PhD Defense
Singular drift SDEs: density estimates and weak discretization

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Notations

We consider the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d, \tag{E}$$

where

- $b \in L^r([0, T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d))$ is some *singular* drift.
- Z_t is an α-stable symmetric process $\alpha \in (1, 2]$.

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We denote by $\mathcal{L}^\alpha$ the generator of the noise and p_α its density

$$p_\alpha(t, z) := \begin{cases} C_\alpha t^{-\frac{d}{\alpha}} \left(1 + \frac{|z|}{t^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} & \text{if } \alpha \in (1, 2) \\ (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|z|^2}{2t}\right) & \text{if } \alpha = 2, \end{cases}, \quad t > 0, z \in \mathbb{R}^d$$

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Self-similarity index α^{-1}

Heuristics for weak well-posedness

For $b \in \mathbb{B}^{\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d) \approx \mathcal{C}^{\beta}$, we work on the PDE

$$(\partial_t + b \cdot \nabla + \mathcal{L}^{\alpha}) u(t,x) = f(t,x) \text{ on } [0,T] \times \mathbb{R}^d \tag{1}$$

where f is some smooth source term.

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Heuristics for weak well-posedness - $\beta > 0$

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where f is some smooth source term.

- If $\beta > 0$, the condition $\alpha + \beta > 1$ is required to give a meaning to ∇u
  - [CZZ21] : $b \in \mathcal{C}^\beta, \alpha \in (0,2)$: weak WP under $\beta + \alpha > 1$.
  - [Pri12] : $b \in \mathcal{C}^\beta, \alpha \in (1,2)$: strong WP under $\beta + \frac{\alpha}{2} > 1$.
  - [Por94], [PP95] : $b \in L^p \hookrightarrow \mathbb{B}^{-d/p}_{\infty,\infty}$, $\alpha \in (1,2)$, weak WP under $\frac{d}{p} < \alpha - 1$.
  - [KR05] : $b \in L^q_t - L^p_x, \alpha = 2$, strong WP under $\frac{d}{p} + \frac{2}{q} < 1$.

Heuristics for weak well-posedness - $\beta < 0$

For $b \in \mathbb{B}^{\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d) \approx \mathcal{C}^{\beta}$, we work on the PDE

$$(\partial_t + b \cdot \nabla + \mathcal{L}^{\alpha})u(t,x) = f(t,x) \text{ on } [0,T] \times \mathbb{R}^d \tag{3}$$

where f is some smooth source term.

- If $\beta > 0$, the condition $\alpha + \beta > 1$ is required to give a meaning to ∇u
- If $\beta < 0$, in order to make sense of $b \cdot \nabla u$, we need $\beta + (\beta + \alpha - 1) > 0$

$$\text{i.e. } \beta > \frac{1-\alpha}{2}$$

- [FIR17] : $b \in \mathcal{C}^{\beta}, \alpha = 2$, WP (virtual solutions) under $\beta > -\frac{1}{2}$.
- [ABM20] : $b \in \mathcal{C}^{\beta}, \alpha \in (1,2)$, 1D strong WP under $\beta > \frac{1-\alpha}{2}$.
- [CdRM22] : $b \in L^r - \mathbb{B}^{\beta}_{p,q}$, weak WP under

$$2\beta + \alpha - \frac{d}{p} - \frac{\alpha}{r} > 1$$

Discretization of singular drift SDEs

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Aim : approach (E) with

$$X^h_{t_{k+1}} = X^h_{t_k} + hb(t_k, X^h_{t_k}) + (Z_{t_{k+1}} - Z_{t_k}).$$

Or, equivalently, its continuous in time version :

$$\mathrm{d}X^h_t = b(\tau^h_t, X^h_{\tau^h_t})\mathrm{d}t + \mathrm{d}Z_t,$$

where $\tau^h_t = h\lfloor t/h \rfloor$, i.e. if $t \in [t_k, t_{k+1}), \tau^h_t = t_k$.

