Biorthogonal multiresolution analysis on a triangle and applications

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Abstract

We present in this paper new constructions of biorthogonal multiresolution analysis on the triangle Δ. We use direct method based on the tensor product to construct dual scaling spaces on Δ. Next, we construct the associated wavelet spaces and we prove that the associated wavelets have compact support and preserve the original regularity. Finally, we describe some regular results which are very useful to establish the norm equivalences. As applications, we prove that the wavelet bases constructed in this paper are adapted for the study of the Sobolev spaces $H_0^s(\Delta)$ and $H^s(\Delta)$ $(s \in \mathbb{N})$ and are easy to implement.

Keywords: Multiresolution analysis, Projection operator, Scaling filter, Riesz Basis, Dual system, Scaling space, Wavelet, Sobolev space, Norm equivalence.

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1. Introduction

Given two multiresolution analyses $V_j(\mathbb{R})$ and $V_j^*(\mathbb{R})$, the following assertions are equivalent:

$$L^2(\mathbb{R}) = V_0(\mathbb{R}) \oplus (V_0^*(\mathbb{R}))^\perp. \tag{1.1}$$

There is a bounded projection operator P_0 on $L^2(\mathbb{R})$ such that

$$RanP_0 = V_0(\mathbb{R}) \text{ and } KerP_0 = (V_0^*(\mathbb{R}))^\perp. \tag{1.2}$$

There are scaling functions φ for $(V_j(\mathbb{R}))$ and φ^* for $(V_j^*(\mathbb{R}))$ such that

$$\langle \varphi(x) | \varphi^*(x-k) \rangle = \delta_{k,0}. \tag{1.3}$$

There are scaling filters m_0 for $(V_j(\mathbb{R}))$ and m_0^* for $(V_j^*(\mathbb{R}))$ such that

$$m_0(\xi)\bar{m}_0^*(\xi) + m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi) = 1. \tag{1.4}$$

We then speak of biorthogonal multiresolution analysis introduced by J.C. Feauveau [18] and developed by A. Cohen et al. [9].

Moreover to the dual scaling functions φ, φ^* (with associated filters m_0, m_0^*) we may associate dual wavelets ψ, ψ^* defined by

$$\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\bar{m}_0^*(\frac{\xi}{2}+\pi)\hat{\varphi}(\frac{\xi}{2}+\pi)\hat{\varphi}(\frac{\xi}{2}) \tag{1.5}$$

and

$$\hat{\psi}^*(\xi) = e^{-i\frac{\xi}{2}}\bar{m}_0(\frac{\xi}{2}+\pi)\hat{\varphi}^*(\frac{\xi}{2}). \tag{1.6}$$

The functions $\psi(x-k), k \in \mathbb{Z}$, are then a Riesz basis for the wavelet space $W_0(\mathbb{R}) = V_1(\mathbb{R}) \cap (V_0^*(\mathbb{R}))^\perp$ and the functions $\psi^*(x-k), k \in \mathbb{Z}$, are a Riesz basis for the dual wavelet space $W_0^*(\mathbb{R}) = V_1^*(\mathbb{R}) \cap (V_0(\mathbb{R}))^\perp$ such that

$$\langle \psi(x) | \psi^*(x-k) \rangle = \delta_{k,0}. \tag{1.7}$$

As usual, we define $\psi_{j,k}$ and $\psi_{j,k}^*$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ by

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \tag{1.8}$$

and

$$\psi_{j,k}^*(x) = 2^{j/2}\psi^*(2^j x - k). \tag{1.9}$$

We have of course the biorthogonality relationship

$$\langle \psi_{j,k} | \psi^*_{\ell,p} \rangle = \delta_{j,\ell} \delta_{k,p}. \tag{1.10}$$

The construction of biorthogonal wavelet bases has been considered by many researchers ([8], [9], [14], [19], [20], [21] and [22]). The Biorthogonal formalism is favored for operator equations in practical computations and allows a commutation property between scale projectors and derivation [24].

We cannot define in the same way multiresolution analyses on general bounded domains or manifolds. The problem is that, in bounded domains, classical invariance by dilation and translation are preserved for dilation, on the other hand they lost in part their meaning for translation.

The search for wavelet bases on bounded domains and more complicated manifolds has been an active field for many years, since the 90's. Several approaches have been explored in wavelet literature. The first approach is the direct method which is based on the usual tensor product of wavelets on the interval and restrictions of integer shifts of scaling functions and wavelets to the domain ([4], [5], [21], [23], [28], [29] and [31]). The second approach is the decomposition method. It was introduced by Z. Ciesielski and T. Figiel in 1982 ([6] and [7]) to construct spline bases of generalized Sobolev spaces $W^k_p(M)$ ($k \in \mathbb{Z}$ and $1 < p < \infty$) where M is a compact Riemannian manifold. This method is based on wavelets on a unit cube by taking tensor products of wavelets on the interval and writing the domain or manifold as a disjoint union of parametric images of this cube. This construction satisfies the lifting scheme which is simply a linear transformation of the wavelets [31]. The third approach uses a multilevel decomposition of finite element spaces. This approach can be more tempting if one wants to combine wavelet properties with the structural simplicity of finite element spaces.

It is clear that the constructions of wavelet bases on bounded domains are related to wavelets on the interval. The problem of existence of an orthonormal basis of $L^2([0,1])$ allowing the characterization of $C^s([0,1])$ and having simple algorithms was treated by Y. Meyer [28]. There are related constructions of wavelets on the interval as well by P. Auscher [2], A. Cohen et al. in 1992 [10]. All these constructions are based on Meyer's work and gave a polynomial extension outside the interval. In 1993, A. Jouini and P.G.

Lemarié-Rieusset [21] defined a multiresolution analysis on the interval and introduced new associated wavelet spaces.

In 1992, A. Jouini et al. [22] used the decomposition approach to construct on a two-dimensional open bounded set biorthogonal wavelet bases adapted for the study of Sobolev spaces H^1 and H_0^1. This approach was used again in 1999 by A. Cohen and R. Schneider [15] to construct biorthogonal wavelet bases $(\psi_\lambda, \widetilde{\psi}_\lambda)_{\lambda \in \nabla}$ of $\mathbb{L}^2(\Omega)$ where Ω is a bounded domain of $\mathbb{R}^d$ $(d \in \mathbb{N})$; these bases were shown to be bases of Sobolev spaces $H^s(\Omega)$ for $-\frac{1}{2} < s < \frac{3}{2}$. There are related constructions as well by C. Canuto and coworkers in [3] and by R. Masson in [27]. In 2003, A. Jouini and P.G. Lemarié-Rieusset [23] studied the L-shaped domain L. They used the direct approach to construct orthogonal wavelet bases and the decomposition method to construct biorthogonal wavelet bases. These bases have simple expressions and the specific geometry of the domain allows to get higher regularity namely the study of the Sobolev spaces $H^k(L)$ $(k \in \mathbb{Z})$. This construction turns out to be well adapted to the wavelet setting due to the simple geometry of the L-Shaped domain. In 2007, A. Jouini and M. Kratou [20] used the decomposition method to construct biorthogonal wavelets on a compact Riemannian manifold with dimension n. These bases were also adapted for the study of the Sobolev spaces H^1 and H_0^1. Recently (in 2011), N. Ajmi, A. Jouini and P.G. Lemarié-Rieusset [1] constructed two orthonormal multiresolution analyses on the triangle Δ. In the first one, they described a direct method to define an orthonormal multiresolution analysis which is adapted for the study of the Sobolev spaces $H_0^s(\Delta)$ $(s \in \mathbb{N})$. In the second one, they added boundary conditions for constructing an orthonormal multiresolution analysis which is adapted for the study of the Sobolev spaces $H^s(\Delta)$ $(s \in \mathbb{N})$. The associated wavelets preserve the original regularity and are easy to implement.

