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Singular drift stochastic differential equations: density estimates and weak discretization

Equations différentielles stochastiques à dérives singulières : estimées de densités et discrétisation faible

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Résumé

Cette thèse est dédiée à l'étude des équations différentielles stochastiques, dont la dérive, singulière, appartient à un espace de Hölder, de Lebesgue ou de Besov, dirigées par un processus stable symétrique. Dans ce cadre irrégulier, le caractère bien-posé faible des équations ne peut être obtenu via les théorèmes usuels de l'analyse stochastique, et il repose sur des effets de régularisation par le bruit. Notre objectif est d'établir le caractère bien-posé de ces équations, d'obtenir des estimées sur les densités de leurs solutions et de fournir des taux de convergence quantitatifs pour l'erreur faible associée à leur discrétisation.

Précisément, nous définissons des schémas de discrétisation de type Euler pour des équations différentielles stochastiques à dérive singulière à bruit additif et donnons des taux de convergence pour l'erreur faible associée aux densités. Nos contributions principales consistent à obtenir des taux de convergence qui se fondent sur la régularité en temps de l'équation aux dérivées partielles parabolique sous-jacente, permettant ainsi de relaxer les hypothèses sur la régularité en espace de la dérive. Ce faisant, nous soulignons le rôle fondamental que joue "l'écart à la singularité", qui est défini comme la marge restante dans la condition d'existence et unicité d'une solution faible à l'équation considérée, et qui conditionne la régularité en temps de la densité de la solution.

Si les estimées sur la régularité en temps de la densité existent, comme dans le cadre Besov où nous prouvons ces estimées indépendamment ou dans le cadre Hölder, elles peuvent être utilisées pour en déduire les résultats sur l'erreur faible. Si ce n'est pas le cas, il est possible de travailler à la place sur la densité de l'équation discrétisée, ce qui permet, par passage à la limite, d'obtenir le caractère bien posé de l'équation différentielle stochastique ainsi que des estimées sur sa densité comme nous le faisons dans le cadre Lebesgue.

Nous mettons également en évidence l'importance de la randomisation de l'argument en temps de la dérive lors de la discrétisation. Cette technique, qui peut être vue comme une forme de régularisation par le bruit, permet de se passer d'hypothèses sur la régularité en temps de la dérive.

Nos méthodes sont valables dans le cadre Brownien et dans le cadre d'un bruit strictement stable symétrique de mesure spectrale régulière.

Title: Singular drift stochastic differential equations: density estimates and weak discretization

Keywords: Singular drifts, heat kernel estimates, Euler scheme, stochastic differential equations (SDE), regularization by noise

Abstract

This thesis is dedicated to the study of stochastic differential equations with singular drifts belonging to Hölder, Lebesgue or Besov spaces, driven by symmetric stable noises. In this singular setting, well-posedness cannot be derived from standard Itô calculus results, and we rely on regularization by noise effects to obtain solutions to these equations. Our aim is to study their weak well-posedness, to derive estimates on the density of the solutions and to provide quantitative weak convergence rates for their discretization.

Namely, we provide discretization schemes of Euler type for singular drift SDEs with additive noise and compute convergence rates for the associated weak error on densities. Our main contributions in this part consist in obtaining rates which are based on the time-regularity of the associated parabolic partial differential equation, therefore allowing to relax assumptions on the spatial regularity of the drift. Doing so, we underline the fundamental role of the "gap to singularity", which can be defined as the margin left in the weak well-posedness condition of a singular drift SDE, and which conditions the regularity of the density of the solution.

If estimates on the time regularity of the density are available, like in the Besov case where we prove these estimates separately or in the Hölder case, they can be used to compute weak error rates. Otherwise, it is possible to work with the density of the discretized equation instead. By passing to the limit, as we do in the Lebesgue setting, we then obtain well-posedness of the SDE as well as heat kernel estimates on its density.

We highlight as well the importance of randomizing the time-argument when defining the schemes, which can also be seen as some sort of regularization by noise phenomenon, in order to avoid any assumptions on the time regularity of the drift.

Our techniques are valid in both the Brownian setting and the strictly stable symmetric, smooth spectral measure setting.

Notations

- $\mathcal{C}(A,B)$: space of continuous functions from A to B
- $\mathcal{C}^\beta$: Hölder space of regularity β
- $\mathcal{C}_b^\infty$: space of smooth bounded functions
- $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class and $\mathcal{S}'(\mathbb{R}^d)$ its dual
- $\mathcal{D}(A,B)$: space of càdlàg functions from A to B
- $\mathbb{B}^\beta_{p,q}$: Besov space with regularity β and integrability indexes p,q
- L^r : Lebesgue space with exponent r
- $A \lesssim B$ if there exists a constant C (possibly depending on the current parameters) such that $A \leq CB$
- $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$
- $\mathcal{L}^\alpha$ generator of the α-stable process (the exact nature of which depends on the context)
- P_t^α semi-group associated with $\mathcal{L}^\alpha$
- p_α : density of the α-stable process (the exact nature of which depends on the context)
- $\tau_s^h = h\lfloor s/h \rfloor \in (s-h, s]$ is the last grid point before time s
- $\star$ denotes the spatial convolution.
- For $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\phi(0) \neq 0$, we set $\phi(D)f = \mathcal{F}^{-1}\left(\phi \times \mathcal{F}(f)\right) = \mathcal{F}^{-1}(\phi) \star f$, where $\mathcal{F}$ denotes the Fourier transform.
- For $p \in [1,+\infty]$, we always denote by $p' \in [1,+\infty]$ s.t. $\frac{1}{p} + \frac{1}{p'} = 1$ its conjugate.
- $\mathbb{N}$ is the set of natural numbers (including 0) and $\mathbb{N}^*$ is the set of non-zero natural numbers

Contents

Part I

Introduction

Chapter 1

Singular stochastic differential equations

1.1 Well-posedness of singular drift SDEs

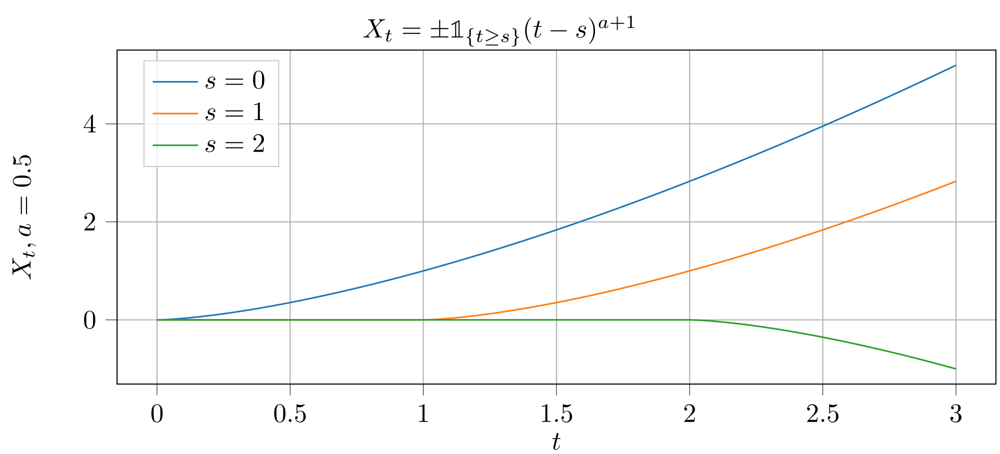
Consider the ordinary differential equation (ODE)

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t, \qquad X_0 = x \in \mathbb{R}^d. \tag{1.1.1}$$

The Cauchy-Lipschitz theorem states that if b is continuous in time and locally Lipschitz in its spatial variable (with Lipschitz constant independant of time), then (1.1.1) admits a unique local solution. The Peano existence theorem allows to go below Lipschitz regularity and ensures local existence of solutions whenever $b \in \mathcal{C}^\beta$, $\beta \in [0, 1)$ is (Hölder) continuous. In this regime, the Peano example gives a counter-example to uniqueness: let us consider the one-dimensional ODE

$$\mathrm{d}X_t = |X_t|^a \mathrm{sgn}(X_t)\,\mathrm{d}t, \qquad X_0 = 0, \tag{1.1.2}$$

for some $a \in (0, 1)$. Then, (1.1.2) admits infinitely many explicit solutions as for all $s > 0$, the following process solves (1.1.2):



This gives an intuition about why uniqueness might break down: in this example, solutions stay at $y = 0$ for any amount of time and then take off at time $s \geq 0$, either towards positive or negative values of y. However, if one were to give them a small push at time 0, one would expect that there is only one possible path for the solution to follow. This heuristic is actually made rigorous in the work of Flandoli and Delarue [DF14]. In this work, it is proved that the limit of a vanishing Brownian viscosity in (1.1.2) is a symmetric probability measure weighting the maximal solutions of (1.1.2), thus restoring a kind of uniqueness.

Going below continuity, we even lose existence. Consider the ODE

$$dX_t = -\text{sgn}(X_t)\,dt, \qquad X_0 = 0 \tag{1.1.3}$$

with the convention $\text{sgn}(0) = 1$. This ODE has bounded drift, yet existence fails: $\frac{dX_t^2}{dt} = -2X_t\text{sgn}(X_t) \leq 0$ and $X_0^2 = 0$ so for all $t \leq 0$, $X_t^2 = 0$ and in turn $X_t = 0$, which does not solve (1.1.3).

To tackle the previous issues and inspired by the Peano example, one way to restore some sort of well-posedness is to introduce randomness in the equation. Let us introduce the stochastic differential equation (SDE)

$$dX_t = b(t, X_t)\,dt + dZ_t, \tag{1.1.4}$$

where (Z_t) is a semi-martingale (for example a standard Brownian motion or a stable process). This equation, which may have a singular (i.e. Hölder, Lebesgue or Besov) drift, is the main object of this thesis. We will be interested in its well-posedness, its discretization as well as in estimates on its underlying density. It was proved by Veretennikov in [Ver80] that this equation, with Brownian noise and bounded drift, admits a unique strong solution. In general terms, for a process (X_t) to be called a solution to (1.1.4), we will require that for all $t \geq 0$,

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + (Z_t - Z_0)$$

holds almost surely. The solution is said to be a *strong* solution if it is progressively measurable with respect to the sigma algebra $\sigma(Z_s, 0 \leq s \leq t)$ generated by the noise (roughly speaking, the solution is a function of the input noise). It is said to be a *weak* solution if any two solutions (possibly with different noises defined on distinct probability spaces) have the same probability distribution. For strong well-posedness, the key notion is pathwise uniqueness, which holds if, for two solutions X^1 and X^2 defined on the same probability space,

$$\mathbb{P}\left(\sup_{t\geq 0} |X_t^1 - X_t^2| = 0\right) = 1.$$

Together with weak existence, pathwise uniqueness gives strong uniqueness and thus strong well-posedness through the Yamada-Watanabe theorem.

