal to 0 on $(0,t_0)$ for some $t_0 \in [0,T_2]$, then for all $t \geq t_0$ we have

$$\|\vec{w}(t)\|_{ML^{2,\infty}} \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{ML^{2,\infty}} (\sup_{0<s<t} \|\vec{X}\|_3 + \sqrt{t-t_0} \sup_{0<s<t} \|\vec{Y}\|_\infty)$$

which is enough (if $C\alpha < 1$) to grant that $\vec{w} = 0$. The estimate of $\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{Y}\otimes\vec{w})\, ds$ is obvious: since pointwise multiplication by a L^∞ function operates on $ML^{2,\infty}$ and since the norm of $ML^{2,\infty}$ is shift-invariant, we have $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{Y}\otimes\vec{w})\|_{ML^{2,\infty}} \leq C\frac{1}{\sqrt{t-s}}\|\vec{Y}(s)\|_\infty \; \|\vec{w}(s)\|_{ML^{2,\infty}}$. For the estimate on $\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{X}\otimes\vec{w})\, ds$, we write that $\vec{X}\otimes\vec{w} \in L^\infty((0,T_2),(ML^{6/5,\infty})^9)$ so that it is enough to check that $\vec{f} \to \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\, ds$ is bounded from $L^\infty((0,T_2),(ML^{6/5,\infty})^9)$ to $L^\infty((0,T_2),(ML^{2,\infty})^3)$. We are going to prove in an equivalent way that $\vec{f} \to A(\vec{f}) = \int_0^{T_2} e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\, ds$ is bounded from $L^\infty((0,T_2),(ML^{6/5,\infty})^9)$ to $(ML^{2,\infty})^3$. We write

$$A(\vec{f}) = \; A(\vec{f}1_{[-2,2]^3}(y-x_0)) + \; A(\vec{f}(1-1_{[-2,2]^3}(y-x_0))) = \vec{f_1} + \vec{f_2}$$

Now, we estimate $\vec{f_1}$ by checking that A operates boundedly from $L^\infty((0,T_2),(L^{6/5,\infty})^9)$ to $(L^{2,\infty})^3$ and $\vec{f_2}$ by checking that $f \to \; A(\vec{f}(1-1_{[-2,2]^3}(y-x_0)))$ operates boundedly from $L^\infty((0,T_2),(M^{1,1})^9)$ to $(L^\infty(x_0+[0,1]^3)^3)$. More precisely, we have estimates that for $1 < q < 6$ $\|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}\|_{L^{6,1}} \leq C_q\|\vec{g}\|_q \; s^{-(3/q+1/2)/2}$ which ensure that $\int_0^\infty \|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}\|_{L^{6,1}}\, ds \leq C\|\vec{g}(x)\|_{L^{2,1}}$ (because each $g \in L^{2,1}$ can be decomposed into $g = \sum_{n\in\mathbb{N}} g_n$ with $\sum_{n\in\mathbb{N}} \|g_n\|_1^{1/2}\|g_n\|_\infty^{1/2} \leq C\|g\|_{L^{2,1}}$) and by duality we see that A operates boundedly from $L^\infty((0,T_2),(L^{6/5,\infty})^9)$ to $(L^{2,\infty})^3$; on the other hand we have that $|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}| \leq C\int_{\mathbb{R}^3}\frac{\sqrt{s}}{|x-y|^4}|\vec{g}(y)|\, dy$ and this gives the control on $\vec{f_2}$.

Global solutions in L^3.

We now consider the case of a global L^3 solution $\vec{u} \in \mathcal{C}([0,\infty),(L^3)^3)$. We split $\vec{u}_0$ in $\vec{v}_0 + \vec{w}_0$ where $\vec{\nabla}.\vec{v}_0 = \vec{\nabla}.\vec{w}_0 = 0$, $\vec{v}_0 \in (L^2)^3$ and $\|\vec{w}_0\|_3 < \epsilon$ where $\epsilon > 0$ is small enough. Then, we write $\vec{W}$ the global Kato soultion in $\mathcal{C}([0,\infty),(L^3)^3)$ of the Navier–Stokes equations with $\vec{w}_0$ as initial value; $\vec{v} = \vec{u} - \vec{W}$ is solution of the equation

$$\begin{cases} \partial_t\vec{v} = \Delta\vec{v} - \vec{\nabla}.(\vec{v}\otimes\vec{v}) - \vec{\nabla}.(\vec{W}\otimes\vec{v}) - \vec{\nabla}.(\vec{v}\otimes\vec{W}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{v} = 0 \\ \vec{v}(0,.) = \vec{v}_0 \end{cases}$$