The decomposition approach turns out to have principal limitations and it does not induce Sobolev Spaces H^s when $|s| \geq 3/2$. The basic difficulty is that function spaces on general bounded domains or compact Riemannian manifolds are usually defined in terms of open covering and associated charts, not in terms of partitions of the manifold. Moreover, more regular spaces are more complicated to consider since regularity is directly related to the size of the support. The idea of considering overlapping functions does not work since quite small overlapping domains cause several stability problems, in particular in the orthonormal process. Finally, there are not general criteria available in wavelet theory that tell under which conditions one has uniform estimates and norm equivalences on bounded domains or manifolds with specific geometry. In particular, we do not have on the triangle regular biorthogonal wavelet bases which have compact support, give uniform estimates and are easy to implement.

The other approaches described in wavelet literature as decomposition method or the tensorization of Meyer's Lemma cause problems in computation or implementation. The first one gives complicated wavelets defined as charts with a limitation in regularity and the second one gives only a generating system which is not independent in the case of a triangle. Then, we have more coefficients in numerical analysis defined as stability constants and the functions are not located near the borders. The direct method used in this paper constitutes a very important method for the study of many problems of mathematics and physics because we have the exact number of wavelets which have many applications as computation and numerical simulation for elliptic problems or image processing (see [25] and [26]) and we give a good description of scaling functions and associated wavelets specially near the boundaries. The biorthogonal formalism gives a great flexibility and it is easy to implement. Such a construction has unfolded their full computation efficiently in numerical and applied analysis. The non linear approximation is an important concept to adaptative approximation and the properties of the present wavelet bases provide a rigorous analysis for dynamical systems. More precisely, this paper is concerned with constructions in an elementary way of biorthogonal wavelet bases on a triangle . These constructions are based on the usual tensor product of the orthogonal scaling functions and wavelets of I. Daubechies [16]. The bases constructed here are regular, have compact support and allow fast algorithms. Moreover, they are adapted for the study of some important functional spaces in numerical analysis as Sobolev spaces.

Section 2 is devoted to the description of biorthogonal multiresolution analyses $V_j(I)$ and $V_j^*(I)$ on the interval I. These analyses will be useful for the remainder of the work.

In section 3, we define and study wavelet bases on the interval. Our construction is based on Meyer's Lemma (Lemma 3.1). This construction is very important to realize the main goal of this paper.

In section 4, we shall use a direct method based on the results A. Jouini and P.G. Lemarié-Rieusset ([21] and [23]) to define a biorthogonal multiresolution analysis $(V_j(\Delta), V_j^*(\Delta))$ on a triangle Δ.

In section 5, we study and construct the associated wavelet bases on the triangle Δ. This construction is complicated and technical due to the geometry of the triangle. In the first part, we study two particular cases ($N = 1$ and $N = 2$). These examples permit to illustrate the constructions of wavelet bases of this paper and to explain clearly the central problem between the tensor product and the geometry of the domain. In the second part, we give a description of the wavelet spaces.

In the last section, we prove some regularity results which give uniform estimates for extension operators on the scaling spaces. These results are very important to characterize regular spaces namely Sobolev spaces $H^s(\Delta)$ and $H_0^s(\Delta)(s \in \mathbb{N})$ in terms of discrete norm equivalences.

We recall that all bases constructed in this work have compact support and the same regularity as for Daubechies bases [16].

NOTATIONS. We denote by
-MRA : Multiresolution analysis
-OMRA : Orthogonal multiresolution analysis
-BMRA : Biorthogonal multiresolution analysis.

2. The spaces $V_j(I)$ and $V_j^*(I)$

We start from the orthogonal multiresolution $(V_j(\mathbb{R}))$ of I. Daubechies, having some Sobolev regularity H^{s_N} with $s_N = (1 - ln3/ln4)N + o(N)$ and spanned by dilates and translates at scale 2^j of a scaling function φ with compact support equal to $[0, 2N - 1]$.

Y. Meyer [28] showed that the restrictions to the interval $[0, 1]$ of the scaling functions $\varphi_{j,k}$, $-2N + 2 \leq k \leq 2^j - 1$, constitute a basis of a multiresolution analysis, noted $V_j([0, 1])$. More precisely, we have the following Lemma.

lemma 2.1. *Let j_0 be the smallest integer satisfying $2^{j_0} \geq 4N - 4$. Then, for $j \geq j_0$, the functions $\varphi_{j,k/[0,1]}$, $2 - 2N \leq k \leq 2^j - 1$, form a Riesz basis of $V_j([0, 1])$.*

We denote by $v_j([0,1])$ the space generated by the functions $\varphi_{j,k}$ with support being completely contained in [0,1]. A. Jouini and P.G. Lemarié-Rieusset [21] defined a multiresolution analysis on the interval as the following.

Definition 2.1. *A sequence $\{V_j\}_{j\geq j_0}$ of closed subspaces of $L^2([0,1])$ is called a multiresolution analysis on $L^2([0,1])$ associated with $V_j(\mathbb{R})$ if we have*
i) $\forall j \geq j_0,\ \ v_j([0,1]) \subset V_j \subset V_j([0,1])$.
ii) $\forall j \geq j_0,\ V_j \subset V_{j+1}$.

Remark 2.1. *If I is a bounded interval of $\mathbb{R}$ the space $V_j(I)$ is defined as the space of restrictions to I of elements of $V_j(\mathbb{R})$. More precisely, we may keep only the indexes k such that $(2^{-j}k, 2^{-j}(k+2N-1)) \cap I \neq \emptyset$.*

In the general case of Remark 2.1, we have the following results from [23].

lemma 2.2. *Let $I = [\alpha, \beta]$. For $j \in \mathbb{Z}$, let α_j the smallest integer which is greater than $2^j\alpha - 2N + 1$ and let β_j the greatest integer which is smaller than $2^j\beta$. The functions $(\varphi_{j,k})_{/I}$, $\alpha_j \leq k \leq \beta_j$ are linearly independent, and thus they are a basis for $V_j(I)$.*

lemma 2.3. *Under the assumptions of Lemma 2.2, there exists a constant $c(j, I)$ such that for all sequences $(\lambda_k)_{\alpha_j \leq k \leq \beta_j}$ we have the inequality*

$$c(j, I) \sum_{\alpha_j \leq k \leq \beta_j} |\lambda_k|^2 \leq \int_\alpha^\beta |\sum_{k\in\mathbb{Z}^2} \lambda_k \varphi_{j,k}|^2 dx \leq \sum_{\alpha_j \leq k \leq \beta_j} |\lambda_k|^2. \tag{2.2}$$

If α or β is not a dyadic number, we may have lim $inf_{j\to+\infty} c(j, I) = 0$: we have $c(j, I) \leq \min(\int_\alpha^{2^{-j}\alpha_j} |\varphi|^2 dx,\ \int_{2^{-j}\beta_j}^\beta |\varphi|^2 dx)$. On the other hand, when α and β are dyadic numbers, $c(j, I)$ does not depend on j when j is big enough.