For weak solutions, this notion is not suitable and uniqueness refers to uniqueness *in law* for the underlying processes. Let us as well mention that if we have strong existence and weak uniqueness, it is possible to obtain strong uniqueness, which is sometimes refered to as the *dual* Yamada-Watanabe theorem.

In this thesis, we are mainly interested in *weak* aspects: we prove weak well-posedness results and related estimates and our results on discretization schemes are related to the rate of convergence of the time marginal laws of the process.

Let us now investigate more precisely what is required for well-posedness of (1.1.4). Roughly speaking, we expect to be able to give a rigorous meaning to (1.1.4) if, formally, the integral

$$I_t := \int_0^t [b(s, Z_s) - b(0, Z_0)]\;ds$$

is *more regular* than the noise process Z_t in the sense that its typical time scale, $\mathbb{E}[|I_t|]$, should be smaller. This leads to the following condition: if we assume that b is time-homogeneous and β-Hölder continuous in space and Z_t is an α-stable process, then (formally), $b(Z_\cdot) \in \mathcal{C}^{\frac{\beta}{\alpha}}$ and the condition reads

$$1 + \frac{\beta}{\alpha} > \frac{1}{\alpha} \iff \alpha + \beta > 1. \tag{1.1.5}$$

If this is the case, it is possible to construct a solution to (1.1.4), the typical time scale of which will be the same as that of the noise.

This is indeed the threshold which appears in the seminal article by Tanaka, Tsuchiya and Watanabe [TTW74]. The authors consider therein the scalar case, and proved that strong uniqueness holds for bounded β-Hölder drifts under this condition, while giving a uniqueness counter example when $\beta + \alpha < 1$. A critical multidimensional case ($\alpha = 1$, $\beta = 0$ continuous drift) in a time-inhomogeneous setting was investigated in [Kom84], in which weak uniqueness is derived, with the driving noise having absolutely continuous spectral measure w.r.t. the Lebesgue measure on the sphere. This was later extended to an arbitrary non-degenerate spectral measure in [CdRMP20b].

Note that most regularization by noise results involving a stable noise usually impose a non-degeneracy condition on the noise, i.e. it should act on all directions. In [Wat07], Watanabe investigates the exact link between the support of the non-degenerate spectral measure of the process and the estimates one can obtain on its density. Whenever the spectral measure of the noise is not equivalent to (i.e. bounded from above and below by) the Lebesgue measure on the sphere, global estimates on the density (such as those presented in Section 2.2) are delicate to obtain, making the equivalence assumption standard to computing heat kernel estimates. Let us mention that a branch of the regularization by noise literature focuses on more general non-degenerate spectral measure such as cylindrical α-stable processes (see e.g. [CdRM22a], [CZZ21], [CHZ20], [CdRMP20a]). However, these last results do no concern pointwise estimates on the density of the underlying process.

Having in mind that weak (or strong) well-posedness is often investigated through the corresponding parabolic PDE, recalling that the associated expected parabolic gain is $\beta + \alpha$, the condition $\beta + \alpha > 1$ coincides with the regularity required to define the gradient of the fundamental solution to the PDE. The aforementioned regularity gain is often obtained through Schauder-type estimates. We can mention [MP14] (bounded drift, stable-like generators), [CdRMP20a] (unbounded drift, general stable generators including e.g. the cylindrical one). These estimates naturally lead to weak uniqueness in the multidimensional setting for (1.1.4) through the martingale problem, which precisely requires a control of the gradient of the solution of the PDE.

In [FJM24], we extend the condition (1.1.5) to SDEs with time-inhomogeneous drifts which belong to the Lebesgue space $L^q([0,T], L^p(\mathbb{R}^d)) := \left\{ f : [0,T] \times \mathbb{R}^d : \|t \mapsto \|f(t,\cdot)\|_{L^p}\|_{L^q([0,T])} < \infty \right\} =: L^q - L^p$ with Z_t being a symmetric non-degenerate d-dimensional α-stable process, whose spectral measure is equivalent to the Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$ under the condition

$$\alpha - \left(\frac{d}{p} + \frac{\alpha}{q} \right) > 1, \qquad \alpha \in (1,2). \tag{1.1.6}$$

This is done by investigating the Euler scheme associated with (1.1.4). Namely, we compute estimates on the density of the Euler scheme which are uniform with respect to the time step of the discretization and we let the time step go to 0 to obtain a solution of the martingale problem associated with (1.1.4). This method also allows to immediately deduce heat kernel estimates for the solution as the uniform estimates on the density of the Euler scheme pass to the limit.

This well-posedness result extends the ones derived in [Por94] (scalar) and [PP95] (multidimensional), which adress the strictly stable *time-homogeneous* case. Under the condition $\alpha - d/p > 1$, authors therein construct the density through its parametrix series expansion and show that it solves the corresponding martingale problem. Let us also mention the work [CdRM22a], in which weak well-posedness is proved for distributional drifts in the Besov-Lebesgue space $L^q - \mathbb{B}^{\beta}_{p,r}$ under the condition (see below why a factor two appears in the distributional setting)

$$\alpha + \left(2\beta - \frac{d}{p} - \frac{\alpha}{q} \right) > 1. \tag{1.1.7}$$

In view of this threshold, our well-posedness result can be seen as an extension of this work for $\beta = 0$.

Note that, when considering the embedding $L^p(\mathbb{R}^d) \hookrightarrow \mathbb{B}^{-\frac{d}{p}}_{\infty,\infty}(\mathbb{R}^d)$ (which roughly speaking expresses the trade-off between integrability and regularity) and the time-space scale of the equation, the condition (1.1.6) is consistent with the condition $\alpha+\beta > 1$ appearing in the Hölder case if we see $-d/p-\alpha/q$ as the regularity of an $L^q - L^p$ drift.

The condition (1.1.6) that we obtain can also be seen as the α-stable extension of the Krylov-Röckner condition

$$\frac{d}{p} + \frac{2}{q} < 1 \tag{1.1.8}$$

for Brownian-driven SDEs (see [KR05]), although not guaranteeing *strong* well-posedness in the strictly stable setting ($\alpha < 2$). Indeed, in [KR05], authors make use of the Girsanov theorem (which is specific to the Brownian setting) along with the aforementioned Yamada-Watanabe theorem to derive strong well-posedness from weak existence and strong uniqueness. In a stable setting, in order to derive strong well-posedness, some additional smoothness conditions on the drift is required, expressed for example in terms of Bessel potential spaces by Xie and Zhang in [XZ20]. It was also shown by Priola in [Pri12] that pathwise uniqueness holds in the multidimensional case for general non-degenerate stable generators with $\alpha \geq 1$ for time-homogeneous bounded β-Hölder drifts under the assumption $\beta > 1 - \alpha/2$. Under the same assumption, [CZZ21] proved strong existence and uniqueness for any $\alpha \in (0, 2)$, as well as weak uniqueness whenever $\beta + \alpha > 1$ for time-inhomogeneous drift with non-trivial diffusion coefficient. Those results are usually obtained using the Zvonkin transform (see [Zvo74], [Ver80]), which requires additional regularity on the underlying PDE, which again follows from Schauder-type estimates.

Let us now mention a few recent results which give a more detailed understanding of the threshold (1.1.6). The critical regime for Lebesgue drift equation driven by Brownian motion was investigated by Röckner and Zhao in [RZ25], where strong existence is proved as well as pathwise uniqueness among a class of solution satisfying certain estimates (see as well [Kry23] and [Kry21a] for the homogeneous case). In [Kry21b], Krylov works under the condition

$$\frac{d}{p} + \frac{1}{q} < 1, \tag{1.1.9}$$

which is slightly weaker than (1.1.8). In the subcritical regime (i.e. when (1.1.8) is satisfied), as previously mentioned, the noise dominates the drift in small time. This is not necessarily the case anymore under (1.1.9), meaning that regularization by noise phenomenon might not take place. However, [Kry21b] provides a proof for weak existence of a solution in this regime, although uniqueness may fail. Extending this result to more general, possibly non-markovian, noises, Butkovsky and Gallay prove in [BG23] weak existence in the strictly stable setting (using the John-Nirenberg inequality) under

$$\frac{\alpha - 1}{q} + \frac{d}{p} < \alpha - 1 \tag{1.1.10}$$

and in the fractional Brownian motion setting (using the stochastic sewing lemma of Lê, see [Lê20]) under

$$\frac{1-H}{q} + \frac{Hd}{p} < 1 - H, \tag{1.1.11}$$

where H is the Hurst parameter of the considered fractional Brownian motion. In this second setting, authors also provide a counter-example to show optimality of this condition. Strong well-posedness in these regimes remains an open problem.

Going towards negative regularity for the drift brings additional difficulties. The first challenge is to specify what is intended with "solution" to (1.1.4). To this end, a key tool is the following PDE:

$$(\partial_t + b \cdot \nabla + \mathcal{L}^\alpha)\, u(t, x) = f(t, x) \text{ on } [0, T) \times \mathbb{R}^d, \qquad u(T, \cdot) = g \text{ on } \mathbb{R}^d \tag{1.1.12}$$

for suitable sources f and final conditions g, and where $\mathcal{L}^\alpha$ is the generator of the noise Z. Note that this PDE is only available in the current *Markovian* setting. When using non-Markovian noises such as fractional

Brownian motion, other tools have to be relied upon (mainly the aforementioned stochastic sewing lemma of Khoa Lê, see [Lê20]). When studying (1.1.12), defining the gradient of the solution still requires $\alpha + \beta > 1$, which now imposes $\alpha > 1$. This is anyhow not sufficient: we also need to be able to define $b \cdot \nabla u$ as a distribution. Roughly, since b has spatial regularity β, this imposes $\beta + (\beta + \alpha - 1) > 0 \Longleftrightarrow \beta > \frac{1-\alpha}{2}$ by usual paraproduct rules (note that this is the exact assumption we need if $p = r = +\infty$). This threshold already appears in [BC01] in the diffusive setting ($\alpha = 2$), where strong well-posedness is derived in the scalar case through Dirichlet forms techniques for specifically structured time-homogeneous drifts. The same threshold is exhibited in [FIR17], where the authors introduce the notion of *virtual solutions* to give a meaning to (1.1.4). Those solutions are defined through a Zvonkin-type transform formula, and, while not requiring any specific structure, do not yield a precise dynamics for the SDE. We can also refer to Zhang and Zhao ([ZZ17]) and Athreya, Butkovsky and Mytnik ([ABM20]), who specified the meaning to be given to (1.1.4), in the sense that the drift therein is defined through smooth approximating sequences of the singular b along the solution. Importantly, the limit drift is a Dirichlet process, highlighting once again that (1.1.4) is a *formal* equation. A thorough description of this Dirichlet process was done in the Brownian scalar case by Delarue and Diel in [DD16] and extended by Cannizzaro and Chouk in [CC18] for multidimensional SDEs. Assuming some additional structure on the drift, they manage to go beyond the above threshold and reach $\beta > -\frac{2}{3}$ (still with $p = r = \infty$). This work was extended in the multidimensional strictly stable case, still assuming a specific structure for the drift by Kremp and Perkowski in [KP22], in which weak well-posedness is proved for $\beta > \frac{2-2\alpha}{3}$. Without any structure on the drift, a similar and consistent description of the dynamics for the weak solutions of (1.1.4) in the multidimensional setting is obtained in [CdRM22a] for $\beta > \frac{1-\alpha}{2}$. The case of a non-trivial diffusion coefficient was investigated by Ling and Zhao in [LZ22] with the same thresholds. Note that, in the present work, we chose to work with a trivial diffusion coefficient as the most delicate issue is the handling of the singular drift (see Remark 15 in [CdRM22a] for the handling of a non trivial diffusion coefficient in a Duhamel expansion). We do believe our approach for heat kernels would be robust enough to treat this case. We emphasize that most of the aforementioned results heavily rely on the Schauder-type regularization properties of the PDE (1.1.12).