Since $\vec{v}_0 \in (L^2)^3$ and $\sup_{t>0}\|\vec{W}\|_3 < 2\epsilon$, we know that this problem has a Leray solution $(\vec{z},q)$ such that :

$$\sup_{t>0}\|\vec{z}\|_2 < \infty \text{ and } \int_0^\infty \|\vec{\nabla}\otimes\vec{z}\|_2^2\, ds < \infty$$

and (for some non-negative local measure μ)

$$\partial_t|\vec{z}|^2 + 2|\vec{\nabla}\otimes\vec{z}|^2 = \Delta|\vec{z}|^2 - \vec{\nabla}.(|\vec{z}|^2\vec{z}) - \vec{\nabla}.(|\vec{z}|^2\vec{W}) - 2\vec{\nabla}.((\vec{z}.\vec{W})\;\vec{z}) + 2((\vec{z}.\vec{\nabla})\;\vec{z}).\vec{W} - 2\vec{\nabla}.(q\;\vec{z}) - \mu$$

Moreover, we know that $\vec{u}$ and $\vec{W}$ are suitable, hence that $\vec{v}$ is locally $L^2(H^1)$ and is $\mathcal{C}(L^3)$; we may then apply the Von Wahl uniqueness theorem [WAH] : indeed, we have for $\vec{y} = \vec{z} - \vec{v}$:

$$\partial_t|\vec{y}|^2 + 2|\vec{\nabla}\otimes\vec{y}|^2 = \Delta|\vec{y}|^2 - \vec{\nabla}.(|\vec{z}|^2\vec{z}) - \vec{\nabla}.(|\vec{v}|^2\vec{v}) +$$

$$+2((\vec{z}.\vec{\nabla})\;\vec{z}).\vec{v} + 2((\vec{v}.\vec{\nabla})\;\vec{v}).\vec{z} - \vec{\nabla}.(|\vec{y}|^2\vec{W}) - 2\vec{\nabla}.((\vec{y}.\vec{W})\;\vec{y}) + 2((\vec{y}.\vec{\nabla})\;\vec{y}).\vec{W} - 2\vec{\nabla}.((q-p)\;\vec{y}) - \mu$$

and we write $((\vec{z}.\vec{\nabla})\;\vec{z}).\vec{v} + ((\vec{v}.\vec{\nabla})\;\vec{v}).\vec{z} = ((\vec{y}.\vec{\nabla})\;\vec{z}).\vec{v} + \vec{\nabla}((\vec{v}.\vec{z})\;\vec{v})$; now, all the quantities can be integrated (against $1_{[0,t]}\;\omega(x/R)$ and letting R go to ∞) to give

$$\|\vec{y}(t)\|^2 + 2\int\int_0^t |\vec{\nabla}\otimes\vec{y}|^2\, ds\, dx \leq 2\int\int_0^t ((\vec{y}.\vec{\nabla})\;\vec{y}).\vec{W}\, ds\, dx + 2\int\int_0^t ((\vec{y}.\vec{\nabla})\;\vec{z}).\vec{v}\, ds\, dx$$

For finite T, we split $\vec{v}$ on $[0,T]$ in $\vec{X}+\vec{Y}$ where $\sup_t \|\vec{X}\|_3 < \epsilon$ and $\vec{Y} \in (L^\infty)^3$. Then, we get a Gronwall inequality on $\|\vec{y}\|_2^2$ which allows one to conclude that $\vec{y}=0$. This implies that $\int_0^\infty \|\vec{u}-\vec{W}\|_3^4\, dt < \infty$ while $\sup_t \|\vec{W}\|_3 < 2\epsilon$; thus $\|\vec{u}\|_3$ take small values and the Kato theorem for small datas in L^3 gives us the behaviour of $\vec{u}$ as t goes to ∞.

The generalized Von Wahl uniqueness theorem.

We consider $(\vec{u},p)$ and $(\vec{v},q)$ two local Leray solutions on $(0,T)\times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated to some $\vec{u}_0 \in (E_3)^3$ and we assume that $\vec{u} \in \mathcal{C}([0,T),(E_3)^3)$. Fisrt, we notice that $\partial_t \langle \vec{u}|\vec{v}\rangle = \langle \partial_t \vec{u}|\vec{v}\rangle + \langle \vec{u}|\partial_t \vec{v}\rangle$ since for all $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^3$ we have

$$\begin{array}{cc} \phi\vec{v} \in L^2((0,T),(H^1)^3) & \phi\partial_t\vec{v} \in L^2((0,T),(H^{-1})^3) + L^1((0,T),(L^{3/2})^3) \\ \phi\vec{u} \in L^2((0,T),(H^1)^3)\cap \mathcal{C}([0,T],(L^3)^3) & \phi\partial_t\vec{u} \in L^2((0,T),(H^{-1})^3). \end{array}$$