Discretization of singular drift SDEs

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$$\left\| \sup_{t \in (0,T)} |X_t - X^h_t| \right\|_{L^r(\Omega)}, \qquad r > 1$$

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- Strong error :

$$\left\| \sup_{t\in(0,T)} |X_t - X^h_t| \right\|_{L^r(\Omega)}, \qquad r > 1$$

- Weak error with a test function :

$$|\mathbb{E}_{0,x}[g(X^h_t) - g(X_t)]|$$

Weak error on densities (i.e. $g = \delta_y$) :

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)|$$

A few important results - Gaussian setting ($\alpha = 2$)

Using PDE techniques :

- [BJ20] $b \in L^\infty$ total variation error : $h^{\frac{1}{2}}$.
- [JM24] $b \in L^q - L^p$ under $\frac{d}{p} + \frac{2}{q} < 1$, weak error : $h^{\frac{1-\frac{d}{p}-\frac{2}{q}}{2}}$.

Using the stochastic sewing lemma ([Lê20]) :

- [LL21] $b \in L^q - L^p$ under $\frac{d}{p} + \frac{2}{q} < 1$, strong error : $h^{\frac{1}{2}} \ln(h)$.
- [DGL22] $b \in L^\infty \cap \dot{W}^{\beta}_{m}$, strong error : $h^{\frac{1+\beta}{2}-}$.
- [Hol24] $b \in L^\infty \cap \mathcal{C}^\beta$, weak error on $\mathcal{C}^\beta$ test functions : $h^{\frac{1+\beta}{2}-}$.

Sewing techniques for strong error

In a brownian noise setting, the sewing lemma ([Lê20]) allows to bound expressions of the type

$$\mathbb{E}\left[\left|\int_0^t b(s, X_s^h) - b(s, X_{\tau_s^h}^h)\mathrm{d}s\right|^r\right], \tag{4}$$

which appear when computing strong error rates.

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which appear when computing strong error rates.

- Only works for $\alpha = 2$ (also works in a non-Markovian setting).
- Integrability requirements on b.
- The resulting bound *does not* take advantage of the full parabolic bootstrap of the underlying PDE.

Weak error techniques

Let u be a solution to

$$\left(\partial_s + b(s,x)\cdot\nabla_x + \mathcal{L}^\alpha\right) u(s,x) = 0 \text{ on } [0,t)\times\mathbb{R}^d, \qquad u(t,\cdot) = g \text{ on } \mathbb{R}^d,$$

By Itô's formula,

$$\begin{aligned}\mathcal{E}(g,t,x,h) &= \mathbb{E}_{0,x}[g(X_t^h) - g(X_t)] = \mathbb{E}_{0,x}[u(t,X_t^h) - u(0,x)] \\ &= \mathbb{E}_{0,x}\left[\int_0^t \left(b(r,X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h)\right)\cdot\nabla u(r,X_r^h)\right]\mathrm{d}r.\end{aligned}$$

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Historical approach by [MP91] : for $b\in\mathcal{C}^\beta$

- Schauder estimates for u : $\|\nabla u\|_{L^\infty} < \infty$
- Use the regularity of b to control $\left|b(r,X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h)\right|$

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- Use the regularity of b to control $\left|b(r,X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h)\right|$

$$\mathcal{E}(g,t,x,h) \le C\|\nabla u\|_{L^\infty}\int_0^t \mathbb{E}_{0,x}\left[|X_r^h - X_{\tau_r^h}^h|^\beta\right]\mathrm{d}r \le C\|\nabla u\|_{L^\infty} h^{\frac{\beta}{\alpha}}. \qquad (5)$$

Weak error techniques

Compare the Duhamel formulas for Γ^h and Γ :

$$\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)$$
$$= \mathbb{E}_{0,x}\left[\int_0^t b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s) - b(\tau_s^h, X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X_s^h)\mathrm{d}s\right]$$

When b is singular, the main difficulty lies in controlling

$$b(\cdot,X_s) - b(\cdot,X_{\tau_s^h})$$

and

$$b(s,\cdot) - b(\tau_s^h,\cdot)$$

Weak error techniques - $b(s, X_s) - b(s, X_{\tau_s^h})$

In the splitting of the error, we write the following

$$\mathbb{E}_{0,x}\left[\int_0^t b(s, X_s)\cdot\nabla_y p_\alpha(t-s, y-X_s) - b(s, X_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s, y-X_{\tau_s^h})\mathrm{d}s\right]$$
$$= \int_0^t \int [\Gamma(0, x, s, z) - \Gamma(0, x, \tau_s^h, z)] b(s, z)\cdot\nabla_y p_\alpha(t-s, y-z)\mathrm{d}z\mathrm{d}s$$