1.2 Defining weak solutions to singular drift SDEs

When we consider SDEs with singular drifts, defining solutions becomes a challenge. Let $\Omega_\alpha = \mathcal{D}([0,T], \mathbb{R}^d)$ (the Skorokhod space of càdlàg functions) if $\alpha \in (1,2)$ and $\Omega_2 = \mathcal{C}([0,T], \mathbb{R}^d)$. We will here focus on weak solutions as they are the main object of this thesis. Let us start with the case of a bounded drift.

Definition 1.1 (Weak solution with bounded drift)**.** *A stochastic process $(X_t, \mathcal{F}_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a weak solution to (1.1.4) with initial distribution μ if there exists a $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(Z_t)_{t\geq 0}$ on some (possibly different) $(\Omega', \mathcal{F}', \mathbb{P}')$ such that*

(i) $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$

(ii) For all $t \geq 0$,

$$X_t = X_0 + \int_0^t b(s, X_s)\, \mathrm{d}s + (Z_t - Z_0) \tag{1.2.1}$$

holds almost surely.

As b is bounded, we have no issues making sense of the integral term in (1.2.1). It was proved in [SV97] and [EK86] that this formulation is equivalent to considering the following so-called "martingale problem":

Definition 1.2 (Bounded drift martingale problem)**.** *Let b be a bounded drift.*
A probability measure $\mathbb{P}$ on Ω_α with time-marginals $(P_t)_{t\in[0,T]}$, solves the martingale problem related to $b \cdot \nabla + \mathcal{L}^\alpha$ and the initial probability measure μ if, denoting by $(\xi_t)_{t\in[0,T]}$ the associated canonical process,

(i) $P_0 = \mu$,

(ii) for all $\mathcal{C}^{1,2}$ function f on $[0,T] \times \mathbb{R}^d$ bounded together with its derivatives, the process

$$\left\{ f(t, \xi_t) - \int_0^t \Big((\partial_s + \mathcal{L}^\alpha) f(s, \xi_s) + b(s, \xi_s) \cdot \nabla f(s, \xi_s) \Big)\, \mathrm{d}s - f(0, \xi_0) \right\}_{0\leq t\leq T}, \tag{1.2.2}$$

is a $\mathbb{P}$-martingale.

One can readily see that if there exists a weak solution in the sense of Definition 1.1, by applying Itô's formula, we obtain a solution to the martingale problem of Definition 1.2.

Moreover, uniqueness holds for the martingale problem if and only if uniqueness in law holds for the weak solution, i.e. if the finite dimensional distributions of any two weak solutions are equal.

The choice of the class of f is not critical. We only need it to be rich enough to characterize marginal laws, i.e. a class of functions Φ is sufficient if whenever two probability measures μ_1 and μ_2 satisfy

$$\int \phi \,\mathrm{d}\mu_1 = \int \phi \,\mathrm{d}\mu_2, \qquad \forall \phi \in \Phi,$$

then $\mu_1 = \mu_2$.

When considering Lebesgue drifts, a new difficulty appears: in order to make sense of

$$\mathbb{E}\left[\int_0^T |b(s, X_s)| \,\mathrm{d}s\right] = \int_{\Omega_\alpha} \int_0^T |b(s, \xi_s)| \,\mathrm{d}s P(\mathrm{d}\xi),$$

we need an additional assumption on the solution. Namely, we can give the following definition:

Definition 1.3 (Lebesgue drift martingale problem)**.** *Let $b \in L^q - L^p$ be a Lebesgue drift. A probability measure $\mathbb{P}$ on Ω_α with time-marginals $(P_t)_{t\in[0,T]}$, solves the martingale problem related to $b \cdot \nabla + \mathcal{L}^\alpha$ and the initial probability measure μ if, denoting by $(\xi_t)_{t\in[0,T]}$ the associated canonical process,*

(i) $P_0 = \mu$,

(ii) for a.a. $t \in (0, T]$, $P_t(\mathrm{d}y) = \rho(t, y)\,\mathrm{d}y$ *for some* $\rho \in L^{q'}((0, T], L^{p'}(\mathbb{R}^d))$,

(iii) for all $\mathcal{C}^{1,2}$ function f on $[0, T] \times \mathbb{R}^d$ bounded together with its derivatives, the process

$$\left\{ f(t, \xi_t) - \int_0^t \Big((\partial_s + \mathcal{L}^\alpha) f(s, \xi_s) + b(s, \xi_s) \cdot \nabla f(s, \xi_s) \Big) \,\mathrm{d}s - f(0, \xi_0) \right\}_{0\le t\le T}, \tag{1.2.3}$$

is a $\mathbb{P}$-martingale.

Let us point out that, in the current singular drift setting, condition (ii) which guarantees that

$$\int_{\Omega_\alpha} \int_0^T |b(s, \xi_s)| \,\mathrm{d}s P(\mathrm{d}\xi) < \infty$$

is somehow the minimal one required for all the terms in (1.2.3) to be well defined. Basically, the requirement is that the probability measure which solves the martingale problem has time marginals, which, seen as a function of time and space, admits a density w.r.t. the Lebesgue measure which belongs to the dual space of b. This highlights that in the singular setting, estimates (in this case Lebesgue estimates) on the density are *somehow* necessary to ensure that a measure is a solution to the SDE in the sense of the previous Definition 1.3.

Now, if b is a distribution, (1.2.3) does not necessarily make sense anymore. To reach this setting, we will study the cauchy problem associated with the operator $\partial_t + b \cdot \nabla + \mathcal{L}^\alpha$ in order to formally get rid of the integrand in (1.2.3) and replace it with a source term. This trick allows to make sense of the martingale problem as long as the Cauchy problem is well-posed, therefore lowering the regularity required on the drift. This approach is known as the generalized martingale problem, see [EK86].

For the sake of simplicity, assume that $b \in \mathbb{B}^{\beta}_{\infty,\infty}, \beta < 0$ is a time-homogeneous distributional drift. For the definition of the Cauchy problem in the distributional Besov setting, we need the following conditions on α, β:

$$2\beta + \alpha > 1 \tag{1.2.4}$$

and we will denote

$$\theta := \beta + \alpha, \tag{1.2.5}$$

which corresponds to the parabolic bootstrap induced by the drift. As explained in [CdRM22a], this choice of θ implies that the mapping

$$(t, x) \mapsto \int_t^T P^{\alpha}_{s-t}[G \cdot v](s, x)\, \mathrm{d}s$$

is well defined and belongs to $\mathcal{C}^{0,1}_b([0,T] \times \mathbb{R}^d, \mathbb{R})$ for $G \in \mathbb{B}^{\beta}_{\infty,\infty}$ and $v \in L^{\infty}([0,T], \mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d))$ for some $0 \leq \varepsilon \ll 1$.

Note that, here, we are only trying to give a meaning to the distributional product $G \cdot v$. Roughly speaking, by Bony's paraproduct rule, we need the sum of the regaularities of G and v to be positive. This is only possible if α and β satisfy (1.2.4), hence the definition of the former.

In the more general case of a time-dependent generic Besov drift, we can give the following definition:

Definition 1.4 (Mild solution of the underlying PDE)**.** *Let $b \in L^r - \mathbb{B}^{\beta}_{p,q}$ be a Besov drift. Let $\alpha \in (1, 2]$, $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$.*
For a given $T > 0$, we say that $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a mild solution of the formal Cauchy problem $\mathcal{C}(b, \mathcal{L}^{\alpha}, \phi, g, T)$

$$(\partial_t + b \cdot \nabla + \mathcal{L}^{\alpha})\, f(t, x) = \phi(t, x) \text{ on } [0, T) \times \mathbb{R}^d, \qquad f(T, \cdot) = g \text{ on } \mathbb{R}^d,$$

if it belongs to $\mathcal{C}^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R})$ with $\nabla f \in \mathcal{C}^0_b([0,T], \mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty})$ for any $0 < \varepsilon \ll 1$ and $\theta = \beta + \alpha - \frac{d}{p} - \frac{\alpha}{r}$, and if it satisfies

$$\forall (t, x) \in [0,T] \times \mathbb{R}^d, f(t, x) = P^{\alpha}_{T-t}[g](x) - \int_t^T P^{\alpha}_{s-t}[\phi - b \cdot \nabla f](s, x)\, \mathrm{d}s, \tag{1.2.6}$$

where P^{α}_t denotes the semi-group generated by $\mathcal{L}^{\alpha}$.

Definition 1.5 (Besov drift martingale problem)**.** *Let $b \in L^r - \mathbb{B}^{\beta}_{p,q}$ be a Besov drift.*
We say that a probability measure $\mathbb{P}$ on Ω_{α} equipped with its canonical filtration is a solution of the martingale problem associated with $(b, \mathcal{L}^{\alpha}, x)$ for $x \in \mathbb{R}^d$ if, denoting by $(\xi_t)_{t \in [0,T]}$ the associated canonical process,

(i) $\mathbb{P}(\xi_0 = x) = 1$,

(ii) $\forall \phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d, \mathbb{R}))$, *$g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ with $\nabla g \in \mathbb{B}^{\theta-1}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\theta = \beta + \alpha - d/p - \alpha/r$, the process*

$$\left\{ f(t, \xi_t) - \int_0^t \phi(s, \xi_s)\, \mathrm{d}s - f(0, x) \right\}_{0 \leq t \leq T}$$

is a (square-integrable if $\alpha = 2$) martingale under $\mathbb{P}$ where f is the mild solution of the Cauchy problem $\mathcal{C}(b, \mathcal{L}^{\alpha}, \phi, g, T)$.