This gives for some non-negative local measure μ:

$$\begin{cases} \partial_t|\vec{v}|^2 + 2|\vec{\nabla}\otimes\vec{v}|^2 = \Delta|\vec{v}|^2 - \vec{\nabla}.(|\vec{v}|^2\vec{v}) - 2\vec{\nabla}.(q\ \vec{v}) - \mu \\ \partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2 = \Delta|\vec{u}|^2 - \vec{\nabla}.(|\vec{u}|^2\vec{u}) - 2\vec{\nabla}.(p\ \vec{u}) \\ \partial_t\langle\vec{u}|\vec{v}\rangle + 2\langle\vec{\nabla}\otimes\vec{u}|\vec{\nabla}\otimes\vec{v}\rangle = \\ = \Delta\langle\vec{u}|\vec{v}\rangle - \vec{\nabla}.(\langle\vec{u}|\vec{v}\rangle\vec{u}) + \langle((\vec{u}-\vec{v}).\vec{\nabla})(\vec{v}-\vec{u})|\vec{u}\rangle + \frac{1}{2}\vec{\nabla}.(|\vec{u}|^2(\vec{u}-\vec{v})) - \vec{\nabla}.(p\ \vec{v}) - \vec{\nabla}.(q\ \vec{u}) \end{cases}$$

Hence, we get :

$$\partial_t|\vec{v}-\vec{u}|^2 + 2|\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2 =$$

$$= \Delta|\vec{v}-\vec{u}|^2 - \vec{\nabla}.(|\vec{v}-\vec{u}|^2\vec{u}) - \vec{\nabla}.(\langle\vec{v}-\vec{u}|\vec{v}+\vec{u}\rangle(\vec{v}-\vec{u})) - 2\vec{\nabla}.((q-p)\ (\vec{v}-\vec{u})) - \mu$$

with $q-p = \mathcal{R}(\vec{v}\otimes(\vec{v}-\vec{u}) + (\vec{v}-\vec{u})\otimes\vec{u})$. Now, we do the same estimations as in section 3 for $\alpha(t,\vec{u}-\vec{v})$ and $\beta(t,\vec{u}-\vec{v})$ and we get :

$$\alpha(t,\vec{u}-\vec{v}) \leq \alpha(0,\vec{u}-\vec{v}) + C_1\int_0^t \alpha(s,\vec{u}-\vec{v})\, ds + C_2(\vec{u},\vec{v})(\int_0^t \alpha(s,\vec{u}-\vec{v})^3\, ds)^{1/6}(\beta(t,\vec{u}-\vec{v}) + \int_0^t \alpha(s,\vec{u}-\vec{v})\, ds)^{1/2}$$

$$2\beta(t,\vec{u}-\vec{v}) \leq \alpha(0,\vec{u}-\vec{v}) + C_1\int_0^t \alpha(s,\vec{u}-\vec{v})\, ds + C_2(\vec{u},\vec{v})(\int_0^t \alpha(s,\vec{u}-\vec{v})^3\, ds)^{1/6}(\beta(t,\vec{u}-\vec{v}) + \int_0^t \alpha(s,\vec{u}-\vec{v})\, ds)^{1/2}$$

where $C_2(\vec{u},\vec{v}) \leq C_3(1+T)^{3/2}\{(\int_0^T \alpha(s,\vec{u})^3 ds)^{1/3} + \beta(T,\vec{u}) + (\int_0^T \alpha(s,\vec{v})^3 ds)^{1/3} + \beta(T,\vec{v})\}$. Then, we can get rid of $\beta(t,\vec{u}-\vec{v})$ and, since $\alpha(0,\vec{u}-\vec{v})=0$, get

$$\alpha(t,\vec{u}-\vec{v}) \leq C_4(T)(\int_0^t \alpha(s,\vec{u}-\vec{v})^3\ ds)^{1/3}$$

which gives $\alpha(t,\vec{u}-\vec{v})=0$ and, thus, $\vec{u}=\vec{v}$.

Thus, Theorem 2 is proved.

7. Global existence of uniformly locally square integrable solutions.

We may now finish the proof of Theorem 1.

Step 1 : Local resolution in E_2.

We consider the initial value $\vec{u}_0$ in $(E_2)^3$ and we solve the Navier–Stokes equations as explained in Section 5. We then have a local Leray solution $\vec{u}^{(0)}$ on $(0,T_0)\times\mathbb{R}^3$.

Step 2 : Local resolution in E_3.

Assume we have a local Leray solution $\vec{u}^{(N)}$ on some $(S_N, T_N) \times \mathbb{R}^3$. Then, due to Proposition 5, we may choose $S_{N+1} \in (\sup(S_N, T_N - 1), T_N)$ such that

$$\vec{u}^{(N)}(S_{N+1}) \in (E_3)^3 \text{ and } \lim_{s > S_{N+1}, s \to S_{N+1}} \|\vec{u}^{(N)}(s) - \vec{u}^{(N)}(S_{N+1})\|_{E_2} = 0$$

Then, due to Theorem 2, we may find $U_{N+1} > S_{N+1}$ and a solution $\vec{v}^{(N+1)} \in \mathcal{C}([S_{N+1}, U_{N+1}], (E_3)^3)$ with $\vec{v}^{(N+1)}(S_{N+1}) = \vec{u}^{(N)}(S_{N+1})$.