Definition 2.2. *Let φ be a compactly supported orthonormal scaling function with support $[0, 2N-1]$. The associated Meyer border functions are defined in the following way:*

i) [left border functions] for $1 \leq p \leq 2N-2$, the functions $\varphi_p^{[l]}$ belong to the linear span of the functions $\varphi(x-k)_{|(0,+\infty)}$ with $-2N+2 \leq k \leq -1$ and satisfy $\displaystyle\int_0^\infty \varphi(x-k)\varphi_p^{[l]}(x)dx = \delta_{k,-p}$.

ii) *[right border functions] for* $1 \leq p \leq 2N-2$*, the functions* $\varphi_p^{[r]}$ *belong to the linear span of the functions* $\varphi(x-k)_{|(-\infty,0)}$ *with* $-2N+2 \leq k \leq -1$ *and satisfy* $\int_{-\infty}^{0} \varphi(x-k)\varphi_p^{[r]}(x)dx = \delta_{k,-p}$.

We have from [21] the following definition of biorthogonal multiresolution analysis on the interval.

Definition 2.3. *A sequence* (V_j, V_j^*) *of closed subspaces of* $L^2([0,1])$ *associated with a biorthogonal multiresolution analysis* $(V_j(\mathbb{R}), V_j^*(\mathbb{R}))$ *of* $L^2(\mathbb{R})$ *is called a biorthogonal multiresolution analysis of* $L^2([0,1])$ *if*
i) $v_j([0,1]) \subset V_j \subset V_j([0,1])$ *and* $v_j^*([0,1]) \subset V_j^* \subset V_j^*([0,1])$.
ii) $V_j \subset V_{j+1}$ *and* $V_j^* \subset V_{j+1}^*$.
iii) $L^2([0,1]) = V_j \oplus (V_j^*)^{\perp}$.

Proposition 2.1. *We denote by* $(\varphi^*_{(j,k)})_{\alpha_j \leq k \leq \beta_j}$ *the dual system of the basis* $(\varphi_{(j,k)})_{\alpha_j \leq k \leq \beta_j}$. *If* α *and* β *are dyadic numbers and if moreover* j_0 *is the smallest integer* j *such that* $2^j\alpha$ *and* $2^j\beta$ *belong to* $\mathbb{Z}$ *and* $2^j(\beta-\alpha) \geq 2N-1$*, then for* $j \geq j_0$ *we have* $\alpha_j = 2^j\alpha - 2N + 2$ *and* $\beta_j = 2^j\beta - 1$*, and*

i) *[interior functions] for* $2^j\alpha \leq k \leq 2^j\beta - 2N + 1$*, we have* $\varphi^*_{(j,k)} = \varphi_{(j,k)} = \varphi_{j,k}$

ii) *[left border functions] for* $2^j\alpha - 2N + 2 \leq k \leq 2^j\alpha - 1, k = 2^j\alpha - p$*, we have* $\varphi^*_{(j,k)}(x) = 2^{j/2}\varphi_p^{[l]}(2^j(x-\alpha))$

iii) *[right border functions] for* $2^j\beta - 2N + 2 \leq k \leq 2^j\beta - 1,\ k = 2^j\beta - p$*, we have* $\varphi^*_{(j,k)}(x) = 2^{j/2}\varphi_p^{[r]}(2^j(x-\beta))$*. In particular ,* $c(j,I) = c(j_0,I)$.

Thus the functions $(\varphi^*_{(j,k)})_{\alpha_j \leq k \leq \beta_j}$ are a basis for $V_j^*(I)$.

3. Wavelet bases on the interval

The construction of wavelet bases on the interval has been extensively discussed in wavelet literature (see [4], [10] and [21]). All these constructions started from the orthonormal multiresolution analysis of I. Daubechies or spline bases.

We start again from the orthogonal multiresolution $(V_j(\mathbb{R}))$ of I. Daubechies. The moments of the related wavelet ψ satisfy $\int x^k \psi(x)dx = 0$ for $0 \leq k \leq N-1$. We normalize the wavelet ψ by taking its support equal to $[0, 2N-1]$.

Y. Meyer [28] showed that the complementary part of $V_j([0,1])$ in $V_{j+1}([0,1])$, noted $W_j([0,1])$, is automatically of dimension 2^j. He proved that the restrictions of the extreme wavelets $\psi_{j,k}$, $-2N+2 \leq k \leq -N$ and $2^j - 2N + 1 \leq k \leq 2^j - 1$ belong to $V_j([0,1])$. Then, by omitting these functions, we obtain a generating system of $(2^{j+1} + 2N - 2)$ vectors of $V_{j+1}([0,1])$, hence we have the following Meyer's Lemma.

lemma 3.1. *Let j_0 be the smallest integer satisfying $2^{j_0} \geq 4N - 4$. Then, for $j \geq j_0$, the functions $\varphi_{j,k/[0,1]}$, $2 - 2N \leq k \leq 2^j - 1$, (which form a Riesz basis of $V_j([0,1])$) and the functions $\psi_{j,k/[0,1]}$, $-N+1 \leq k \leq 2^j - N$, constitute a Riesz basis for $V_{j+1}([0,1])$.*

Definition 3.1. *Let φ be a compactly supported orthonormal scaling function with support $[0, 2N-1]$. The associated Meyer border wavelets are defined in the following way:*

i) left border scaling functions the family $(\varphi_p^{\{l\}})_{1\leq p\leq 2N-2}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[l]})_{1\leq p\leq 2N-2}$.

ii) right border scaling functions the family $(\varphi_p^{\{r\}})_{1\leq p\leq 2N-2}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[r]})_{1\leq p\leq 2N-2}$.

iii) left border wavelets the family $(\varphi_p^{\{l\}})_{1\leq p\leq 2N-2} \cup (\psi_q^{\{l\}})_{1\leq q\leq N-1}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[l]})_{1\leq p\leq 2N-2} \cup (\psi(x+q)_{|(0,+\infty)})_{1\leq q\leq N-1}$.

iv) right border wavelets the family $(\varphi_p^{\{r\}})_{1\leq p\leq 2N-2} \cup (\psi_q^{\{r\}})_{1\leq q\leq N-1}$ is the Gram-Schmidt orthormalization of the family $(\varphi_p^{[r]})_{1\leq p\leq 2N-2-2} \cup (\psi(x-2+N+q)_{|(-\infty,0)})_{1\leq q\leq N-1}$.