Let us mention that in the distributional setting, the solution process has to be understood as a Dirichlet process in the following sense: assuming that $b \in \mathbb{B}^{\beta}_{\infty,\infty}, \beta < 0$ is a time-homogeneous drift, Chaudru de Raynal and Menozzi proved in [CdRM22a] that the martingale problem of Definition 1.5 is well-posed. They also introduce the following notion:

Definition 1.6 (Weak solution with distributional drift)**.** *A weak solution to the formal distributional drift SDE* (1.1.4) *is a pair* (X, Z) *of adapted processes on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* Z *is an* α*-stable process under* $\mathbb{P}$ *and, almost surely under* $\mathbb{P}$*,*

$$X_t = X_0 + \int_0^t \mathcal{F}(s, X_s, \mathrm{d}s) + Z_t, \qquad \mathbb{E}\left|\int_0^t \mathcal{F}(s, X_s, \mathrm{d}s)\right| < \infty. \tag{1.2.7}$$

where, for any $0 \leq v \leq s \leq T, x \in \mathbb{R}^d$*,*

$$\mathcal{F}(v, x, s - v) := \int_v^s \int b(r, y) p_\alpha(r - v, y - x) \, \mathrm{d}y \, \mathrm{d}r \tag{1.2.8}$$

and the integral in (1.2.7) *is understood as an* L^ℓ *stochastic Young integral for any* $\ell \in (1, \alpha]$*.*

Importantly, the existence of such object is derived through an appropriate extention of the martingale problem which also keeps track of the noise, which is used to reconstruct the drift.

This result bears a huge importance in the scope of developping the associated numerical scheme. It gives the means to discretize the equation as long as we can compute $\mathcal{F}$ numerically in some way. The discretization scheme that we use in Chapter 6 is inspired by this representation.

Other definitions of solutions have been proposed, such as virtual solutions, by Flandoli, Issoglio and Russo in [FIR17], where the solution is defined through a kind of Zvonkin transform. Those are shown in the same paper to be equivalent to defining solutions through mollification of the drift.

Once again, since the distributional setting imposes to study the Cauchy problem associated with the underlying PDE. Estimating the density, when possible, is the most complete approach in this scope. The next section is dedicated to this issue.

1.3 Heat kernel estimates

When studying well-posedness, a natural question is to obtain estimates on the time marginals of the solution. The first such estimates were obtained in a purely analytical setting by Aronson in [Aro67], in which he considers an operator L in divergence form

$$Lf(x) = \frac{1}{2}\mathrm{div}\,(a(x)\nabla f(x)) + b(x) \cdot \nabla f(x), \qquad f \in C_0^\infty(\mathbb{R}^d), \tag{1.3.1}$$

with a, b bounded measurable and a uniformly elliptic (i.e. $\exists \Lambda \leq 1 : \forall (x, \xi) \in (\mathbb{R}^d)^2, \Lambda^{-1}|\xi|^2 \leq a(x)\xi \cdot \xi \leq \Lambda|\xi|^2$) and obtains bounds on the fundamental solution p of the associated Cauchy problem $(\partial_t + L)f = 0$ with a Dirac mass as terminal condition. Namely, for all $T > 0$, there exists constants $C > 1$ depending on $\Lambda, T, \|b\|_\infty$ and $c \in (0, 1]$ depending on Λ such that if p is the fundamental solution associated with L, for all $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\frac{C^{-1}}{t^{\frac{d}{2}}} \exp\left[-c^{-1}\frac{|y - x|^2}{t}\right] \leq p(t, x, y) \leq \frac{C}{t^{\frac{d}{2}}} \exp\left[-c\frac{|y - x|^2}{t}\right]. \tag{1.3.2}$$

This expresses the fact that the fundamental solution associated with L is controlled from above and below by the fundamental solutions of constant coefficients equations (i.e. the rescaled densities of the noise in probabilistic terms). This type of bounds will be refered to as Aronson bounds. We will also be interested in obtaining estimates on the derivatives of p and their Hölder modulus in space and time.

For stable noises, which are the main topic of interest in this thesis, the seminal work of Kolokoltsov [Kol00] addresses the case of a stable-driven SDE with smooth bounded drift where the driving noise has a Lévy measure which is equivalent to the isotropic α-stable measure, with smooth density. This work was extended in various directions for stable-driven SDEs for which weak well-posedness can be obtained. Kulik proved continuity of the density as well as its time derivative for a Hölder drift in [Kul19] and [KK18] for a rotationnaly invariant α-stable noise using the parametrix method. In [MZ22], authors cover the whole range $\alpha \in (0,2)$ with Hölder unbounded drift using flow techniques to account for the unboundedness of the drift.

In the scope of distributional drift heat kernels estimates, the sole result we were able to gather is due to Perkowski and van Zuijlen, [PvZ22]. In a Brownian setting, using the Littlewood-Paley characterization of Besov spaces, they managed to derive two-sided gaussian heat kernel estimates of type (1.3.2) in a time-inhomogeneous setting with time-continuous drift in $\mathbb{B}^{\beta}_{\infty,1}, \beta > -\frac{1}{2}$. They also derive gradient estimates w.r.t. the backward variable. The constants therein explicitly depend on the Littlewood-Paley decomposition of the drift.

In Chapter 3, we consider an SDE with additive strictly stable noise (although we expect the results to hold with multiplicative noise) which has a time-inhomogeneous distributional drift in $L^r - \mathbb{B}^{\beta}_{p,q}$, and we compute heat kernel estimates for its density. To this end, we rely on its first order parametrix expansion (i.e. the Duhamel formula) and on a proper mollification of the drift. We prove the following result:

Theorem 1.1 (Heat kernel estimates for Besov SDE with stable noise)**.** *Fix the parameters $T > 0$ and $\Theta := \{\alpha, d, \beta, r, p, q, \|b\|_{L^r - \mathbb{B}^{\beta}_{p,q}}\}$. Take $b \in L^r([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d))$ and assume $\gamma := \alpha - 1 + 2\beta - \alpha/q - d/p > 0$. Consider the solution $\mathbb{P}$ to the martingale problem associated with $(b, \mathcal{L}^{\alpha}, x)$ starting at time s and denote $(x_t)_{t\in[s,T]}$ the associated canonical process. For all $t \in (s,T]$, x_t admits a density $\Gamma(s,x,t,\cdot)$ such that there exists $C := C(T,\Theta,\rho) \geq 1$ such that for all $y \in \mathbb{R}^d$,*

$$C^{-1} p_\alpha(t-s, y-x) \leq \Gamma(s,x,t,y) \leq C p_\alpha(t-s, y-x), \tag{1.3.3}$$

$$|\nabla_x \Gamma(s,x,t,y)| \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}} p_\alpha(t-s, y-x), \tag{1.3.4}$$

$$\forall (y,y') \in \mathbb{R}^d, \qquad |\Gamma(s,x,t,y) - \Gamma(s,x,t,y')| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}} \left(p_\alpha(t-s, y-x) + p_\alpha(t-s, y'-x)\right), \tag{1.3.5}$$

$$\forall (y,y') \in \mathbb{R}^d, \qquad |\nabla_x \Gamma(s,x,t,y) - \nabla_x \Gamma(s,x,t,y)| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \left(p_\alpha(t-s, x-y) + p_\alpha(t-s, x-y')\right), \tag{1.3.6}$$

for any $\rho \in (-\beta, -\beta + \gamma/2)$, where $\gamma := 2\beta + \alpha - 1 - \frac{\alpha}{r} - \frac{d}{p} > 0$ is the "gap to singularity".

Moreover, for any $\varepsilon \in (0, -\beta)$ and $\rho \in (-\beta, -\beta + \varepsilon/2)$, for $t' > t$ such that $(t'-t) < t/2$,

$$\left\| \frac{\Gamma(s,x,t,\cdot) - \Gamma(s,x,t',\cdot)}{p_\alpha(t-s, \cdot - x)} \right\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \leq C \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon-\beta}{\alpha}}} \tag{1.3.7}$$

Note that (1.3.7) was actually not present in [Fit23] but is included in the associated Chapter 3 in the current work as it follows from the same proof and the same procedure.

The lower bound in (1.3.3) is rather straightforward to obtain in the strictly stable setting because the polynomial nature of the bounds allows to obtain the lower bound from the same procedure as the upper bound. When dealing with a Brownian noise, chaining techniques have to be used to account for the exponential tails of the gaussian distribution in order to have related variances for the upper and lower bounds (see e.g. the monograph of Bass [Bas98]).

In the Lebesgue setting, taking the limit of the discretization scheme, we obtain the following in Chapter 4 ([FJM24] for the published version):

Theorem 1.2 (Weak existence and density estimates for Lebesgue drift SDEs with strictly stable noise)**.** *Assume $b \in L^q - L^p$ and $\alpha - 1 - d/p - \alpha/q$. The stochastic differential equation* (1.1.4) *admits a weak* *solution such that for each $t \in (0,T]$, X_t admits a density $y \mapsto \Gamma(0,t,x,y)$ w.r.t. the Lebesgue measure such that $\exists C := C(b,T) < \infty : \forall t \in (0,T], \forall (x,y) \in (\mathbb{R}^d)^2$,*

$$\Gamma(0,x,t,y) \leq C p_\alpha(t, y-x), \tag{1.3.8}$$

and this density is the unique solution to the following Duhamel representation among functions of $(t,y) \in [0,T] \times \mathbb{R}^d$ satisfying (1.3.8):

$$\forall t \in (0,T],\ \forall y \in \mathbb{R}^d,\ \Gamma(0,x,t,y) = p_\alpha(t,y-x) - \int_0^t \int_{\mathbb{R}^d} \Gamma(0,x,r,z) b(r,z).\nabla_y p_\alpha(t-r, y-z)\,\mathrm{d}z\,\mathrm{d}r. \tag{1.3.9}$$

Furthermore, there exists a unique solution to the martingale problem related to $b \cdot \nabla + \mathcal{L}^\alpha$ starting from x at time 0 in the sense of Definition 1.2.3.

Finally, let us define the "gap to singularity" as

$$\gamma := \alpha - 1 - \Big(\frac{d}{p} + \frac{\alpha}{q}\Big) > 0. \tag{1.3.10}$$

Then, Γ has the following regularity in the forward spatial variable: $\forall t \in (0,T], \forall (x,y,y') \in (\mathbb{R}^d)^3$,

$$|\Gamma(0,x,t,y) - \Gamma(0,x,t,y')| \leq C \frac{|y-y'|^\gamma \wedge t^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}}} \left(p_\alpha(t,y-x) + p_\alpha(t,y'-x) \right). \tag{1.3.11}$$

Those two results once again highlight the fundamental role of the gap to singularity.

1.4 Discretization schemes for singular SDEs

In this section, we are interested in the discretization of the previously studied SDEs. In order to make a panorama of the associated literature, we will for a moment consider a more general setting which includes non-trivial diffusion coefficients. Namely, for a driving process Z with stability index $\alpha > 1$, we consider dynamics of the form

$$\mathrm{d}X_t = b(t,X_t)\,\mathrm{d}t + \sigma(t, X_{t^-})\,\mathrm{d}Z_t, \tag{1.4.1}$$

and the X_t^h the following discretization

$$X^h_{t_{k+1}} = X^h_{t_k} + hb(t_k, X^h_{t_k}) + \sigma(t_k, X^h_{t_k})(Z_{t_{k+1}} - Z_{t_k}), \tag{1.4.2}$$

provided those equations are well defined. There are mainly two types of error in the literature. The strong error focuses on the difference between the trajectory of a strong solution and a given discretization. In this setting, we are interested in obtaining almost sure estimates or L^p averaged estimates. We are interested in quantifying the convergence of

$$\sup_{t\in(0,T]} |X_t - X_t^h| \qquad \text{or} \qquad \mathbb{E}\left[\sup_{t\in(0,T]} |X_t - X_t^h|^p\right]^{\frac{1}{p}} \tag{1.4.3}$$

for suitable exponents $p \geq 1$. In the case of smooth coefficients and non-trivial diffusion coefficient, the strong error rate in the Brownian setting is $1/2$.