Step 3 : Resolution in $E_3 + L^2$.

Using the approximation lemma, we know that we may split $\vec{u}^{(N)}(S_{N+1})$ into $\vec{X}_{N+1} + \vec{Y}_{N+1}$ with $\vec{\nabla}.\vec{X}_{N+1} = \vec{\nabla}.\vec{Y}_{N+1} = 0$, $\vec{Y}_{N+1} \in (L^2)^3$ and $\|\vec{X}_{N+1}\|_{E_3} < \epsilon$, where we choose ϵ small enough to ensure that we may find a Kato solution $\vec{X}^{(N+1)} \in \mathcal{C}([S_{N+1}, S_{N+1} + 2], (E_3)^3)$ with $\vec{X}^{(N+1)}(S_{N+1}) = \vec{X}_{N+1}$. Then, using Theorem 3, we know that we may find a Leray solution $\vec{Y}^{(N+1)}$ of

$$\begin{cases} \partial_t \vec{Y} = \Delta \vec{Y} - \vec{\nabla}.(\vec{Y} \otimes \vec{Y}) - \vec{\nabla}.(X^{(\vec{N}+1)} \otimes \vec{Y}) - \vec{\nabla}.(\vec{Y} \otimes X^{(\vec{N}+1)}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{Y} = 0 \\ \vec{Y}(S_{N+1}, .) = \vec{Y}_{N+1} \end{cases}$$

Then, $\vec{u}^{(N+1)} = \vec{X}^{(N+1)} + \vec{Y}^{(N+1)}$ is a local Leray solution on $(S_{N+1}, T_{N+1}) \times \mathbb{R}^3$ (with $T_{N+1} = S_{N+1} + 2$) of the Navier–Stokes initial value problem associated to $\vec{u}^{(N)}(S_{N+1})$.

Step 4 : Identifications of solutions.

If we use the generalized Von Wahl uniqueness theorem (Theorem 2), we find that on $[S_{N+1}, U_{N+1}]$, we have $\vec{u}^{(N)} = \vec{v}^{(N+1)} = \vec{u}^{(N+1)}$.

Step 5 : Construction of $\vec{u}$.

By induction on N, we have a sequence of local Leray solutions $\vec{u}^{(N)}$ on $(S_N, U_{N+1}) \times \mathbb{R}^3$ such that :
i) $S_0 = 0$ and $\vec{u}^{(0)}(0) = \vec{u}_0$
ii) $\vec{u}^{(N)} = \vec{u}^{(N+1)}$ on $(S_{N+1}, U_{N+1}) \times \mathbb{R}^3$
iii) $S_N + 1 < S_{N+1} < U_{N+1} < S_N + 2$ so that $(0, \infty) = \cup_{N \in \mathbb{N}} (S_N, U_{N+1})$

This defines a unique distribution $\vec{u}$ on $(0, \infty) \times \mathbb{R}^3$. Now, it is obvious that we have the desired estimates on $\vec{u}$, as it can be seen by using a non-negative partition of unity adapted to $[0, U_1) \cup \cup_{N \geq 1} (S_N, U_{N+1})$.

Thus, Theorem 1 is proved.

8. X_r spaces and Morrey–Campanato spaces.

In this section, we prove that the X_r spaces are very close to the Morrey–Campanato spaces $M^{2,3/r}$. We shall give the proof in $\mathbb{R}^n$ so that the space X_r of pointwise multipliers which are continuous from $H^r(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ will be close to $M^{2,n/r}$. First, we recall the definition of Morrey–Campanato spaces :

Definition 7 : [Morrey–Campanato spaces]

a) For $1 < p \leq q \leq \infty$ the Morrey-Campanato space $M^{p,q}$ is defined by:: $f \in M^{p,q}$ if and only if f is locally L^p and $\|f\|_{M^{p,q}} < \infty$ where

$$\|f\|_{M^{p,q}} = \sup_{x \in \mathbb{R}^n} \sup_{0 < R < 1} R^{n/q - n/p} \|f(y) 1_{B(x,R)}(y)\|_{L^p(dy)}$$

For $p = 1$, we require that f is a locally finite measure such that

$$\sup_{x\in\mathbb{R}^n}\ \sup_{0<R<1} R^{n(1/q-1)} \int_{B(x,R)} d|f|(y) < \infty$$

b) The closure of $\mathcal{S}$ in $M^{p,p}$ is called E_p.

It is easy to check that (for $p < \infty$) f belongs to E_p if and only if f is locally L^p and f vanishes at infinity in the sense that $\lim_{x_0\to\infty} \int_{|x-x_0|<1} |f(x)|^p\, dx = 0$.