Then, Meyer's lemma reads as:

Proposition 3.1. *Let j such that $2^j \geq 2N-1$. Then* *i) A Hilbertian basis for* $V_j((0,1))$ *is given by the family* $(\varphi^{\perp}_{j,k})_{-2N+2\le k\le 2^j-1}$*, with*

- *interior functions for* $0 \le k \le 2^j - 2N + 1,\ \varphi^{\perp}_{j,k} = \varphi_{j,k}$
- *left border functions for* $-2N+2 \le k \le -1,\ k = -p,\ \varphi^{\perp}_{j,k} = 2^{j/2}\varphi^{\{l\}}_p(2^j x)$
- *right border functions for* $2^j - 2N + 2 \le k \le 2^j - 1,\ k = 2^j - p,\ \varphi^{\perp}_{j,k} = 2^{j/2}\varphi^{\{r\}}_p(2^j(x-1))$

ii) A Hilbertian basis for $W_j((0,1))$ *is given by the family* $(\psi^{\perp}_{j,k})_{-N+1\le k\le 2^j-N}$*, with*

- *interior wavelets for* $0 \le k \le 2^j - 2N + 1,\ \psi^{\perp}_{j,k} = \psi_{j,k}$
- *left border wavelets for* $-N+1 \le k \le -1,\ k = -q,\ \psi^{\perp}_{j,k} = 2^{j/2}\psi^{\{l\}}_q(2^j x)$
- *right border wavelets for* $2^j - 2N + 2 \le k \le 2^j - N,\ k = 2^j - N + 1 - q,\ \psi^{\perp}_{j,k} = 2^{j/2}\psi^{\{r\}}_p(2^j(x-1))$.

A. Jouini and P.G. Lemarié-Rieusset [21] proposed a new wavelet space $W_j([0,1])$ by keeping the wavelets with support being completely contained in [0,1] and replacing the collection of the wavelets on the borders 0 and 1. We have the second important result from [21].

Proposition 3.2. *Let* j_0 *be the smallest integer satisfying* $2^{j_0} \ge 4N - 4$*. For* $j \ge j_0$*, we denote*

$$\begin{aligned} X_j = \ & Vect\,\{\psi_{j,k}, 0 \le k \le 2^j - 2N + 1; \varphi_{j+1,2k+1}, \\ & \quad 0 \le k \le N-2; \varphi_{j+1,2k}, 2^j - 2N + 2 \le k \le 2^j - N\}. \end{aligned} \tag{3.1}$$

Then
i) $\dim X_j = 2^j$.
ii) There exists an integer J *such that for every* $j \ge J$*,* $V_{j+1} = V_j \oplus X_j$.

4. The spaces $V_j(\Delta)$ and $V^*_j(\Delta)$

Starting from the orthogonal multiresolution analysis of I. Daubechies, we define $V_j(\mathbb{R}^2)$ the multiresolution analysis associated to the separable scaling function $\varphi \otimes \varphi$: $V_j(\mathbb{R}^2)$ is the tensor product $V_j(\mathbb{R}^2) = V_j(\mathbb{R})\widehat{\otimes}V_j(\mathbb{R})$.

The next domain we shall consider is the triangle $\Delta = \{(x,y) \in [-1,1] \times [0,1],\ y \leq 1 - |x|\}$. In the following, we study a multiresolution analysis on Δ.

Definition 4.1. *The space $V_j(\Delta)$ is defined as the space of restrictions to Δ of elements of $V_j(\mathbb{R}^2)$.*

We have an obvious generating family of $V_j(\Delta)$.

Proposition 4.1. *For $2^j \geq 4N-4$, $V_j(\Delta)$ has the following basis: the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j - 2N + 2 + p$, $0 \leq p \leq 2^j - 2$ and $-2N+2 \leq k_2 \leq p$; the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2N+1+p$, $0 \leq p \leq 2N-1$ and $-2N+2 \leq k_2 \leq 2^j - 1$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 1+p$, $0 \leq p \leq 2^j - 2$ and $-2N+2 \leq k_2 \leq 2^j - 2 - p$.*

It is clear that Lemma 2.1 and Lemma 2.2 prove that the system described in Proposition 4.1 is linearly independent. If we look now at the supports of these functions, we can split these families into the following sets:

i) interior functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 2N - 1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1 + 2^j - 2N + 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-N+1 \leq k_1 \leq 2^j - 4N + 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 4N + 2$

ii) edge functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j - 2N + 3 \leq k_1 \leq -2^j + 2N - 2$ and $1 \leq k_2 \leq k_1 + 2^j + 2N - 2$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 2N - 1 \leq k_1 \leq 2^j - 4N + 2$ and $2 - 2N \leq k_2 \leq -1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 2N - 1 \leq k_1 \leq -2N$ and $k_1 + 2^j - 2N + 2 \leq k_2 \leq k_1 + 2^j + 2N - 2$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2N+1 \leq k_1 \leq -N-1$ and $k_1 + 2^j - 2N + 2 \leq k_2 \leq -k_1 + 2^j - 4N + 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-N+2 \leq k_1 \leq 0$ and $-k_1 + 2^j - 4N + 3 \leq k_2 \leq k_1 + 2^j - 2N$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 4N + 2$ and $-k_1 + 2^j - 4N + 3 \leq k_2 \leq -k_1 + 2^j - 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $2^j - 4N + 3 \leq k_1 \leq 2^j - 2$ and $1 \leq k_2 \leq -k_1 + 2^j - 1$

iii) exterior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j - 2N + 2 \leq k_1 \leq -2^j + 2N - 2$ and $2 - 2N \leq k_2 \leq 0$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $2^j - 4N + 3 \leq k_1 \leq 2^j - 1$ and $2 - 2N \leq k_2 \leq 0$

iv) interior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2N+1 \leq k_1 \leq -N$ and $-k_1 + 2^j - 4N + 2 \leq k_2 \leq 2^j - 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-N+1 \leq k_1 \leq 0$ and $k_1 + 2^j - 2N + 1 \leq k_2 \leq 2^j - 1$.

We define $v_j(\Delta)$ the space of elements of $V_j(\mathbb{R}^2)$ with support in Δ. It is clear that the interior functions described in i) form an orthonormal basis of $v_j(\Delta)$ and we have $V_j(\Delta) = v_j(\Delta) \oplus X_j(\Delta)$ where $X_j(\Delta)$ is the space generated by the border functions (edge and corner functions) described above.