On the other hand, the weak error is defined as the difference between the expectation of a functional of the process and its discretization. In this thesis, we will consider weak errors of the form

$$\mathcal{E}(f,t,x,h) := \mathbb{E}_{0,x}\left[f(X_t^h) - f(X_t)\right], \tag{1.4.4}$$

for f belonging to a suitable class of test functions, and where the meaning of the expectation subscript is $\mathbb{E}_{0,x}[\cdot] := \mathbb{E}[\cdot | X_0^h = X_0 = x]$. In our results, we will focus on the case of a Dirac test function, meaning that we study directly the difference between the density of the solution to the SDE and the density of the discretization, provided they do exist.

Deriving convergence results for the weak error in the chosen Markovian setting involves studying the PDE

$$(\partial_s + b(s,x)\cdot\nabla + \mathcal{L})\, u(s,x) = 0 \text{ on } [0,T)\times\mathbb{R}^d, \qquad u(T,\cdot) = f \text{ on } \mathbb{R}^d, \tag{1.4.5}$$

where $\mathcal{L}$ is the generator of the martingale part of the equation (which is $\mathcal{L}^\alpha$ whenever we work with a trivial diffusion coefficient). When the coefficients of (1.1.4) and the test function f are smooth (and, importantly, with a non-trivial diffusion coefficient), the seminal paper of Talay and Tubaro [TT90] gives a convergence rate of order 1 in h in the Brownian case. Let us also mention the work of Protter and Talay [PT97], which deals with an SDE with no drift and a non-trivial diffusion coefficient. Assuming that the general Lévy noise Z that they consider has either finite high order moments or bounded jumps, they prove a convergence rate of order 1 in h for the standard Euler scheme. Similar results were obtained for the densities in [KM02] and [KM10] respectively in the Brownian and pure-jump stable settings. With β-Hölder coefficients and again a smooth f in (1.4.4), the work of Mikulevicius and Platen [MP91] proves a convergence in $h^{\frac{\beta}{2}}$ in the Brownian case. This result was extended to densities by Konakov and Mammen in [KM17]. In these works, when applying Itô's formula, authors use the regularity of the coefficients to treat terms of the form

$$[b(r,X_r^h) - b(t_k, X_{t_k}^h)]\nabla u(r, X_r^h), r\in(t_k,t_{k+1})$$

and

$$\mathrm{Tr}\left[[\sigma\sigma^*(r,X_r^h) - \sigma\sigma^*(t_k,X_{t_k}^h)]\nabla^2 u(r,X_r^h)\right].$$

Namely, for u solving (1.4.5) with smooth terminal condition f, applying Itô's formula, the error writes, considering an additive noise for simplicity, with $\tau_s^h := h\lfloor s/h\rfloor$,

$$\begin{aligned}\mathcal{E}(f,t,x,h) &= \mathbb{E}_{0,x}[f(X_t^h) - f(X_t)] = \mathbb{E}_{0,x}[u(t,X_t^h) - u(0,x)]\\ &= \mathbb{E}_{0,x}\left[\int_0^t \left(b(r,X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h)\right)\cdot\nabla u(r,X_r^h)\,\mathrm{d}r\right].\end{aligned}$$

The authors then use classic Schauder type estimates, see e.g. [Fri64], to control $\|\nabla u\|_{L^\infty}$ (and $\|\nabla^2 u\|_{L^\infty}$ whenever we have multiplicative noise). From the β-Hölder continuity of the drift, the following bound is then derived

$$|\mathcal{E}(f,t,x,h)| \le C\|\nabla u\|_{L^\infty}\int_0^t \mathbb{E}_{0,x}\left[|X_r^h - X_{\tau_r^h}^h|^\beta\right]\,\mathrm{d}r \le C\|\nabla u\|_{L^\infty} h^{\frac{\beta}{2}}. \tag{1.4.6}$$

The above final rate then comes from the magnitude of the increment of the Euler scheme on one time step in the $L^\beta(\mathbb{P})$ norm. However, one can see that this essentially consists in using strong error analysis techniques to derive a weak error rate, which does not necessarily seem adequate. We insist that all the previously quoted results are for an SDE with multiplicative noise which is as well β-Hölder continuous in space. In that setting, the results are believed to be sharp. However, for an additive noise as in (1.1.4), one of the main contribution of this thesis is that the rate can be significantly improved by exploiting the full parabolic bootstrap associated with the PDE (1.4.5).

In the current setting of singular drift additive SDEs driven by a stable noise, it appears that the most reasonable scheme to use is a standard Euler scheme as defined in (1.4.2), as opposed to considering higher-order schemes with a regularized drift, which would lead to additional numerical context-dependent issues. In the scope of the singular drift weak error, we are led to average the results over a large number M of simulated trajectories to compute $\mathbb{E}[f(X_t^h)]$. From a simulation viewpoint, we are thus also limited by the asymptotic behavior in M of this quantity when computing Monte-Carlo estimations. For Brownian-driven SDEs, a central limit theorem applies, provided finiteness of the second order moment of the solution. When the noise is an α-stable process, $\alpha < 2$, the solution does not have a moment of second order. Anyhow, a stable central limit theorem still applies, although with a lower rate of convergence $1 - 1/\alpha$ (see [Zol86]).

In the (possibly fractional) Brownian singular setting, one way to derive strong or weak results is to use the stochastic sewing lemma introduced in [Lê20], which allows to quantify the discretization error along rough functionals of the (fractional) Brownian path. The main contribution of the sewing lemma consists in bounding L^r norms of the form

$$\mathbb{E}\left[\left|\int_0^t b(s, X_s^h) - b(s, X_{\tau_s^h}^h)\,\mathrm{d}s\right|^r\right], \tag{1.4.7}$$

that is, the strong error associated with local differences of the path along an irregular function with suitable integrability properties.

Importantly, as this approach does not rely on the underlying PDE, it also works for SDEs driven by an additive fractional Brownian motion (i.e. in a *non-Markovian* setting), as was done by Gerencsér, Dareiotis and Butkovsky in [BDG21]. Therein, authors derive a strong error rate of almost $1/2 + \beta H$, where H is the Hurst parameter of the noise. In the specific case of a β-Hölder continuous drift and terminal condition f, in the work [Hol24], the author improves the convergence rate from [MP91] to $h^{\frac{\beta+1}{2}-\varepsilon}$, $\varepsilon > 0$, still using the stochastic sewing lemma. We extended this result to the pure-jump setting $\alpha \in (1, 2]$ and to a more general class of test functions by working on densities, achieving as well $\varepsilon = 0$ in [FM24]. Let us also mention the work [LL21], which proves a *strong* (i.e. on trajectories) rate of convergence of order 1/2 (up to a logarithmic factor) in the Brownian setting for $L^q - L^p$ drifts under the Krylov-Röckner type condition $d/p + 2/q < 1$. However, the use of stochastic sewing techniques still does not allow to take advantage of the parabolic bootstrap associated with the fundamental solution of (1.4.5) when the test function is rough, e.g. Dirac masses leading to the weak error on densities.

In our works, we precisely focus on these types of errors of the form $\mathcal{E}(\delta_y, t, x, h)$ (where δ_y is the Dirac mass at point y). From Itô's formula, (1.4.4) and (1.4.5), this formally writes

$$\mathcal{E}(\delta_y, t, x, h) = \mathbb{E}_{0,x}\left[\int_0^t \Big(b(r, X_r^h) - b(U_{\tau_r^h/h}, X_{\tau_r^h}^h))\Big) \cdot \nabla_z \Gamma(r, z, t, y)|_{z = X_r^h}\,\mathrm{d}r\right]. \tag{1.4.8}$$

To analyze the corresponding error, a new idea was introduced in [BJ22]. The drift was therein assumed to be merely measurable and bounded so that no rate could be *a priori* derived from the difference in (1.4.8). The point then consists in using the regularity of the solution to (1.4.5) instead of that of b. Writing

$$\begin{aligned}\mathbb{E}_{0,x}[b(r, X_r^h) \cdot \nabla\Gamma(r, X_r^h, t, y) - b(r, X_{\tau_r^h}^h) \cdot \nabla\Gamma(r, X_{\tau_r^h}^h, t, y)]&\\ = \int [\Gamma^h(0, x, r, z) - \Gamma^h(0, x, \tau_r^h, z)] b(r, z) \cdot \nabla\Gamma(r, z, t, y)\,\mathrm{d}z&\end{aligned} \tag{1.4.9}$$

one can exploit some additional (or-bootstrapped) regularity of Γ^h in its forward time variable. Namely, it was proved in the Brownian setting of [BJ22] that for a bounded drift, this regularity was of order 1/2. This technique is robust enough to be adapted to a vast range of singular drifts (Lebesgue, Hölder, Besov) and provides a significant improvement, corresponding to the expected regularity deriving from the parabolic bootstrap in the forward variable, even in the Hölder case when some regularity is available on the drift. Indeed, it appears that in those three settings, the forward time regularity of Γ^h (or Γ) is of order γ/α, where γ is what we call the "gap to singularity" and changes depending on the context:

$$\begin{aligned}\text{Holder} \qquad & \gamma = \alpha - 1 + \beta,\\ \text{Lebesgue} \qquad & \gamma = \alpha - 1 - \frac{d}{p} - \frac{\alpha}{q},\\ \text{Besov} \qquad & \gamma = \alpha - 1 + 2\beta - \frac{d}{p} - \frac{\alpha}{q}.\end{aligned}$$

Note that in the Hölder and Lebesgue cases, the gap to singuarity reads as the sum of $\alpha - 1$ and the regularity of the drift (β in the Hölder case and $-d/p - \alpha/q$ in the Lebesgue case). This is still the case in the Besov setting (up to an additional β, which is specific to the distributional setting), highlighting the continuity in our methods and results. Let us as well mention that for suitable and smooth test functions for the weak error in the kinetic case, Hao, Lê and Ling are able to derive a rate of order 1/2 through iterated Duhamel expansions in [HLL24]. This approach would also extend to the current non-degenerate case.

Note as well that, if estimates on the time regularity of the density Γ are already available, like in the Besov case where we prove these estimates separately in [Fit23] or in the Hölder case where they already existed, they can be used to compute weak error rates using the aforementioned method. Otherwise, it is possible to work with the density Γ^h of the discretized equation instead. By passing to the limit, as we do in the Lebesgue setting, we then obtain well-posedness of the SDE as well as heat kernel estimates on its density.