The space $X_r(\mathbb{R}^n)$ of pointwise multipliers which map L^2 into H^{-r} is defined in the following way :

Definition 8 : For $0 \le r < n/2$, we define the space $X_r(\mathbb{R}^n)$ as the space of functions which are locally square integrable on $\mathbb{R}^n$ and such that pointwise multiplication with these functions maps boundedly $H^r(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. The norm of X_r is given by the operator norm of pointwise multiplication:

$$\|f\|_{X_r} = \sup\{\ \|fg\|_2\ /\ \|g\|_{H^r} \le 1\}$$

We have the following comparison result between X_r and Morrey–Campanato spaces :

Theorem 4 : (Comparison theorem)
For $2 < p \le n/r$ and $0 < r$ we have $M^{p,n/r} \subset X_r \subset M^{2,n/r}$.

Proof : The injection $X_r \subset M^{2,n/r}$ is easy : indeed, since $H^r.X_r \subset L^2$, we have that $X_r \subset L^2_{loc}$. Moreover, we have (choosing ω equal to 1 on the unit ball of $\mathbb{R}^n$) that for $R < 1$:

$$\|f(y)1_{B(x,R)}(y)\|_{L^2(dy)} \le \|f\|_{X_r}\|\omega((x-y)/R)\|_{H^r(dy)} \le C\ \|f\|_{X_r}R^{n/2-r}$$

For the injection $M^{p,n/r} \subset X_r$, we shall consider a dual formulation of the problem, introducing the pre–dual of $M^{p,n/r}$.

Definition 9 : [The pre-dual $N^{p',q'}$]
For $1 \le q' \le p'$, we define $N^{p',q'}$ as the subspace of $L^{q'}(\mathbb{R}^n)$ of functions f which can be decomposed into an atomic series $f = \sum_{k\in\mathbb{N}} g_k$ where the functions g_k are in $L^{p'}$ with compact support and satisfy the following inequalities for the diameter d_k of the support of g_k : $d_k \le 1$ and $\sum_{k\in\mathbb{N}} d_k^{n/q'-n/p'}\ \|g_k\|_{p'} < \infty$. $N^{p',q'}$ is normed by $\|f\|_{N^{p',q'}} = \inf_{f=\sum_{k\in\mathbb{N}} g_k} \sum_{k\in\mathbb{N}} d_k^{n/q'-n/p'}\ \|g_k\|_{p'}$.

Remark : It is obvious that for $q' < p' < p''$ we have $N^{p'',q'} \subset N^{p',q'} \subset L^{q',1} \subset L^{q'}$ since $\|g_k\|_{p'} \le (C\ d_k^n)^{1/p'-1/p''}\ \|g_k\|_{p''}$.

Lemma 13 : The dual space of $N^{p',q'}$ for $1 \le q' \le p'$ is $M^{p,q}$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.
Proof : It is obvious since $L^{p'}_{comp} \subset N^{p',q'}$ and thus $(N^{p',q'})^* \subset L^p_{loc}$.

Due to Lemma 13, we may reformulate Theorem 4 in the following proposition :

Proposition 6 : Pointwise multiplication is a bounded bilinear operator from $L^2(\mathbb{R}^n) \times H^r$ to $N^{p',q'}$ for $1 \le q' \le p' < 2$ and $1/q' = 1 - r/n$.

Pointwise multiplication by L^2 maps obviously H^r to $L^{q'}$ because of the Sobolev imbedding theorem ($H^r \subset L^s$ with $1/s = 1/2 - r/n$); it is well known that it maps more precisely H^r into the Lorentz space $L^{q',1}$ [LIO] since the sharp Sobolev imbedding theorem tells that $H^r \subset L^{s,2}$.

Now, what we have to do, is just to prove a still sharper Sobolev imbedding estimate. We are going to introduce an atomic decomposition for functions in the Sobolev space H^r which will be transformed, by multiplication with a function in L^2, into an atomic decomposition in $N^{p',q'}$.

The key idea for decomposing $v \in H^r$ is to start from an atomic decompostion of $(\sqrt{-\Delta})^r v \in L^2$, where L^2 is to be considered as the real Hardy space $\mathcal{H}^2$ [STE], than to apply fractional integration $(\sqrt{-\Delta})^{-r}$ to each atom of the decomposition of $(\sqrt{-\Delta})^r v$ to get an atom of the decomposition of v. However, in order to preserve the compactness of the supports of the atoms, it will be more convenient to work with the decomposition of v on compactly supported orthonormal wavelets [KAL] [MEY 1] and to define the atomic decomposition through the wavelet coefficients instead of using $(\sqrt{-\Delta})^r v$.