We shall now construct a space $V_j^*(\Delta)$ which is in duality with $V_j(\Delta)$ for the scalar product on Δ. We remark at first that $v_j(\Delta) \subset V_j^*(\Delta)$. Then, it is enough to construct a dual system of $X_j(\Delta)$. The interior functions are already orthogonal to $V_j(\Delta)$ and orthonormal. We begin by the duality of the edge functions. Then, we proceed to the dualization of the exterior corner functions. Due to the control of the supports of the scaling functions involved in those computations, we see in those computations the global geometry of the open set and for each corner, the computations are the same as if we were in the case of Lemma 2.2, and we find functions provided by tensor products (more precisely, we find the corner elements of the tensor product as in Proposition 2.1). Finally, we construct in the same way a dual system of the interior corner functions. We get a dual space $X_j^*(\Delta)$ of $X_j(\Delta)$. We write $V_j^*(\Delta) = (v_j(\Delta) \oplus X_j^*(\Delta)) \cap H_0^s(\Delta)$. Then, we have the following result.

Proposition 4.2. *For j such that $2^j \geq 4N-4$, the spaces $V_j(\Delta)$ and $V_j^*(\Delta)$ form a biorthogonal multiresolution analysis of $L^2(\Delta)$.*

To simplify notations, we denote by $\phi_{j,k_1,k_2/\Delta}$ the Riesz basis of $V_j(\Delta)$ and ϕ^*_{j,k_1,k_2} the Riesz basis of $V_j^*(\Delta)$ where $(k_1,k_2) \in M_j$ and $\text{card} M_j = \dim V_j(\Delta) = \dim V_j^*(\Delta) = 2^{2j} + (6N-5)2^j + (2N-2)^2$. All these functions are regular (same regularity as Daubechies scale function). We denote by P_j (resp P_j^*) the projection operator on $V_j(\Delta)$ (resp $V_j^*(\Delta)$) parallel to $(V_j^*(\Delta))^\perp$ (resp $(V_j(\Delta))^\perp$). Thus, we have:

$$P_j f = \sum_{(k_1,k_2)\in M_j} < f/\phi^*_{j,k_1,k_2} > \phi_{j,k_1,k_2/\Delta} \tag{4.1}$$

and

$$P_j^* f = \sum_{(k_1,k_2)\in M_j} < f/\phi_{j,k_1,k_2} >_\Delta \phi^*_{j,k_1,k_2} \tag{4.2}$$

where $< f/g >_\Delta = \int_\Delta f\overline{g}dx$.

5. The spaces $W_j(\Delta)$ and $W_j^*(\Delta)$

Recall first that wavelet spaces are given by $W_j(\Delta) = V_{j+1}(\Delta) \cap (V_j^*(\Delta))^\perp$ and $W_j^*(\Delta) = V_{j+1}^*(\Delta) \cap (V_j(\Delta))^\perp$. The construction of wavelet spaces is very technical and complicated in biorthogonal case. N. Ajmi, A. Jouini and P.G. Lemarié-Rieusset [1] show the complexity of this construction even in orthogonal case because the tensorization of Meyer's Lemma (Lemma 3.1) gives in our case only a generating system of $V_{j+1}(\Delta)$ which is not linearly independent. Moreover, the regularity of the bases is directly related to the length of the support.
To explain this point, we study at first two particular cases ($N = 1$ and $N = 2$). We consider the Haar basis (which corresponds to the case $N = 1$). Proposition 4.1 shows that $V_j(\Delta)$ has the following basis: the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq p \leq 2^j - 1$ and $0 \leq k_2 \leq p$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq p \leq 2^j - 1$ and $0 \leq k_2 \leq 2^j - 1 - p$. We can split these families into the following sets:

i) interior functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2}$ with $-2^j + 1 \leq k_1 \leq -1$ and $0 \leq k_2 \leq k_1 + 2^j - 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2}$ with $0 \leq k_1 \leq 2^j - 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 2$

ii) edge functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 1 \leq k_1 \leq -2$ and $k_2 = k_1 + 2^j$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 2$ and $k_2 = 2^j - 1 - k_1$

iii) exterior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j$ and $k_2 = 0$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 2^j - 1$ and $k_2 = 0$

iv) interior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-1 \leq k_1 \leq 0$ and $k_2 = 2^j - 1$.

We have $\dim V_j(\Delta) = 2^{2j} + 2^j$. We study now the space $W_j(\Delta)$. The construction of wavelets here is more simple due to small support of the Haar basis. We have $\dim W_j(\Delta) = 3 \times 2^{2j} + 2^j$. Let $X_j(\Delta)$ be a supplement of $V_j(\Delta)$ into $V_{j+1}(\Delta)$, then $X_j(\Delta)$ has the following Riesz basis:

i) the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p$ and $0 \leq p \leq 2^j - 1$ and the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq k_2 \leq 2^j - p - 1$ and $0 \leq p \leq 2^j - 1$

ii) the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p$ and $0 \leq p \leq 2^j - 1$ and the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq k_2 \leq 2^j - p - 1$ and $0 \leq p \leq 2^j - 1$

iii) the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p-1$ and the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p-1$, $0 \leq k_2 \leq 2^j - p - 1$ and $1 \leq p \leq 2^j - 1$.

We have exactly $(3 \times 2^{2j} + 2^j)$ functions which are linearly independent because the third collection has a support in the interior of Δ and the boundary functions are in the sets i) and ii). More precisely, We can split these families into the following sets:

i) interior functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$, $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-2^j+1 \leq k_1 \leq -1$ and $0 \leq k_2 \leq k_1+2^j-1$; $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$, $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $0 \leq k_1 \leq 2^j - 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 2$

ii) edge functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$, with $-2^j+1 \leq k_1 \leq -2$ and $k_2 = k_1 + 2^j$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 2$ and $k_2 = 2^j - 1 - k_1$

iii) exterior corner functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j$ and $k_2 = 0$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 2^j - 1$ and $k_2 = 0$

iv) interior corner functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-1 \leq k_1 \leq 0$ and $k_2 = 2^j - 1$.

It remains to realize orthogonality for the scalar product of $L^2(\Delta)$ of the $(4x2^j - 8)$ edge functions, the four exterior corner functions and the four interior corner functions with $V_j^*(\Delta)$ by using proposition 2.1. Then, we get a nice basis for $W_j(\Delta)$. Now, to construct a Riesz basis for the wavelet space $W_j^*(\Delta)$, we consider interior wavelets described in i) (which are orthogonal) and we add dual system of the $4x2^j$ functions (edge functions, exterior corner functions and interior corner functions) by using Proposition 3.1.