On the other hand, we also have to account for terms involving $b(r, X_r) - b(\tau_r^h, X_r)$. One way to achieve the expected convergence rate is to make strong assumptions on the time regularity of b: we would need $b(\cdot, z)$ to be γ/α-Hölder to match the rate that is achieved with other error terms (with, again, γ depending on the context). However, it is possible to handle those terms without any assumptions on the drift's time regularity by randomizing the time argument in the discretization schemes. This allows for a convenient use of the Fubini theorem in the error analysis (see (4.2.10) below). This averaging procedure can somehow be seen as well as a regularization by noise phenomenon.

From the above techniques (forward time regularity of Γ^h or Γ and time randomization), a rate of order $\frac{1-\left(\frac{d}{p}+\frac{\alpha}{q}\right)}{2} > 0$ is derived in [JM24a] in the Brownian setting, for a Lebesgue drift in $L^q - L^p$ for the difference of the densities $\Gamma - \Gamma^h$. In [FJM24], we extended this result to the strictly stable setting, with (Z_t) being a symmetric non-degenerate d-dimensional α-stable process with spectral measure equivalent to the Lebesgue measure on the sphere $\mathbb{S}^{d-1}$ with a smooth density.

To present our result, let us first introduce the scheme that we used. Since we consider a potentially unbounded drift coefficient, it is natural to introduce a cutoff for the discretization scheme. For a time step size h, the cutoff we consider is the following:

- If $p = q = \infty$, we simply take, for almost all $(t, y) \in [0, T] \times \mathbb{R}^d, b_h(t, y) = b(t, y)$.
- Otherwise, we set

$$b_h(t, y) := \frac{\min\left\{|b(t, y)|, Bh^{-\frac{d}{\alpha p} - \frac{1}{q}}\right\}}{|b(t, y)|} b(t, y) \mathbb{1}_{|b(t,y)|>0}, \qquad (t, y) \in [0, T] \times \mathbb{R}^d, \tag{1.4.10}$$

  for some constant $B > 0$ which can be chosen freely as long as it does not depend on h nor T.

The idea behind this cutoff level is to make sure the contribution of the drift does not dominate over that of the stable noise on each time step of the scheme. Note that, as such, the cutoffed drift might not be defined for all starting points x but only for *almost* all x. This issue disappears after one time step since the driving noise introduces a density. To bypass this issue, one can set the drift to zero on the first time step without impacting our results.

We then define a discretization scheme with n time steps over $[0, T]$, with constant step size $h := T/n$. We recall that, $\forall k \in \{1, ..., n\}, t_k := kh$ and $\forall s > 0, \tau_s^h := h\lfloor \frac{s}{h} \rfloor \in (s - h, s]$, which is the last grid point before time s. Namely, if $s \in [t_k, t_{k+1}), \tau_s^h = t_k$.

In order to avoid assumptions on the drift b beyond integrability and measurability, we are led to randomize the evaluations of b_h in the time variable. For each $k \in \{0, ..., n-1\}$, we will draw a random variable U_k according to the uniform law on $[t_k, t_{k+1}]$, independently of each other and the noise $(Z_t)_{t\geq 0}$. We can then define a step of the Euler scheme as

$$X_{t_{k+1}}^h = X_{t_k}^h + hb_h(U_k, X_{t_k}^h) + (Z_{t_{k+1}} - Z_{t_k}), \tag{1.4.11}$$

and its time interpolation as the solution to

$$\mathrm{d}X_t^h = b_h(U_{\lfloor \frac{t}{h} \rfloor}, X_{\tau_t^h}^h) \,\mathrm{d}t + \mathrm{d}Z_t. \tag{1.4.12}$$

As b_h is bounded, the scheme (1.4.12) is well defined and admits a density in positive times. We will denote by $\Gamma^h(0, x, t, \cdot)$ this density at time $t \in (0, T]$ when starting from x at time 0. Recall that Γ denotes the density of the solution to the SDE.

Theorem 1.3 (Convergence Rate for the stable-driven Euler-Maruyama scheme with $L^q - L^p$ drift)**.** *Assume that $\gamma := \alpha - 1 - d/p - \alpha/q > 0$. There exists a constant $C < \infty$ s.t. for all $h = T/n$ with $n \in \mathbb{N}^*$, and all $t \in (0,T]$, $(x,y) \in (\mathbb{R}^d)^2$*

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \le Ch^{\frac{\gamma}{\alpha}} p_\alpha(t, y-x).$$

Comparing this rate to that of [LL21], although $1/\alpha$ is lost due to the gradient in (1.4.9) (time singularity induced by the gradient of the density of the noise), one sees that the convergence rate displays explicitly the "gap to singularity" $\alpha - 1 - (d/p + \alpha/q)$ or Serrin condition in that setting (critical stable parabolic scaling in Lebesgue spaces).

We later proved a similar result for Hölder bounded drifts in [FM24], covering at the same time the Brownian setting and the case of a symmetric isotropic d-dimensional α-stable driving noise. In this work, the scheme is defined in the following way: we again use a discretization scheme with n time steps over $[0,T]$, with constant step size $h := T/n$. We define a step of the Euler scheme, starting from $X_0 = x$, as

$$X^h_{t_{k+1}} = X^h_{t_k} + hb(U_k, X^h_{t_k}) + (Z_{t_{k+1}} - Z_{t_k}),\ k \in \mathbb{N}, \tag{1.4.13}$$

where the $(U_k)_{k\in\mathbb{N}}$ are, again, independent random variables, independent as well from the driving noise, s.t. $U_k \overset{(\text{law})}{=} \mathcal{U}([t_k, t_{k+1}])$, i.e. U_k is uniform on the time interval $[t_k, t_{k+1}]$. We consider the corresponding time interpolation defined as the solution to

$$\mathrm{d}X^h_t = b(U_{\tau^h_t/h}, X^h_{\tau^h_t})\,\mathrm{d}t + \mathrm{d}Z_t. \tag{1.4.14}$$

Again, as b is bounded, equation (1.4.14) is well-posed and X^h_t admits a density for $t > 0$. We will denote by $\Gamma^h(0,x,t,\cdot)$ this density at time $t \in (0,T]$ when starting from x at time 0. We obtain the following result:

Theorem 1.4 (Convergence Rate for the stable-driven Euler scheme with $L^\infty_t \mathcal{C}^\beta_x$ drift)**.** *Denoting by Γ and Γ^h the respective densities of the SDE* (1.1.4) *and its Euler scheme defined in* (1.4.13)*, there exists a constant $C := C(d,b,\alpha,T) < \infty$ s.t. for all $h = T/n$ with $n \in \mathbb{N}^*$, and all $t \in (0,T]$, $x,y \in \mathbb{R}^d$,*

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \le C\big(1 + t^{-\frac{\beta}{\alpha}}\big)h^{\frac{\gamma}{\alpha}} p_\alpha(t, y-x), \tag{1.4.15}$$

where $\gamma = \beta + \alpha - 1 > 0$ is again the "gap to singularity".

Let us mention that if one is interested in the weak error for some test function f, $\mathcal{E}(f,x,t,h) := \mathbb{E}_{0,x}[f(X^h_t) - f(X_t)]$, as soon as f is $\delta \in [\beta, 1]$-Hölder (not necessarily bounded) then, a rate can be derived as a consequence of the convergence of $|\Gamma(s,x,t,y) - \Gamma^h(s,x,t,y)|$ using a simple cancellation argument:

$$\mathcal{E}(f,x,t,h) = \int_{\mathbb{R}^d} (\Gamma^h - \Gamma)(0,x,t,y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} (\Gamma^h - \Gamma)(0,x,t,y)\big(f(y) - f(x)\big)\,\mathrm{d}y,$$
$$|\mathcal{E}(f,x,t,h)| \lesssim h^{\frac{\gamma}{\alpha}}\big(1 + t^{-\frac{\beta}{\alpha}}\big) \int_{\mathbb{R}^d} p_\alpha(t,y-x)|x-y|^\delta\,\mathrm{d}y \lesssim \big(1 + t^{-\frac{\beta}{\alpha}}\big)t^{\frac{\delta}{\alpha}} h^{\frac{\gamma}{\alpha}}.$$

Precisely, the smoothness of f allows to absorb the time-singularity from (1.4.15) in small time.

Importantly, when comparing the Hölder rate to the Lebesgue one, if we interpret $-\left(\frac{d}{p} + \frac{\alpha}{q}\right)$ in the latter as the regularity loss[1], there is continuity of the rate of the convergence w.r.t. the regularity of the drift. Continuity w.r.t. the stability index α also holds when comparing Theorem 1.4 to the results in [Hol24] (and getting rid of the ε in the rate therein), thus extending the former to a more general class of test functions and noises.

[1]actually this exponent naturally appears as the negative regularity parameter when embedding the time-space Lebesgue space in a Besov space with infinite integrability indexes (which can be identified with a usual Hölder space when the regularity index is positive), see e.g. [Saw18].

To introduce the scheme associated with the formal Besov-drift SDE, one first needs to recall that the precise meaning to be given to the SDE, following [CdRJM22] in the pure-jump setting, inspired by [DD16] in the Brownian setting, is:

$$X_t = x + \int_0^t \mathfrak{b}(s, X_s,\, \mathrm{d}s) + Z_t, \tag{1.4.16}$$

where for all $(s,z) \in [0,T] \times \mathbb{R}^d, h > 0$,

$$\mathfrak{b}(s,z,h) := \int_s^{s+h} \int b(u,y) p_\alpha(u-s, z-y)\, \mathrm{d}y\, \mathrm{d}u = \int_s^{s+h} P^\alpha_{u-s} b(u,z)\, \mathrm{d}u, \tag{1.4.17}$$

$p_\alpha(v,\cdot)$ denoting the density of the α-stable driving noise $(Z_v)_{v\geq 0}$ at time v and P^α the associated semi-group. For this result, we will assume that (Z_t) is a symmetric isotropic d-dimensionnal α-stable process whose spectral measure is equivalent to that of the Lebesgue measure on the sphere $\mathbb{S}^{d-1}$ with smooth density. The integral in (1.4.16) is intended as a nonlinear Young integral obtained by passing to the limit in a suitable procedure aimed at reconstructing the drift (see again [CdRJM22]). The resulting drift in (1.4.16) is, *per se*, a Dirichlet process (as it had already been indicated in the literature, see e.g. [ABM20] and references therein). Importantly, the dynamics in (1.4.16) also naturally provides a corresponding approximation scheme to be analyzed. Note that, in order to give a precise meaning to the integral appearing in (1.4.16), we need the following condition:

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2 \right) \qquad \beta \in \left(\frac{1-\alpha+\frac{2d}{p}+\frac{2\alpha}{r}}{2}, 0 \right), \tag{1.4.18}$$

which is more stringent than that associated with well-posedness. Interestingly enough, this condition does not appear elsewhere in the analysis since we only consider the time marginals of the process.