Let us recall that, if $\phi \in L^2(\mathbb{R})$ is a compactly supported orthonormal scaling function such that $\phi, \phi', \ldots, \phi^{(m)}$ are bounded functions for some $m > r$, , if $\varphi = \phi \otimes \ldots \otimes \phi$ is the n-dimensional scaling function associated to ϕ and if $\psi_\epsilon, \epsilon = 1, \ldots, 2^n - 1$ are the associated separable wavelets, then we have, decomposing f into

$$f = \sum_{k \in \mathbb{Z}^n} c_k\ \varphi(x-k) + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \sum_{1 \leq \epsilon \leq 2^n - 1} c_{j,k,\epsilon}\ 2^{nj/2}\ \psi_\epsilon(2^j x - k)$$

and letting $Q_{j,k}$ be the dyadic cube $Q_{j,k} = \{x /\ 2^j x - k \in [0,1]^n\}$, the following equivalence of norms:

$$\|f\|_{H^r} \approx \|(c_k)\|_{l^2} + (\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \sum_{1 \leq \epsilon \leq 2^n - 1} 4^{jr} |c_{j,k,\epsilon}|^2)^{1/2}$$

$$\approx \|(c_k)\|_{l^2} + \sum_{1 \leq \epsilon \leq 2^n - 1} \|\big(\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} 4^{jr}\ |c_{j,k,\epsilon}|^2\ 2^{nj} 1_{Q_{j,k}}(x)\big)^{1/2}\|_2$$

and

$$\forall p \in]1, \infty[\quad \|f\|_p \approx \|(c_k)\|_{l^p} + \sum_{1 \leq \epsilon \leq 2^n - 1} \|\big(\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |c_{j,k,\epsilon}|^2\ 2^{nj} 1_{Q_{j,k}}(x)\big)^{1/2}\|_p$$

In order to prove Proposition 6, we take $u \in L^2$ and $v \in H^r$ and we write

$$v = w_0 + \sum_{1 \leq \epsilon \leq 2^n - 1} w_\epsilon = \sum_{k \in \mathbb{Z}^n} c_k\ \varphi(x-k) + \sum_{1 \leq \epsilon \leq 2^n - 1} \big(\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} c_{j,k,\epsilon}\ 2^{nj/2}\ \psi_\epsilon(2^j x - k)\big)$$

Lemma 14 : $u\ w_0 \in N^{p',q'}$.
Proof :: This is obvious. We have $\varphi \in L^s$ where $1/s + 1/2 = 1/p'$, and thus :

$$\sum_{k \in \mathbb{Z}^n} |c_k|\ \|u\ \varphi(x-k)\|_{p'} \leq\ \|\varphi\|_s\ \|(c_k)\|_{l^2}\ \big(\sum_{k \in \mathbb{Z}^n} \int\ |u|^2 1_{Supp\ \varphi}(x-k)\ dx\big)^{1/2} \leq C\ \|u\|_2 \|v\|_{H^r}$$

Lemma 15 : For $\epsilon = 1, \ldots, 2^n - 1$, $u\ w_\epsilon \in N^{p',q'}$.
Proof :: To avoid notational burden, we drop the index ϵ.

We define $W(x) = \big(\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} 4^{jr}\ |c_{j,k}|^2\ 2^{nj} 1_{Q_{j,k}}(x)\big)^{1/2}$. For $N \in \mathbb{Z}$, we consider $E_N = \{x /\ W(x) > 2^N\}$. Then, $2^{j(r+n/2)}\ |c_{j,k}| > 2^N$ implies $Q_{j,k} \subset E_N$. We write $F_N = \{(j,k) \in \mathbb{N} \times \mathbb{Z}^n /\ Q_{j,k} \subset E_N\}$ and we get $c_{j,k} \neq 0 \Rightarrow \exists N\ Q_{j,k} \in F_N$ and $Q_{j,k} \notin F_{N+1}$.Thus, we may write:

$$w = \sum_{N \in \mathbb{Z}} \big(\sum_{Q_{j,k} \in F_N - F_{N+1}} c_{j,k}\ 2^{nj/2}\ \psi(2^j x - k)\big) = \sum_{N \in \mathbb{Z}} w_N$$

Last, we consider for a given N the collection H_N of maximal dyadic cubes in $F_N - F_{N+1}$, and we may thus decompose w_N into:

$$w_N = \sum_{Q\in H_N} \big(\sum_{Q_{j,k}\in F_N - F_{N+1}, Q_{j,k}\subset Q} c_{j,k}\ 2^{nj/2}\ \psi(2^j x - k)\big) = \sum_{\in H_N} w_{N,Q}$$

This is the atomic decomposition we are looking for. Indeed, we first notice that:

$$\big(\sum_{Q_{j,k}\in F_N - F_{N+1}, Q_{j,k}\subset Q} 4^{jr} |c_{j,k}|^2\ 2^{nj}\ 1_{Q_{j,k}}(x)\big)^{1/2} \leq C\ 2^N 1_{\tilde{Q}}(x)$$

where $\tilde{Q}_{j,k} = \{x/\ 2^j x - k \in [-M, M]^n\}$ with a constant M depending only on the support of ψ. In particular, we get that $Supp\ w_{N,Q} \subset \tilde{Q}$ and that for $1 < s < \infty$ we have $\|w_{N,Q}\|_s \leq C_s\ 2^N\ d_Q^{r+n/s}$ where d_Q is the diameter of Q.