We study now the case $N = 2$. It is clear that this case is more complicated than the first one because the wavelets described in the third collection iii) does not have a support in the interior of Δ. Proposition 4.1 shows that $V_j(\Delta)$ has the following basis : the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $-2 \leq p \leq 2^j - 4$ and $-2 \leq k_2 \leq p+2$, the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $-2 \leq k_2 \leq 2^j - 1$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $1 \leq p \leq 2^j - 1$ and $-2 \leq k_2 \leq 2^j - 1 - p$. We have $\dim V_j(\Delta) = 2^{2j} + 7 \times 2^j + 4$. We describe now a basis of the associated space $W_j(\Delta)$. The construction of wavelets here is different from the case of the Haar basis ($N = 1$). We have $\dim W_j(\Delta) = 3 \times 2^{2j} + 7 \times 2^j$. Let $X_j(\Delta)$ be a supplement of $V_j(\Delta)$ into $V_{j+1}(\Delta)$, then $X_j(\Delta)$ has the following Riesz basis:

i) the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $-1 \leq k_2 \leq p+1$ and $-1 \leq p \leq 2^j - 4$, the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $-1 \leq k_2 \leq 2^j - 2$ and the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p$, $-1 \leq k_2 \leq 2^j - p - 2$ and $1 \leq p \leq 2^j - 2$

ii) the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $-2 \leq k_2 \leq p+2$ and $-1 \leq p \leq 2^j - 4$, the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $-2 \leq k_2 \leq 2^j - 1$ and the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $-2 \leq k_2 \leq 2^j - 1 - p$ and $1 \leq p \leq 2^j - 2$

iii) the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p - 1$ and $1 \leq p \leq 2^j - 4$, the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $0 \leq k_2 \leq 2^j - 4$, the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq k_2 \leq 2^j - 4 - p$ and $1 \leq p \leq 2^j - 4$ and the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ where $(k_1, k_2) \in \{(-2^j+1, -1), (-2, 2^j-3), (-1, 2^j-3), (2^j-4, -1)\}$.

We have exactly $(3 \times 2^{2j} + 7 \times 2^j)$ functions which are linearly independent due to Lemma 3.1.We can split these families into the following sets:

i) interior functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$, $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-2^j + 3 \leq k_1 \leq -2$ and $0 \leq k_2 \leq k_1 + 2^j - 3$; $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$, $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-1 \leq k_1 \leq 2^j - 6$ and $0 \leq k_2 \leq -k_1 + 2^j - 6$

ii) edge functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-2^j \leq k_1 \leq -2^j + 2$ and $1 \leq k_2 \leq k_1 + 2^j + 1$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-2^j + 3 \leq k_1 \leq 2^j - 6$ and $k_2 = -1$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-2^j + 3 \leq k_1 \leq -4$ and $k_1 + 2^j - 2 \leq k_2 \leq k_1 + 2^j + 1$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -3$ and $2^j - 5 \leq k_2 \leq 2^j - 4$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = 0$ and $2^j - 5 \leq k_2 \leq 2^j - 4$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 6$ and $-k_1 + 2^j - 5 \leq k_2 \leq -k_1 + 2^j - 2$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $2^j - 5 \leq k_1 \leq 2^j - 2$ and $1 \leq k_2 \leq -k_1 + 2^j - 2$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j - 1 \leq k_1 \leq -2^j + 2$ and $1 \leq k_2 \leq k_1 + 2^j + 2$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 3 \leq k_1 \leq 2^j - 6$ and $-2 \leq k_2 \leq -1$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 3 \leq k_1 \leq -4$ and $k_1 + 2^j - 2 \leq k_2 \leq k_1 + 2^j + 2$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -3$ and $2^j - 5 \leq k_2 \leq 2^j - 4$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 0$ and $2^j - 5 \leq k_2 \leq 2^j - 4$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 6$ and $-k_1 + 2^j - 5 \leq k_2 \leq -k_1 + 2^j - 1$;

$\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $2^j - 5 \leq k_1 \leq 2^j - 2$ and $1 \leq k_2 \leq -k_1 + 2^j - 1$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + 2$ and $k_2 = 1$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-2^j + 3 \leq k_1 \leq -4$ and $k_1 + 2^j - 2 \leq k_2 \leq k_1 + 2^j - 1$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -3$ and $2^j - 5 \leq k_2 \leq 2^j - 4$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = 0$ and $2^j - 5 \leq k_2 \leq 2^j - 4$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 6$ and $-k_1 + 2^j - 5 \leq k_2 \leq -k_1 + 2^j - 4$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 \leq 2^j - 5$ and $k_2 = 1$

iii) exterior corner functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-2^j - 1 \leq k_1 \leq -2^j + 2$ and $-1 \leq k_2 \leq 0$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $2^j - 5 \leq k_1 \leq 2^j - 2$ and $-1 \leq k_2 \leq 0$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j - 1 \leq k_1 \leq -2^j + 2$ and $-2 \leq k_2 \leq 0$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $2^j - 5 \leq k_1 \leq 2^j - 2$ and $-2 \leq k_2 \leq 0$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + 1$ and $-1 \leq k_2 \leq 0$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + 2$ and $k_2 = 0$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = 2^j - 5$ and $k_2 = 0$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = 2^j - 4$ and $-1 \leq k_2 \leq 0$

iv) interior corner functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq -2$ and $-k_1 + 2^j - 6 \leq k_2 \leq 2^j - 2$; $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-1 \leq k_1 \leq 0$ and $k_1 + 2^j - 3 \leq k_2 \leq 2^j - 2$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq -2$ and $-k_1 + 2^j - 6 \leq k_2 \leq 2^j - 1$; $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-1 \leq k_1 \leq 0$ and $k_1 + 2^j - 3 \leq k_2 \leq 2^j - 1$; $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-2 \leq k_1 \leq -1$ and $2^j - 4 \leq k_2 \leq 2^j - 3$.

Orthogonalization for the scalar product of $L^2(\Delta)$ of the $(28 \times 2^j - 110)$ edge functions, the forty six exterior corner functions and the twenty eight interior corner functions with $V_j^*(\Delta)$ by using proposition 2.1 gives a Riesz basis for $W_j(\Delta)$. Now, to construct a Riesz basis for the wavelet space $W_j^*(\Delta)$, we consider the $(3 \times 2^{2j} - 21 \times 2^j + 36)$ interior wavelets described in i) (which are orthogonal) and we add dual system of the $(28 \times 2^j - 36)$ functions (edge functions, exterior corner functions and interior corner functions) by using Proposition 3.1.

Theorem 5.1. *Let $2^{j_0} \geq 4N - 4$. Then:*

a) there exist $(3 \times 2^{2j} + (6N-5)2^j)$ functions Ψ_{j,k_1,k_2} such that the functions $\phi_{j,k_1,k_2/\Delta}$ for $V_j(\Delta)$ where $(k_1, k_2) \in M_j$ and Ψ_{j,k_1,k_2} where $(k_1, k_2) \in M_{j+1} \setminus M_j$, form a Riesz basis for $V_{j+1}(\Delta)$,

b) there exist $(3 \times 2^{2j} + (6N-5)2^j)$ functions Ψ^*_{j,k_1,k_2} such that the functions $\phi^*_{j,k_1,k_2/\Delta}$ for $V_j^*(\Delta)$ where $(k_1, k_2) \in M_j$ and Ψ^*_{j,k_1,k_2} where $(k_1, k_2) \in M_{j+1} \setminus M_j$, form a Riesz basis for $V^*_{j+1}(\Delta)$.