We can now define the related Euler scheme X^h, starting from $X^h_0 = x$, on the time grid as

$$X^h_{t_{i+1}} = X^h_{t_i} + \mathfrak{b}(t_i, X^h_{t_i}, h) + Z_{t_{i+1}} - Z_{t_i}. \tag{1.4.19}$$

We have precisely used the quantity $\mathfrak{b}(t_i, X^h_{t_i}, h)$ defined in (1.4.17) as an approximation of the nonlinear Young integral $\int_{t_i}^{t_{i+1}} \mathfrak{b}(s, X^h_s,\, \mathrm{d}s)$, which served to define the limit dynamics (1.4.16) for the SDE, with a time argument corresponding to the chosen time step.

We extend the dynamics of the scheme in continuous time as follows

$$X^h_t = X^h_{\tau^h_t} + \mathfrak{b}(\tau^h_t, X^h_{\tau^h_t}, t-\tau^h_t) + Z_t - Z_{\tau^h_t}. \tag{1.4.20}$$

For this scheme, we proved the following result:

Theorem 1.5 (Convergence Rate for the stable-driven Euler scheme with Besov drift)**.** *Assume that*

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2 \right) \qquad \beta \in \left(\frac{1-\alpha+\frac{d}{p}+\frac{\alpha}{r}}{2}, 0 \right)$$

and let the drift b be an element of $L^r\left([0,T], \mathbb{B}^\beta_{p,q}(\mathbb{R}^d,\mathbb{R}^d)\right)$ for some $r \in [1,\infty]$.

Denoting by Γ and Γ^h the respective densities of the SDE (1.1.4) *and its Euler scheme defined in* (1.4.20)*, for all $\varepsilon > 0, \rho > -\beta$ there exists a constant $C := C(d,b,\alpha,T,\varepsilon,\rho) < \infty$ s.t. for all $h = T/n$ with $n \in \mathbb{N}^*$, and all $t \in (0,T]$, $(x,y) \in (\mathbb{R}^d)^2$,*

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \leq C h^{\frac{\gamma-\varepsilon}{\alpha}} p_\alpha(t, y-x), \tag{1.4.21}$$

where $\gamma = \alpha + 2\beta - \frac{d}{p} - \frac{\alpha}{r} - 1 > 0$ is the "gap to singularity" in the Besov case.

The layout of this manuscript is as follows: in Chapter 2, we give a summary of the main tools linked to the stable distribution and related estimates in Lebesgue and Besov spaces, whose properties are also detailed therein. In Chapter 3, we show heat kernel estimates for stable-driven SDEs with distributional drift (see Theorem 1.1). In Chapter 4, we show the weak convergence rate of Euler schemes for additive SDEs with Hölder drift of Theorem 1.4. In Chapter 5, we prove well-posedness of an SDE with Lebesgue drift and stable additive noise, study the associated weak discretization (see Theorem 1.3) and provide heat kernel estimates for the underlying density. Finally, in Chapter 6, we define an Euler scheme for distributional SDEs of the type studied in Chapter 3 and provide the associated weak error rate appearing in Theorem 1.5.

Chapter 2

Technical tools

In this Chapter, we collect some of the main tools which are common to most of the upcoming works.

2.1 Definition and basic properties of Besov spaces

We first recall that denoting by $\mathcal{S}'(\mathbb{R}^d)$ the dual space of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, for $\ell, m \in (0,+\infty]$, $\vartheta \in \mathbb{R}$, the Besov space $\mathbb{B}^{\vartheta}_{\ell,m}$ can be characterized with

$$\mathbb{B}^{\vartheta}_{\ell,m} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f\|_{\mathbb{B}^{\vartheta}_{\ell,m}} := \|\mathcal{F}^{-1}(\phi\mathcal{F}(f))\|_{L^\ell} + \mathcal{T}^{\vartheta}_{\ell,m}(f) < \infty \right\},$$

$$\mathcal{T}^{\vartheta}_{\ell,m}(f) := \begin{cases} \left(\int_0^1 \frac{\mathrm{d}v}{v} v^{(n-\vartheta/\alpha)m} \|\partial_v^n \tilde{p}_\alpha(v,\cdot) \star f\|^m_{L^\ell} \right)^{\frac{1}{m}} & \text{for } 1 \le m < \infty, \\ \sup_{v \in (0,1]} \left\{ v^{(n-\vartheta/\alpha)} \|\partial_v^n \tilde{p}_\alpha(v,\cdot) \star f\|_{L^\ell} \right\} & \text{for } m = \infty, \end{cases} \tag{2.1.1}$$

with $\star$ denoting the spatial convolution, n being any non-negative integer (strictly) greater than ϑ/α, the function ϕ being a $\mathcal{C}^\infty_0$-function (infinitely differentiable function with compact support) such that $\phi(0) \neq 0$, and $\tilde{p}_\alpha(v,\cdot)$ denoting the density function at time v of the d-dimensional isotropic stable process.

For our analysis we will rely on the following important inequalities:

- Product rule: for all $\vartheta \in \mathbb{R}$, $(\ell, m) \in [1,+\infty]^2$ and $\rho > \max\left(\vartheta, d\left(\frac{1}{\ell} - 1\right)_+ - \vartheta\right)$, $\forall (f,g) \in \mathbb{B}^{\rho}_{\infty,\infty} \times \mathbb{B}^{\vartheta}_{\ell,m}$,

$$\|f \cdot g\|_{\mathbb{B}^{\vartheta}_{\ell,m}} \le \|f\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|g\|_{\mathbb{B}^{\vartheta}_{\ell,m}}. \tag{2.1.2}$$

  See Theorem 4.37 in [Saw18] for a proof.

- Duality inequality: for all $\vartheta \in \mathbb{R}$, $(\ell, m) \in [1,+\infty]^2$, with m' and ℓ' respective conjugates of m and ℓ, and $(f,g) \in \mathbb{B}^{\vartheta}_{\ell,m} \times \mathbb{B}^{-\vartheta}_{\ell',m'}$,

$$\left| \int f(y) g(y) \mathrm{d}y \right| \le \|f\|_{\mathbb{B}^{\vartheta}_{\ell,m}} \|g\|_{\mathbb{B}^{-\vartheta}_{\ell',m'}}. \tag{2.1.3}$$

  See Proposition 6.6 in [LR02] for a proof.

- Young inequality: for all $\vartheta \in \mathbb{R}$, $(\ell, m) \in [1,+\infty]^2$, for any $\delta \in \mathbb{R}$ and for $(\ell_1, \ell_2) \in [1,\infty]^2$ and $(m_1, m_2) \in (0,\infty]^2$ such that

$$1 + \frac{1}{\ell} = \frac{1}{\ell_1} + \frac{1}{\ell_2} \qquad \text{and} \qquad \frac{1}{m} \le \frac{1}{m_1} + \frac{1}{m_2},$$

there exists C such that, for $f \in \mathbb{B}^{\vartheta-\delta}_{\ell_1,m_1}$ and $g \in \mathbb{B}^{\delta}_{\ell_2,m_2}$,

$$\|f \star g\|_{\mathbb{B}^{\vartheta}_{\ell,m}} \le C\|f\|_{\mathbb{B}^{\vartheta-\delta}_{\ell_1,m_1}} \|g\|_{\mathbb{B}^{\delta}_{\ell_2,m_2}}. \tag{2.1.4}$$

See Theorem 2.2 in [KS21] for a proof.

Proposition 2.1 (Smooth approximation of Besov functions)**.** *Let $b \in L^r - \mathbb{B}^{\vartheta}_{\ell,m}$ with $\vartheta \in (-1,0]$ and $(\ell,m) \in [1,\infty]^2$. There exists a sequence of smooth bounded time-space functions $(b^n)_{n\in\mathbb{N}}$ such that*

$$\forall \tilde\vartheta < \vartheta, \qquad \|b - b^n\|_{L^{\tilde r}-\mathbb{B}^{\tilde\vartheta}_{\ell,n}} \underset{n\to\infty}{\longrightarrow} 0 \tag{2.1.5}$$

with $\tilde r = r$ if $r < \infty$ and for any $\tilde r < \infty$ otherwise. Moreover, there exists $\kappa \ge 1$:

$$\sup_{n\in\mathbb{N}} \|b^n\|_{L^{\tilde r}-\mathbb{B}^{\beta}_{p,q}} \le \kappa\|b\|_{L^{\tilde r}-\mathbb{B}^{\vartheta}_{\ell,m}}. \tag{2.1.6}$$

2.2 Estimates for the stable kernel

Let us denote by $\mathcal{L}^\alpha$ the generator of the driving noise Z and $p_\alpha : \mathbb{R}_+\backslash\{0\} \times \mathbb{R}^d \to \mathbb{R}_+$ its density. In the case $\alpha = 2$, $\mathcal{L}^\alpha$ is the usual normalized Laplacian $\frac{1}{2}\Delta$. The noise is a Brownian Motion and its gaussian marginal densities are explicit.

When $\alpha \in (1,2)$, in whole generality, the generator of a symmetric stable process writes, $\forall \phi \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$ (smooth compactly supported functions),

$$\begin{aligned}\mathcal{L}^\alpha\phi(x) &= \text{p.v.}\int_{\mathbb{R}^d} [\phi(x+z) - \phi(x)]\,\nu(\,\mathrm{d}z)\\ &= \text{p.v.}\int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}} [\phi(x+\rho\xi) - \phi(x)]\,\mu(\,\mathrm{d}\xi)\frac{\mathrm{d}\rho}{\rho^{1+\alpha}}\end{aligned}$$

(see [Sat99] for the polar decomposition of the stable Lévy measure) where μ is a symmetric measure on the unit sphere $\mathbb{S}^{d-1}$. We will here restrict to the case where μ is symmetric and

$$C^{-1}m(\,\mathrm{d}\xi) \le \mu(\,\mathrm{d}\xi) \le Cm(\,\mathrm{d}\xi),$$

i.e. it is equivalent to the Lebesgue measure on the sphere. Indeed, in that setting Watanabe (see [Wat07], Theorem 1.5) and Kolokoltsov ([Kol00], Propositions 2.1–2.5) showed that if $C^{-1}m(\,\mathrm{d}\xi) \le \mu(\,\mathrm{d}\xi) \le Cm(\,\mathrm{d}\xi)$, the following estimates hold: there exists a constant C depending only on α, d, s.t. $\forall v \in \mathbb{R}_+\backslash\{0\}, z \in \mathbb{R}^d$,

$$C^{-1}v^{-\frac{d}{\alpha}}\left(1 + \frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \le p_\alpha(v,z) \le Cv^{-\frac{d}{\alpha}}\left(1 + \frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}. \tag{2.2.1}$$

On the other hand let us mention that the sole non-degeneracy condition $\exists\kappa \ge 1 : \forall\lambda \in \mathbb{R}^d$

$$\kappa^{-1}|\lambda|^\alpha \le \int_{\mathbb{S}^{d-1}} |\lambda\cdot\xi|^\alpha\mu(\,\mathrm{d}\xi) \le \kappa|\lambda|^\alpha,$$

does not allow to derive *global* heat kernel estimates for the noise density.