We then fix $s, t < \infty$ such that $1/2 + 1/s + 1/t = 1/p'$ and we write (taking σ as $1/\sigma = 1/2 + 1/s$):

$$d_Q^{n/q' - n/p'}\ \|u\ w_{N,Q}\|_{p'} \leq C \|u\ 1_{\tilde{Q}}\|_\sigma\ 2^N\ d_Q^{n/q' - n/p'}\ d_Q^{r+n/t} = C \|u\ 1_{\tilde{Q}}\|_\sigma\ 2^N\ d_Q^{n(1-1/\sigma)}$$

and thus

$$\sum_N \sum_{Q\in H_N} d_Q^{n/q' - n/p'}\ \|u\ w_{N,Q}\|_{p'} \leq C\ \big(\sum_N \sum_{Q\in H_N} \|u\ 1_{\tilde{Q}}\|_\sigma^\sigma\ 2^{N\ (2-\sigma)}\big)^{1/\sigma}\ \big(\sum_N \sum_{Q\in H_N} d_Q^n\ 2^{2N}\big)^{1-1/\sigma}$$

This latter inequality may be rewritten as :

$$\sum_N \sum_{Q\in H_N} d_Q^{n/q' - n/p'}\ \|u\ w_{N,Q}\|_q \leq C\ \|u\ \big(\sum_N \sum_{Q\in H_N} 2^{N\ (2-\sigma)}\ 1_{\tilde{Q}}\big)^{1/\sigma}\|_\sigma\ \big(\sum_N \sum_{Q\in H_N} d_Q^n\ 2^{2N}\big)^{1-1/\sigma}$$

and thus we get:

$$\|u\ w\|_{N^{p',q'}} \leq\ C\ \|u\|_2\ \|\big(\sum_N \sum_{Q\in H_N} 2^{4N/(s+2)}\ 1_{\tilde{Q}}\big)^{1/2+1/s}\|_s\ \big(\sum_N \sum_{Q\in H_N} d_Q^n\ 2^{2N}\big)^{1/2-1/s}$$

$$= C\ \|u\|_2\ \|\ \sum_N \sum_{Q\in H_N} 2^{4N/(s+2)}\ 1_{\tilde{Q}}\|_{1+s/2}^{1/2+1/s}\ \big(\sum_N \sum_{Q\in H_N} d_Q^n\ 2^{2N}\big)^{1/2-1/s}$$

We then use the classical result that for any sequence of positive real numbers α_j and any sequence of cubes Q_j we have $\|\sum_j\ \alpha_j\ 1_{\tilde{Q}_j}\|_{1+s/2} \leq C\ \|\sum_j\ \alpha_j\ 1_{Q_j}\|_{1+s/2}$: indeed, let L^τ be the dual space of $L^{1+s/2}$, then we have for $g \in L^\tau$ and $x \in Q_j$: $\frac{1}{|Q_j|}\int_{\tilde{Q}_j} |g(y)|\ dy \leq C g^*(x)$ (where g^* is the Hardy-Littlewood maximal function of g), hence:

$$\int \sum_j\ \alpha_j\ 1_{\tilde{Q}_j}(y)\ |g(y)|\ dy \leq C \int \sum_j\ \alpha_j\ 1_{Q_j}\ g^*(y)\ dy \leq C'\ \|\sum_j\ \alpha_j\ 1_{Q_j}\|_{1+s/2}\ \|g\|_\tau$$

Now, we use the inequalities $\sum_N \sum_{Q\in H_N}\ 2^{4N/(s+2)}\ 1_Q \leq C\ W(x)^{2/(1+s/2)}$ and $\sum_N \sum_{Q\in H_N}\ d_Q^n\ 2^{2N} \leq C\ \int W(x)^2\ dx$ to conclude:

$$\|u\ w\|_{N^{p',q'}} \leq\ C\ \|u\|_2\ \|W\|_2 \leq C'\|u\|_2\ \|w\|_{H^r}$$

.

9. The case of a singular initial value.

In this final section, we consider the case of singular initial values. Such problems have been discussed by many authors, and we shall focus mainly on the problem of adding a singular term to a locally L^2 vector field. As usual, the singular values will be required to belong to some Besov spaces included in $B_\infty^{-1,\infty}$.

Definition 10 : (Besov spaces)

For a Banach space of distributions E whose norm is shift-invariant ($\|f(x-x_0)\|_E = \|f\|_E$), we define the Besov space $B_E^{s,q}$ as the space of distributions $f \in \mathcal{S}'$ such that $S_0 f \in E$ and $(2^{js}\|\Delta_j f\|_E)_{j\in\mathbb{N}} \in l^q(\mathbb{N})$.

Remark : We shall always assume that $\|f * g\|_E \leq \|f\|_E\|g\|_1$, which is a consequence of the shift-invariance of $\| \ \|_E$ if the test functions are dense in E or in the predual of E. We then have classically the following equivalent definition of Besov spaces on E:

Lemma 16 : If $0 < s \leq 1$, if E is a shift-invariant Banach space of distributions and if $T \in (0,\infty)$, then $f \in B_E^{-s,q}$ if and only if $(t^{s/2}\|e^{t\Delta}f\|_E)_{0<t<T} \in L^q((0,T),\frac{dt}{t})$.