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$$= \int_0^t \int [\Gamma(0,x,s,z) - \Gamma(0,x,\tau_s^h,z)] b(s,z)\cdot\nabla_y p_\alpha(t-s,y-z)\mathrm{d}z\mathrm{d}s$$

Using bootstrap techniques on the PDE, we can prove

$$|\Gamma(0,x,s,z) - \Gamma(0,x,\tau_s^h,z)| \leq C\frac{(s-\tau_s^h)^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}} p_\alpha(\tau_s^h, z-x) \leq C\frac{h^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}} p_\alpha(\tau_s^h, z-x)$$

where γ is the *gap to singularity* :

$$\gamma = \alpha - 1 - \frac{d}{p} - \frac{\alpha}{q} > 0 \qquad \text{Lebesgue, } b \in L^q - L^p$$
$$\gamma = \alpha + \beta - 1 > 0 \qquad \text{Holder, } b \in \mathcal{C}^\beta$$
$$\gamma = 2\beta + \alpha - 1 - \frac{d}{p} - \frac{\alpha}{q} > 0 \qquad \text{Negative Besov, } b \in L^q - \mathbb{B}^\beta_{p,r}$$

Weak error techniques - Heat kernel estimates - Besov case

Theorem ([F.,2023])

Let $\mathbb{P}$ be the solution to (E) *and* $(x_t)_{t\in[s,T]}$ *the associated canonical process. Then,* $\forall t \in (s, T]$, x_t *admits a density* $\Gamma(s,t,x,\cdot)$ *s.t.* $\exists C \geq 1 : \forall (x,y) \in \mathbb{R}^d$,

$$
\begin{aligned}
C^{-1}p_\alpha(t-s,y-x) \leq \Gamma(s,t,x,y) &\leq Cp_\alpha(t-s,y-x),\\
|\nabla_x \Gamma(s,t,x,y)| &\leq \frac{C}{(t-s)^{\frac{1}{\alpha}}}p_\alpha(t-s,y-x),\\
|\Gamma(s,t,x,y)-\Gamma(s,t,x,y')| &\leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}}\left(p_\alpha(t-s,y-x)\right.\\
&\qquad \left.+p_\alpha(t-s,y'-x)\right),\\
|\nabla_x\Gamma(s,t,x,y)-\nabla_x\Gamma(s,t,x,y')| &\leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left(p_\alpha(t-s,x-y)\right.\\
&\qquad \left.+p_\alpha(t-s,x-y')\right),
\end{aligned}
$$

for any $\rho \in (-\beta, -\beta+\gamma/2)$, *where* $\gamma := 2\beta+\alpha-1-\alpha/r-d/p > 0$

Weak error techniques - Heat kernel estimates - Besov case

Theorem ([F.,2025])

Moreover, for any $\varepsilon \in (0, -\beta)$ *and* $\eta \in (-\beta, -\beta + \varepsilon/2)$, *for* $t' > t$ *such that* $(t' - t) < t/2$,

$$\left\| \frac{\Gamma(s, x, t, \cdot) - \Gamma(s, x, t', \cdot)}{p_\alpha(t - s, \cdot - x)} \right\|_{\mathbb{B}^{\eta}_{\infty,\infty}} \leq C \frac{(t' - t)^{\frac{\gamma - \varepsilon}{\alpha}}}{(t - s)^{\frac{\gamma - \varepsilon - \beta}{\alpha}}} \tag{6}$$

Weak error techniques - Heat kernel estimates - Besov case

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Going back to the scheme, we can use this to write the following :

$$\begin{aligned} &\int_0^t \int [\Gamma(0, x, s, z) - \Gamma(0, x, \tau_s^h, z)] b(s, z) \cdot \nabla_y p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \\ &\quad \lesssim \int_0^t \frac{(s - \tau_s^h)^{\frac{\gamma - \varepsilon}{\alpha}}}{(\tau_s^h)^{\frac{\gamma - \varepsilon - \beta}{\alpha}}} \int p_\alpha(\tau_s^h, z - x) b(s, z) \cdot \nabla_y p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \end{aligned}$$

Weak error techniques - $b(s, X_s) - b(s, X_{\tau_s^h})$

It remains to show the boundedness of

$$\int_0^t \int p_\alpha(s, z-x) b(s,z) \nabla_y p_\alpha(t-s, y-z) \mathrm{d}z \mathrm{d}s.$$

If $b \in \mathfrak{H}$ in space, we have to estimate

$$\|p_\alpha(s, \cdot - x) \nabla_y p_\alpha(t-s, y-\cdot)\|_{\mathfrak{H}^*},$$

and recoup singularities in s and $t-s$, which we want to integrate against $\|b(s,\cdot)\|_{\mathfrak{H}}$.