Proof. a) We consider interior wavelets $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$ and $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-2^j + 2N - 1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1 + 2^j - 2N + 1$ or $-N + 1 \leq k_1 \leq 2^j - 4N + 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 4N + 2$, and we complete this system from the collection described above (edge and corner functions). Next, we realize orthogonality for the scalar product of $L^2(\Delta)$ of edge and corner wavelets with $V_j^*(\Delta)$ by using proposition 2.1.
b) We keep interior wavelets $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$ and $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-2^j + 2N - 1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1 + 2^j - 2N + 1$ or $-N + 1 \leq k_1 \leq 2^j - 4N + 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 4N + 2$, and we construct dual system Ψ^*_{j,k_1,k_2} of edge and corner wavelets by using Proposition 3.1. ■

Remark 5.1. *The general idea for constructing wavelets consists to take near wavelets which satisfy Proposition 3.1 or Proposition 3.2. Next, we proceed to dualization of edge functions, exterior corner functions and interior corner functions.*

6. Uniform estimates and Sobolev spaces

Definition 6.1. *Let* $\Delta = \{(x,y) \in [-1,1] \times [0,1],\ y \leq 1 - |x|\}$. *Let us consider, for* $2^j \geq 4N-4$, *the basis for* $V_j(\Delta)$ *(resp the basis for* $V_j^*(\Delta)$*) given by the family* $(\phi_{j,k_1,k_2/\Delta})_{(k_1,k_2)\in M_j})$ *described in (4.1) (resp* $(\phi^*_{j,k_1,k_2)_{(k_1,k_2)}\in M_j}$ *described in (4.2)). Then we define the extension operator* E_j *from* $V_j(\Delta)$ *to* $V_j(\mathbb{R}^2)$ *by the formula*

$$E_j f = \sum_{(k_1,k_2)\in M_j} < f/\phi^*_{j,k_1,k_2} > \phi_{j,k_1,k_2}. \tag{6.1}$$

To establish the main object of this section , we need the following results for extension operators and multiresolution analyses on a triangle.

Proposition 6.1. *Let* $\Delta = \{(x,y) \in [-1,1] \times [0,1],\ y \leq 1 - |x|\}$. *There exists a positive constant* α *such that for all* j *such that* $2^j \geq 4N - 4$ *and all* $f \in V_j(\Delta)$*:*

$$\|E_j f\|^2_{L^2(\mathbb{R}^2)} \leq \alpha \|f\|^2_{L^2(\Delta)}. \tag{6.2}$$

Proof. To prove this important result, we use special triangulations for $[0,1] \times [0,1]$ and $\mathbb{R}^2$ which are adapted to scale because this condition is necessary for projects. In fact, we divide $[0,1] \times [0,1]$ into four triangles defined by: for $0 \leq \eta \leq 3$, $T^\eta = \{(x,y) \in [0,1] \times [0,1] / (-1)^\eta (x-y) \geq 0$ and $(-1)^{E(\frac{\eta}{2})}(x+y-1) \geq 0\}$.

We triangulate $\mathbb{R}^2$ such that

$$\mathbb{R}^2 = \bigcup_{0 \leq \eta \leq 3} \bigcup_{(k_1,k_2) \in \mathbb{Z}^2} T^\eta_{j,k_1,k_2}$$

where

$$T^\eta_{j,k_1,k_2} = \{(x,y)/(2^j x - k_1, 2^j y - k_2) \in T^\eta\}.$$

This triangulation is adapted to scale and also to our triangle $\Delta = \{(x,y) \in [-1,1] \times [0,1], y \leq 1 - |x|\}$ because we have

$$\Delta = \bigcup_{T^\eta_{j,k_1,k_2} \subset \Delta} T^\eta_{j,k_1,k_2}.$$

We put $\phi_{j,k_1,k_2}(x,y) = 2^j \varphi(2^j x - k_1)\varphi(2^j y - k_2)$ and $\phi_{k_1,k_2} = \phi_{0,k_1,k_2}$. Let us write

$$\int\int_\Delta |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy = \sum_{T^\eta_{j,l_1,l_2} \subset \Delta} \int\int_{T^\eta_{j,l_1,l_2}} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy;$$

then,

$$\int\int_{T^\eta_{j,l_1,l_2}} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{k_1,k_2}|^2 dxdy = \int\int_{T^\eta} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{k_1-l_1,k_2-l_2}|^2 dxdy.$$

Let C^η be the set of indexes (k_1,k_2) such that the support of ϕ_{k_1,k_2} has an intersection of non vanishing measure with T^η, C^η_{j,k_1,k_2} the set of indexes (l_1,l_2) such that the support of ϕ_{j,l_1,l_2} has an intersection of non vanishing measure with T^η_{j,k_1,k_2} and C_j the set of indexes (k_1,k_2) such that the support of ϕ_{j,k_1,k_2} has an intersection of non vanishing measure with Δ.
We have $C_j = \bigcup_{T^\eta_{j,k_1,k_2} \subset \Delta} C^\eta_{j,k_1,k_2}$. The family $(\phi_{k_1,k_2|T^\eta})_{(k_1,k_2)\in C^\eta}$ is linearly independent. Then, there exists a positive constant γ such that we have

$$\int\int_{T^\eta} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \beta_{k_1,k_2}\phi_{k_1,k_2}|^2 dxdy \geq \gamma \sum_{(k_1,k_2)\in C^\eta} |\beta_{k_1,k_2}|^2;$$

hence,

$$\int\int_{T^{\eta}_{j,l_1,l_2}} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy \geq \gamma \sum_{(k_1,k_2)\in C^n} |\alpha_{k_1+l_1,k_2+l_2}|^2 = \gamma \sum_{(k_1,k_2)\in C^{\eta}_{j,l_1,l_2}} |\alpha_{k_1,k_2}|^2$$

and then

$$\int\int_{\Delta} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy \geq \gamma \sum_{T^{\eta}_{j,l_1,l_2}\subset\Delta} \sum_{(k_1,k_2)\in C^{\eta}_{j,l_1,l_2}} |\alpha_{k_1,k_2}|^2 \geq \gamma \sum_{(k_1,k_2)\in C_j} |\alpha_{k_1,k_2}|^2. \ \blacksquare$$

We establish now the following important result which completes the precedent proposition to get equivalence norms for Sobolev spaces (or other functional spaces as Besov spaces).

Theorem 6.1. *Let $\Delta = \{(x,y) \in [-1,1]\times[0,1],\ y \leq 1-|x|\}$ and $j_0 \in \mathbb{N}$ such that $2^{j_0} \geq 4N-4$. Let $(V_j(\mathbb{R}^2))_{j\in\mathbb{Z}}$ be a regular multiresolution analysis of $L^2(\mathbb{R}^2)$. We assume that there exists a projection operator A_j onto $V_j(\mathbb{R}^2)$ such that*

i) $A_{j+1} o A_j = A_j o A_j = A_j$

ii) $\|F\|^2_{H^s(\mathbb{R}^2)} \approx \|A_0F\|^2_{L^2(\mathbb{R}^2)} + \sum_{j\geq 0} 2^{2js}\|A_{j+1}F - A_jF\|^2_{L^2(\mathbb{R}^2)}$.