Using Lemma 16, the following theorem is easy to get:

Theorem 5 : (Singular initial value)

A) Let $\vec{u}_0$ belong to the closure of the test functions in $(B_{M^{1,3/r}}^{-1+r,\infty})^3$ such that $\vec{\nabla}.\vec{u}_0 = 0$ and assume that $0<r<1$. Then, there exists a suitable weak solution $\vec{u} \in \cap_{T>0}L^2((0,T),(E_2)^3)$ for the Navier–Stokes equations on $(0,\infty)\times\mathbb{R}^3$ such that $\lim_{t\to 0^+}\|\vec{u}-\vec{u}_0\|_{B_\infty^{-4,\infty}} = 0$.

B) The same conclusion holds when $\vec{u}_0 = \vec{v}_0 + \vec{w}_0$ with $\vec{\nabla}.\vec{v}_0 = \vec{\nabla}.\vec{w}_0 = 0$, $\vec{v}_0 \in (E_2)^3$ and $\vec{w}_0 \in (B_{\tilde{X}_r}^{-1+r,2/(1-r)})^3$ and $0<r<1$.

Remark : Existence for small times of mild solutions for initial datas the closure of the test functions in $(B_{M^{1,3/r}}^{-1+r,\infty})^3$ have been discussed by Kozono and Yamazaki [KOY]. For Besov spaces on Morrey–Campanato spaces, see also [FLT]. Other existence results related to Besov spaces or Morrey–Campanato spaces may be found in [CAN], [FED], [KAT 2] and [TAY].

Proof : Point (A) follows the classical lines of [KAT 1], [KAT 2]. We prove local existence by a contraction argument in $E_T = \{\vec{u} \ / \ \vec{\nabla}.\vec{u} = 0, \sup_{0<t<T} t^{(1-r)/2}\|\vec{u}\|_{M^{1,3/r}} < \infty, \sup_{0<t<T}\sqrt{t}\|\vec{u}\|_\infty < \infty$ and $\lim_{t\to 0}\sqrt{t}\|\vec{u}\|_\infty = 0\}$.This local solution is of course suitable (since $\vec{u}$ and p are smooth in $(0,T)\times\mathbb{R}^3$); moreover $\vec{u}$ belongs for almost all t to $(E_1\cap L^\infty)^3 \subset (E_2\cap E_3)^3$, so that we may consider a point t_0 such that $\vec{u}(t_0)\in(E_3)^3$ and $\lim_{s>t_0, s\to t_0}\|\vec{u}(s)-\vec{u}(t_0)\|_{E_2} = 0$. From this point t_0 we know how to extend $\vec{u}$ into a global suitable weak solution (as we did in Section 7).

The proof of Point (B) begins with solving the Navier-Stokes initial value problem associated to $\vec{w}_0$. We find for some positive T, through the Kato method, a solution $\vec{w}$ in

$$F_T = \{\vec{u} \ / \ \vec{\nabla}.\vec{u} = 0, \vec{u} \in L^{\frac{2}{1-r}}((0,T),(\tilde{X}_r)^3), \sup_{0<t<T}\sqrt{t}\|\vec{u}\|_\infty < \infty \text{ and } \lim_{t\to 0}\sqrt{t}\|\vec{u}\|_\infty = 0\}.$$

Now, we look for a solution $\vec{v}$ of

$$\begin{cases} \partial_t\vec{v} = \Delta\vec{v} - \vec{\nabla}.(\vec{v}\otimes\vec{v}) - \vec{\nabla}.(\vec{w}\otimes\vec{v}) - \vec{\nabla}.(\vec{v}\otimes\vec{w}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{v} = 0 \\ \vec{v}(0,.) = \vec{v}_0 \end{cases}$$

We saw in Section 1 how to find a solution through the Leray method when $\vec{v}_0$ is square integrable. So, one more time, we approximate $\vec{v}_0$ in $(E_2)^3$ by a square integrable divergence free vector field and we prove that at least on a small strip $(0,T_0)\times\mathbb{R}^3$ we have a good uniform control of the $L^\infty((E_2)^3)$ norms. This provides us with a suitable solution $\vec{v}+\vec{w}$ on $(0,T_0)\times\mathbb{R}^3$ which can then be extended into a global solution in the same way as in point (A).

Remark : This paper was initially motivated by the desire of "interpolating" the theorems of Leray and Kato for initial values in L^p with p between 2 and 3. As a matter of fact, this problem was already solved by C. Calderón [CAL] in the very same way as ours in [LEM 2] (splitting the initial value into a L^2 part and a L^3 part). However, the case of locally square integrable initial values, as addressed now in our paper, required new methods.

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