Weak error techniques - $b(s, X_s) - b(\tau_s^h, X_s)$

Natural time-space scaling :
When $b(t, \cdot) \in \mathcal{C}^\beta$ in space, we usually assume $b(\cdot, z) \in \mathcal{C}^{\frac{\beta}{\alpha}}$, i.e.

$$|b(s, X_s) - b(\tau_s^h, X_s)| \leq (s - \tau_s^h)^{\frac{\beta}{\alpha}}$$

Not enough to show a convergence with order

$$\frac{\gamma}{\alpha} = \frac{\beta}{\alpha} + \frac{\alpha - 1}{\alpha}$$

Weak error techniques - $b(s, X_s) - b(\tau_s^h, X_s)$

Time-randomized Euler scheme :

$$X^h_{t_{k+1}} = X^h_{t_k} + hb(U_k, X^h_{t_k}) + (Z_{t_{k+1}} - Z_{t_k}),$$

where $(U_k) \sim \mathcal{U}([t_k, t_{k+1}])$.
Duhamel integral over one time-step for the time-randomized Euler scheme :

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \mathbb{E}\left[b(U_k, X^h_{\tau_s^h}) \cdot \nabla_y p_\alpha(t - U_k, y - X^h_{\tau_s^h})\right] \mathrm{d}s \\
&\quad = \int_{t_k}^{t_{k+1}} \frac{1}{h} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[b(r, X^h_{\tau_s^h}) \cdot \nabla_y p_\alpha(t - r, y - X^h_{\tau_s^h})\right] \mathrm{d}r\mathrm{d}s \\
&\quad = \int_{t_k}^{t_{k+1}} \frac{1}{h} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[b(r, X^h_{\tau_r^h}) \cdot \nabla_y p_\alpha(t - r, y - X^h_{\tau_r^h})\right] \mathrm{d}r\mathrm{d}s \\
&\quad = \int_{t_k}^{t_{k+1}} \mathbb{E}\left[b(r, X^h_{\tau_r^h}) \cdot \nabla_y p_\alpha(t - r, y - X^h_{\tau_r^h})\right] \mathrm{d}r.
\end{aligned}$$

Results - Hölder case , $\alpha \in (1, 2]$

Theorem ([F., Menozzi, 2024])

For $b \in L^\infty - \mathcal{C}^\beta, \beta > 0$*, we define the Euler scheme by*

$$X^h_{t_{k+1}} = X^h_{t_k} + hb(U_k, X^h_{t_k}) + (Z_{t_{k+1}} - Z_{t_k}),$$

where $\forall k, U_k \sim \mathcal{U}(t_k, t_{k+1})$*. Define*

$$\gamma = \beta + \alpha - 1 > 0.$$

Then,

$$|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| \leq Ch^{\frac{\gamma}{\alpha}}(1 + t^{-\frac{\beta}{\alpha}})p_\alpha(t, y - x) \quad (8)$$

Results - Lebesgue case, $\alpha \in (1, 2)$

Theorem ([F., Jourdain, Menozzi, 2024])

For $b \in L^q - L^p$, if

$$\gamma = \alpha - 1 - \frac{d}{p} - \frac{\alpha}{q} > 0,$$

then equation

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d$$

is weakly well-posed (in the sense of the martingale problem).

Theorem ([F., Jourdain, Menozzi, 2024])

For $b \in L^q - L^p$, we define the Euler scheme by

$$X^h_{t_{k+1}} = X^h_{t_k} + hb_h(U_k, X^h_{t_k}) + (Z_{t_{k+1}} - Z_{t_k}),$$

where b_h is a cut-offed b s.t. $|b_h| \le h^{-\frac{d}{\alpha p} - \frac{1}{q}}$ and $\forall k, U_k \sim \mathcal{U}(t_k, t_{k+1})$. Then,

$$|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| \le C h^{\frac{\gamma}{\alpha}} p_\alpha(t, y - x) \tag{9}$$

Besov case, dynamics of the SDE

Proposition (Dynamics of the distributional SDE, [DD16],[CdRM22])