If P_j is a projection operator from $L^2(\Delta)$ onto $V_j(\Delta)$ such that, for a constant β and $j \geq j_0$, P_j satisfies:

$$\|P_jf\|^2_{L^2(\Delta)} \leq \beta\|f\|^2_{L^2(\Delta)} \tag{6.3}$$

then, we have

$$\forall f \in H^s(\Delta),\ \|f\|^2_{H^s(\ \Delta\ )} \approx \|P_{j_0}f\|^2_{L^2(\Delta)} + \sum_{j\geq j_0} 2^{2js}\|P_{j+1}f - P_jf\|^2_{L^2(\Delta)}. \tag{6.4}$$

Proof. The case of the triangle and Proposition 6.1 give $f = F_{/\Delta}$ and $F = (F - A_jF) + A_jF$. We get

$$\begin{aligned} \|P_{j+1}f - P_jf\|^2_{L^2(\Delta)} &= \|(P_{j+1} - P_j)(F - A_jF)_{/\Delta}\|^2_{L^2(\Delta)} \\ &\leq \beta\|F - A_jF\|^2_{L^2(\Delta)} \end{aligned}$$

where β is a positive constant independent of j. Then, we have

$$\sum_{j\geq j_0} 2^{2js}\|P_{j+1}f - P_jf\|^2_{L^2(\Delta)} \leq \beta\sum_{j\geq j_0} 2^{2js}\|F - A_jF\|^2_{L^2(\Delta)}$$

$$
\begin{aligned}
&\le \beta \sum_{j\ge j_0} 2^{2js}\|F - A_j F\|^2_{L^2(\mathbb{R}^2)} \\
&\le \beta \sum_{j\ge j_0} 2^{2js}\| \sum_{p\ge j+1} (A_p - A_{p-1})F\|^2_{L^2(\mathbb{R}^2)} \\
&\le \beta \sum_{j\ge j_0} \{ \sum_{p\ge j+1} 2^{(j-p)s} 2^{2ps} \|(A_p - A_{p-1})F\|^2_{L^2(\mathbb{R}^2)}\}^2.
\end{aligned}
$$

It's a convolution $\ell^1 o \ell^2 \subset \ell^2$, then we get the first inequality. To prove the reverse inequality, we write $f = F_{/\Delta}$ and $F = E_0(P_0 f) + \sum_{j\ge 0} E_{j+1}(P_{j+1}f - P_j f)$ where E_j is the extension operator described in Definition 4.1. Then, we have:

$$
\|f\|^2_{H^s(\Delta)} \le \|F\|^2_{H^s(\mathbb{R}^2)} \approx \|A_0 f\|^2_{L^2(\mathbb{R}^2)} + \sum_{j\ge 0} 2^{2js}\|A_{j+1}F - A_j F\|^2_{L^2(\mathbb{R}^2)}
$$

and

$$
A_{j+1}F - A_j F = \sum_{l\ge j} (A_{j+1} - A_j) E_{l+1}(P_{l+1} f - P_l f).
$$

Then, we get for a constant M:

$$
\begin{aligned}
2^{2js}\|A_{j+1}F - A_j F\|^2_{L^2(\mathbb{R}^2)} &\le \sum_{l\ge j} 2^{js} \|(A_{j+1} - A_j)E_{l+1}(P_{l+1}f - P_l f)\|^2_2 \\
&\le \sum_{l\ge j} M\alpha \|P_{l+1}f - P_l f\|^2_{L^2(\Delta)} 2^{ls} 2^{(j-l)s}.
\end{aligned}
$$

It's a convolution $\ell^2 o \ell^1 \subset \ell^2$, then we get the first inequality. ■

Remark 6.1. *See that Theorem 6.1 described above dosen't depend on the formalism of the given multiresolution analysis (orthogonal or biorthogonal).*

Proposition 6.1 and Theorem 6.1 are very useful for analyzing regular functions on the triangle. Recall that P_j (resp P_j^*) is the projection operator on $V_j(\Delta)$ (resp $V_j^*(\Delta)$) parallel to $(V_j^*(\Delta))^\perp$ (resp $(V_j(\Delta))^\perp$). We denote by Q_j (resp Q_j^*) the projection operator on $W_j(\Delta)$ (resp $W_j^*(\Delta)$) parallel to $(W_j^*(\Delta))^\perp$ (resp $(W_j(\Delta))^\perp$). These projectors are completely described by the scaling function constructed in section 4 and the associated wavelets constructed in section 5. We can now establish the first main result of this section.

Theorem 6.2. *Let $2^{j_0} \geq 4N - 4$. Then:*

a) for $f \in L^2(\Delta)$, we have $\|f\|^2_{L^2(\Delta)} = \|P_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} \|Q_j f\|^2_{L^2(\Delta)}$,

b) for $f \in H^s(\Delta)$, we have $\|f\|^2_{H^s(\Delta)} \approx \|P_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} 4^{sj} \|Q_j f\|^2_{L^2(\Delta)}$.

Proof. a) is a classical result in wavelet theory.
b) the first inequality follows from Proposition 6.1 and the second inequality follows from Theorem 6.1. ■

We use now dual multiresolution analysis and Remark 6.1 to characterize the following spaces.

Theorem 6.3. *Let $2^{j_0} \geq 4N - 4$. Then:*

a) for $f \in L^2(\Delta)$, we have $\|f\|^2_{L^2(\Delta)} = \|P^*_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} \|Q^*_j f\|^2_{L^2(\Delta)}$,

b) for $f \in H^s_0(\Delta)$, we have $\|f\|^2_{H^s(\Delta)} \approx \|P^*_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_*}^{j=\infty} 4^{sj} \|Q^*_j f\|^2_{L^2(\Delta)}$.

Proof. a) is a classical result in wavelet theory.
b) follows from Proposition 6.1 and Theorem 6.1. ■

The wavelet bases on a triangle constructed in this paper allow many concrete numerical examples. In fact, we can use these bases for the study of the image-watermarking robust to the desynchronizations and we improve the general robustness of the scheme by embedding in the wavelet transform domain by using the same method described in [3]. The second example is to perform a Scan-based Wavelet Compression of 3D semi-regular multiresolution meshes [17]. Of course, we can use these bases for applications to gas dynamics and scalar conservation laws or to improve integral formulation in electromagnetism and scale-space approximations ([11], [12], [13]).

7. Conclusion

We used in this paper a direct method based on Lemma 2.2 to construct a biorthogonal multiresolution analysis on Δ. This construction is very technical due to the specific geometry of the triangle. Moreover, it is difficult to analyze more regular spaces since regularity is directly related to the size of the support. The analysis constructed in this paper is adapted for the study of the Sobolev spaces $H^s(\Delta)$ and $H_0^s(\Delta)$ ($s \in \mathbb{N}$). The associated wavelet bases are regular and have compact support. More precisely, they are associated to simple algorithms. Proposition 6.1 and Theorem 6.1 permit to get the norm equivalences in the two cases. We should notice that our construction can be achieved numerically in a satisfactory way only for the first Daubechies scaling functions.

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