Assume

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2 \right) \qquad \beta \in \left(\frac{1-\alpha+\frac{2d}{p}+\frac{2\alpha}{r}}{2}, 0 \right), \tag{GRD}$$

Then, the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d$$

rewrites

$$X_t = X_0 + \int_0^t \mathfrak{b}(s, X_s, \mathrm{d}s) + Z_t,$$

where $\forall 0 \le v \le s \le T$,

$$\mathfrak{b}(v, x, s-v) = \int_v^s \int b(r, y) p_\alpha(r-v, y-x)\mathrm{d}y\mathrm{d}r$$

Besov case, definition of the scheme

Definition of the scheme :

$$\begin{aligned}
X^h_{t_{i+1}} &= X^h_{t_i} + \mathfrak{b}(t_i, X^h_{t_i}, h) + (Z_{t_{i+1}} - Z_{t_i}) \\
&= X^h_{t_i} + \int_{t_i}^{t_{i+1}} \int b(r, y) p_\alpha(r - t_i, y - X^h_{t_i}) \mathrm{d}y \mathrm{d}r + (Z_{t_{i+1}} - Z_{t_i})
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&= X^h_{t_i} + \int_{t_i}^{t_{i+1}} \int b(r,y) p_\alpha(r - t_i, y - X^h_{t_i}) \mathrm{d}y \mathrm{d}r + (Z_{t_{i+1}} - Z_{t_i})
\end{aligned}
$$

$\rightarrow$ case dependent !

Results - Besov case , $\alpha \in (1,2)$

Theorem ([F., Menozzi, Issoglio, 2025])

For $b \in L^r - \mathbb{B}^{\beta}_{p,q}, \beta < 0$, using the previously defined Euler scheme, for all $\varepsilon > 0$,

$$|\Gamma(0,x,t,y) - \Gamma^h(0,x,t,y)| \leq Ch^{\frac{\gamma-\varepsilon}{\alpha}}(1 + t^{-\frac{\beta}{\alpha}})p_\alpha(t, y-x) \tag{10}$$

where $\gamma = 2\beta + \alpha - 1 - d/p - \alpha/r > 0$.

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For $b \in L^r - \mathbb{B}^{\beta}_{p,q}, \beta < 0$, using the previously defined Euler scheme, for all $\varepsilon > 0$,

$$|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| \leq Ch^{\frac{\gamma-\varepsilon}{\alpha}}(1 + t^{-\frac{\beta}{\alpha}})p_\alpha(t, y - x) \tag{11}$$

where $\gamma = 2\beta + \alpha - 1 - d/p - \alpha/r > 0$.

- Strictly stable setting only (although method is robust enough for gaussian case)
- Loss of ε due to the distributional setting
- Gap is the same as for well-posedness

Besov case, remark on the dynamics of the scheme

We are able to define the scheme under the condition

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2\right) \qquad \beta \in \left(\frac{1-\alpha+\frac{d}{p}+\frac{\alpha}{r}}{2}, 0\right), \tag{12}$$

and show tension and convergence of the law of (X_s^h) to some law which solves the SDE.

Besov case, remark on the dynamics of the scheme

We are able to define the scheme under the condition

$$\alpha \in \left(\frac{1 + \frac{d}{p}}{1 - \frac{1}{r}}, 2 \right) \qquad \beta \in \left(\frac{1 - \alpha + \frac{d}{p} + \frac{\alpha}{r}}{2}, 0 \right), \tag{13}$$

and show tension and convergence of the law of (X_s^h) to some law which solves the SDE.
If

$$\beta \in \left(\frac{1 - \alpha + \frac{d}{p} + \frac{\alpha}{r}}{2}, \frac{1 - \alpha + \frac{2d}{p} + \frac{2\alpha}{r}}{2} \right), \tag{14}$$

the laws still converge but we cannot make sense of the Young integral in the previous proposition.

Besov case as a consequence of the Hölder case ?

- Hölder : $|\Gamma(0,x,t,y)-\Gamma^h(0,x,t,y)| \le C[b]_{\mathcal{C}^\beta} h^{\frac{\beta+\alpha-1}{\alpha}} p_\alpha(t,y-x)$
- Negative Besov regularity : let $b \in L^r([0,T],\mathbb{B}^\eta_{p,q}), \eta < 0$ and set

$$\mathfrak{b}(s,z,h) := \frac{1}{h}\int_s^{s+h}\int b(u,y)p_\alpha(u-s,y-z)\mathrm{d}y\mathrm{d}u.$$

Then, $\mathfrak{b}(s,\cdot,h)$ is β-Hölder with $\beta > 0$ and its Hölder modulus

$$|\mathfrak{b}(s,z,h)-\mathfrak{b}(s,z',h)| \le C|z-z'|^\beta h^{-\frac{1}{r}-\frac{\beta}{\alpha}+\frac{\eta}{\alpha}-\frac{d}{\alpha p}}$$

Besov case as a consequence of the Hölder case ?

$$\mathrm{d}X_t^{h_1,h_2} = \mathfrak{b}(U_k, X_{\tau_t^{h_1}}^{h_1,h_2}, h_2)\mathrm{d}t + \mathrm{d}Z_t \qquad \text{Scheme for Holder SDE}$$

$$\uparrow\downarrow (1)$$

$$\mathrm{d}X_t^{h_2} = \mathfrak{b}(t, X_t^{h_2}, h_2)\mathrm{d}t + \mathrm{d}Z_t \qquad \text{Holder SDE}$$

$$\uparrow\downarrow (2)$$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t \qquad \text{Besov SDE}$$

- Hölder result : weak error on densities for (1) with rate $\frac{\eta+\alpha-\frac{\alpha}{r}-\frac{d}{p}-1}{\alpha}$ w.r.t. h_1.
- How to handle (2) is still unclear : taking $h_1 = h_2$ yields a lower rate, may be possible to optimize.

Thank you for your attention

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SDE solutions

Definition (Weak solution with bounded drift)

A stochastic process $(X_t, \mathcal{F}_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a weak solution to (E) with initial distribution μ if there exists a $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(Z_t)_{t\geq 0}$ on some (possibly different) $(\Omega', \mathcal{F}', \mathbb{P}')$ such that

(i) $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$

(ii) For all $t \geq 0$,

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + (Z_t - Z_0) \tag{15}$$

holds almost surely.

SDE solutions

Definition (Bounded drift martingale problem)

Let b be a bounded drift.
A probability measure $\mathbb{P}$ on Ω_α with time-marginals $(P_t)_{t\in[0,T]}$, solves the martingale problem related to $b\cdot\nabla+\mathcal{L}^\alpha$ and the initial probability measure μ if, denoting by $(\xi_t)_{t\in[0,T]}$ the associated canonical process,

(i) $P_0=\mu$,

(ii) for all $\mathcal{C}^{1,2}$ function f on $[0,T]\times\mathbb{R}^d$ bounded together with its derivatives, the process

$$\left\{f(t,\xi_t)-\int_0^t\Big((\partial_s+\mathcal{L}^\alpha)f(s,\xi_s)+b(s,\xi_s)\cdot\nabla f(s,\xi_s)\Big)\mathrm{d}s-f(0,\xi_0)\right\}_{0\le t\le T}, \tag{16}$$

is a $\mathbb{P}$-martingale.

Definition (Lebesgue drift martingale problem)

Let $b \in L^q - L^p$ be a Lebesgue drift.
A probability measure $\mathbb{P}$ on Ω_α with time-marginals $(P_t)_{t\in[0,T]}$, solves the martingale problem related to $b \cdot \nabla + \mathcal{L}^\alpha$ and the initial probability measure μ if, denoting by $(\xi_t)_{t\in[0,T]}$ the associated canonical process,

(i) $P_0 = \mu$,

(ii) for a.a. $t \in (0, T]$, $P_t(\mathrm{d}y) = \rho(t, y)\mathrm{d}y$ for some $\rho \in L^{q'}((0, T], L^{p'}(\mathbb{R}^d))$,

(iii) for all $\mathcal{C}^{1,2}$ function f on $[0, T] \times \mathbb{R}^d$ bounded together with its derivatives, the process

$$\left\{ f(t, \xi_t) - \int_0^t \Big((\partial_s + \mathcal{L}^\alpha) f(s, \xi_s) + b(s, \xi_s) \cdot \nabla f(s, \xi_s) \Big) \mathrm{d}s - f(0, \xi_0) \right\}_{0 \le t \le T}, \tag{17}$$

is a $\mathbb{P}$-martingale.

SDE solutions

Definition (Mild solution of the underlying PDE)

Let $b \in L^r - \mathbb{B}^{\beta}_{p,q}$ be a Besov drift. Let $\alpha \in (1,2]$, $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$.
For a given $T > 0$, we say that $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a mild solution of the formal Cauchy problem $\mathcal{C}(b, \mathcal{L}^\alpha, \phi, g, T)$

$$(\partial_t + b \cdot \nabla + \mathcal{L}^\alpha) f(t,x) = \phi(t,x) \text{ on } [0,T) \times \mathbb{R}^d, \qquad f(T,\cdot) = g \text{ on } \mathbb{R}^d,$$

if it belongs to $\mathcal{C}^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R})$ with $\nabla f \in \mathcal{C}^0_b([0,T], \mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty})$ for any $0 < \varepsilon \ll 1$ and $\theta = \beta + \alpha - \frac{d}{p} - \frac{\alpha}{r}$, and if it satisfies

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, f(t,x) = P^\alpha_{T-t}[g](x) - \int_t^T P^\alpha_{s-t}[\phi - b \cdot \nabla f](s,x) \mathrm{d}s, \quad (18)$$

where P^α_t denotes the semi-group generated by $\mathcal{L}^\alpha$.

SDE solutions

Definition (Besov drift martingale problem)

Let $b \in L^r - \mathbb{B}^{\beta}_{p,q}$ be a Besov drift.
We say that a probability measure $\mathbb{P}$ on Ω_α equipped with its canonical filtration is a solution of the martingale problem associated with $(b, \mathcal{L}^\alpha, x)$ for $x \in \mathbb{R}^d$ if, denoting by $(\xi_t)_{t\in[0,T]}$ the associated canonical process,

(i) $\mathbb{P}(\xi_0 = x) = 1$,

(ii) $\forall \phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d, \mathbb{R}))$, $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ with $\nabla g \in \mathbb{B}^{\theta-1}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\theta = \beta + \alpha - d/p - \alpha/r$, the process

$$\left\{ f(t,\xi_t) - \int_0^t \phi(s,\xi_s)\mathrm{d}s - f(0,x) \right\}_{0\le t\le T}$$

is a (square-integrable if $\alpha = 2$) martingale under $\mathbb{P}$ where f is the mild solution of the Cauchy problem $\mathcal{C}(b, \mathcal{L}^\alpha, \phi, g, T)$.

SDE solutions

Definition (Weak solution with distributional drift)

A weak solution to the formal distributional drift SDE (E) is a pair (X, Z) of adapted processes on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that Z is an α-stable process under $\mathbb{P}$ and, almost surely under $\mathbb{P}$,

$$X_t = X_0 + \int_0^t \mathcal{F}(s, X_s, \mathrm{d}s) + Z_t, \qquad \mathbb{E}\left|\int_0^t \mathcal{F}(s, X_s, \mathrm{d}s)\right| < \infty. \tag{19}$$

where, for any $0 \leq v \leq s \leq T, x \in \mathbb{R}^d$,

$$\mathcal{F}(v, x, s - v) := \int_v^s \int b(r, y) p_\alpha(r - v, y - x) \mathrm{d}y \mathrm{d}r \tag{20}$$

and the integral in (19) is understood as an L^ℓ stochastic Young integral for any $\ell \in (1, \alpha]$.

Test functions 1

Let us mention that if one is interested in the weak error for some test function f,

$$\mathcal{E}(f, x, t, h) := \mathbb{E}_{0,x}[f(X_t^h) - f(X_t)],$$

as soon as f is δ-Hölder (not necessarily bounded) then, a rate can be derived as a consequence of the convergence of $|\Gamma(s, x, t, y) - \Gamma^h(s, x, t, y)|$ using a simple cancellation argument :

$$\begin{aligned}
\mathcal{E}(f, x, t, h) &= \int_{\mathbb{R}^d} (\Gamma^h - \Gamma)(0, x, t, y) f(y) \mathrm{d}y \\
&= \int_{\mathbb{R}^d} (\Gamma^h - \Gamma)(0, x, t, y) \big(f(y) - f(x)\big) \mathrm{d}y, \\
|\mathcal{E}(f, x, t, h)| &\lesssim h^{\frac{\gamma}{\alpha}} \int_{\mathbb{R}^d} p_\alpha(t, y - x) |x - y|^\delta \mathrm{d}y \\
&\lesssim t^{\frac{\delta}{\alpha}} h^{\frac{\gamma}{\alpha}}.
\end{aligned}$$