Lemma 2.1 (Pointwise estimates for the stable kernel)**.** *Let $\alpha \in (1,2]$ and $c \ge 1$. There exists a constant C depending only on α, c and the dimension d such that*

- *Derivatives: $\forall\delta \in \{0,1,2\}, k \in \mathbb{N}, x \in \mathbb{R}^d, t \in \mathbb{R}_+, c_1 > c$*

$$|\partial_t^\delta\nabla^k p_{\alpha,c}(t,x)| \le \frac{C}{t^{\delta+\frac{k}{\alpha}}}p_{\alpha,c_1}(t,x). \tag{2.2.2}$$

- *Moments:* $\forall t \in \mathbb{R}_+, \eta \in [0,\alpha)$,
$$\int_{\mathbb{R}^d} |x|^\eta p_{\alpha,c}(t,x)\,\mathrm{d}x \le \frac{C}{t^{\frac{\eta}{\alpha}}}. \tag{2.2.3}$$

- *Time Hölder regularity:* $\forall \vartheta \in (0,1], x \in \mathbb{R}^d, 0 < t \le t', \delta \in \{0,1,2\}, k \in \mathbb{N}, c_1 > c$,
$$\left|\partial_t^\delta \nabla^k p_{\alpha,c}(t,x) - \partial_{t'}^\delta \nabla^k p_{\alpha,c}(t',x)\right| \lesssim \frac{|t-t'|^\vartheta}{t^{\delta+\vartheta+\frac{k}{\alpha}}} \left(p_{\alpha,c_1}(t,x) + p_{\alpha,c_1}(t',x)\right). \tag{2.2.4}$$

- *Spatial Hölder regularity:* $\forall \vartheta \in (0,1], (x,x') \in (\mathbb{R}^d)^2, t \in \mathbb{R}_+, \delta \in \{0,1,2\}, k \in \mathbb{N}, c_1 > c$,
$$\left|\partial_t^\delta \nabla^k p_{\alpha,c}(t,x) - \partial_t^\delta \nabla^k p_{\alpha,c}(t,x')\right| \lesssim \left(\frac{|x-x'|^\vartheta}{t^{\frac{\vartheta}{\alpha}}} \wedge 1\right) \frac{1}{t^{\delta+\frac{k}{\alpha}}} \left(p_{\alpha,c_1}(t,x) + p_{\alpha,c_1}(t,x')\right). \tag{2.2.5}$$

Lemma 2.2 (Lebesgue estimates for the stable kernel)**.** *Let $\alpha \in (1,2]$ and $c \ge 1$. There exists a constant C depending only on α, c and the dimension d such that*

- *Lebesgue norm of the stable kernel:* $\forall p \ge 1, \forall t \in \mathbb{R}_+$,
$$\|p_{\alpha,c}(t,\cdot)\|_{L^{p'}} \le C t^{-\frac{d}{\alpha p}}. \tag{2.2.6}$$

- *Lebesgue norm of the convolution of stable kernels:* $\forall p \ge 1, \forall (j,k) \in (\mathbb{N})^2, \forall (r,t) \in (\mathbb{R}_+)^2$, $\forall (x,y) \in (\mathbb{R}^d)^2$, $\forall c_1 > c$,
$$\|\nabla^k p_{\alpha,c}(t,\cdot - x)\nabla^j p_{\alpha,c}(r, y-\cdot)\|_{L^{p'}} \le C \frac{1}{t^{\frac{k}{\alpha}} r^{\frac{j}{\alpha}}} \left[t^{-\frac{d}{\alpha p}} + r^{-\frac{d}{\alpha p}}\right] p_{\alpha,c_1}(t+r, y-x). \tag{2.2.7}$$

Lemma 2.3 (Besov estimates for the stable kernel)**.** *Let $\alpha \in (1,2]$ and $c \ge 1$. There exists a constant C depending only on α, c and the dimension d such that*

- *Besov norm of the stable kernel:* $\forall p \ge 1, \forall t \in \mathbb{R}_+$,
$$\|p_{\alpha,c}(t,\cdot)\|_{\mathbb{B}^\vartheta_{\ell,m}} \le C t^{-\frac{d}{\alpha p}}. \tag{2.2.8}$$

- *Besov norm of the convolution of stable kernels:* $\forall \vartheta < 0, \forall (\ell,m) \in [1,\infty]^2, \forall (j,k) \in \mathbb{N}^2, \forall (r,t) \in (\mathbb{R}_+)^2$, $\forall (x,y) \in (\mathbb{R}^d)^2$, $\forall c_1 > c$,
$$\|\nabla^k p_{\alpha,c}(t,\cdot - x)\nabla^j p_{\alpha,c}(r, y-\cdot)\|_{\mathbb{B}^\vartheta_{\ell,m}} \le C \frac{(t+r)^{\vartheta+\zeta}}{t^{\frac{k}{\alpha}} r^{\frac{j}{\alpha}}} \left[\frac{1}{t^{\frac{d}{\alpha \ell'}}} + \frac{1}{r^{\frac{d}{\alpha m'}}}\right] \left[\frac{1}{t^{\frac{\zeta}{\alpha}}} + \frac{1}{r^{\frac{\zeta}{\alpha}}}\right] p_{\alpha,c_1}(t+r, y-x). \tag{2.2.9}$$

Lemma 2.4 (Besov estimates for kernels with controlled Hölder modulus)**.** *Let $\alpha \in (1,2]$ and $c \ge 1$. Let $p : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ such that for some $\kappa > 0$ and some $\eta \in (0,1)$,*
$$\forall t \in \mathbb{R}_+, (x,x') \in \mathbb{R}^d, \quad |p(t,x) - p(t,x')| \le \kappa \left(\frac{|x-x'|^\eta}{t^{\frac{\eta}{\alpha}}} \wedge 1\right) \left(p_{\alpha,c}(t,x) + p_{\alpha,c}(t,x')\right). \tag{2.2.10}$$

Then there exists a constant C depending only on α, c and the dimension d such that for all $0 < \vartheta < \eta, (\ell,m) \in [1,\infty]^2, j \in \mathbb{N}, s < t \in (\mathbb{R}_+)^2$, $(x,y) \in (\mathbb{R}^d)^2$, $c_1 > c$, the following hold:

- *Convolution with a stable kernel*
$$\|p(s,\cdot - x)\nabla^j p_{\alpha,c}(t-s, y-\cdot)\|_{\mathbb{B}^\vartheta_{\ell,m}} \le C \frac{t^{-\frac{\vartheta}{\alpha}}}{(t-s)^{\frac{j}{\alpha}}} \left[1 + \frac{t^{\frac{\eta}{\alpha}}}{s^{\frac{\eta}{\alpha}}} + \frac{t^{\frac{\eta}{\alpha}}}{(t-s)^{\frac{\eta}{\alpha}}}\right] \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right] \bar{p}_{\alpha,c_1}(t, y-x). \tag{2.2.11}$$

- *Convolution with space sensitivity of a stable kernel. For $(y,y')\in\mathbb{R}^d$ such that $|y-y'|\le t^{\frac{1}{\alpha}}$, for all $\rho\in(\vartheta,\eta+\vartheta)$*

$$\begin{aligned}&\left\|p(s,\cdot-x)\left[\nabla^j p_{\alpha,c}(t-s,y-\cdot)-\nabla^j p_{\alpha,c}(t-s,y'-\cdot)\right]\right\|_{\mathbb{B}^{\vartheta}_{\ell,m}}\\&\quad\le C\frac{t^{-\frac{\vartheta}{\alpha}}|y-y'|^{\rho}}{(t-s)^{\frac{\rho+j}{\alpha}}}\left[1+\frac{t^{\frac{\eta}{\alpha}}}{s^{\frac{\eta}{\alpha}}}+\frac{t^{\frac{\eta}{\alpha}}}{(t-s)^{\frac{\eta}{\alpha}}}\right]\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\bar p_{\alpha,c_1}(t,y-x).\end{aligned}\tag{2.2.12}$$

- *Convolution with time sensitivity of a stable kernel. For all $0<r\le s<t$, for all $\rho\in(\vartheta,\eta+\vartheta)$,*

$$\begin{aligned}&\left\|p(s,\cdot-x)\left[\nabla^j p_{\alpha,c}(t-s,y-\cdot)-\nabla^j p_{\alpha,c}(t-r,y-\cdot)\right]\right\|_{\mathbb{B}^{\vartheta}_{\ell,m}}\\&\quad\le C\frac{t^{-\frac{\vartheta}{\alpha}}(r-s)^{\frac{\rho}{\alpha}}}{(t-s)^{\frac{\rho+j}{\alpha}}}\left[1+\frac{t^{\frac{\eta}{\alpha}}}{s^{\frac{\eta}{\alpha}}}+\frac{t^{\frac{\eta}{\alpha}}}{(t-s)^{\frac{\eta}{\alpha}}}\right]\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\bar p_{\alpha,c_1}(t,y-x).\end{aligned}\tag{2.2.13}$$

We will only prove (2.2.11) as the next two estimates follow from the same line, using as well the Hölder sensitivity in time and space of the stable kernel (see (2.2.4) and (2.2.5)).

Proof. Denote $\mathfrak{q}^{s,t}_{x,y}(\cdot):=p(s,\cdot-x)\nabla^j p_{\alpha,c}(t-s,y-\cdot)$, of which we will control the $\mathbb{B}^{\vartheta}_{\ell,m}$ norm using the thermic characterization

$$\|\mathfrak{q}^{s,t}_{x,y}\|_{\mathbb{B}^{\vartheta}_{\ell,m}}=\|\phi(D)\mathfrak{q}^{s,t}_{x,y}\|_{L^\ell}+\mathcal{T}^{\vartheta}_{\ell,m}[\mathfrak{q}^{s,t}_{x,y}].$$

Thermic part

Let us recall the definition of the thermic part and split it in two parts:

$$\begin{aligned}\mathcal{T}^{\vartheta}_{\ell,m}[\mathfrak{q}^{s,t}_{x,y}]^{\ell}&=\int_0^t\frac{\mathrm{d}v}{v}v^{(1-\frac{\vartheta}{\alpha})m}\|\partial_v p_\alpha(v,\cdot)\star\mathfrak{q}^{s,t}_{x,y}(\cdot)\|^m_{L^\ell}+\int_t^1\frac{\mathrm{d}v}{v}v^{(1-\frac{\vartheta}{\alpha})m}\|\partial_v p_\alpha(v,\cdot)\star\mathfrak{q}^{s,t}_{x,y}(\cdot)\|^m_{L^\ell}\\&=:\mathcal{T}^{\vartheta,(0,t)}_{\ell,m}[\mathfrak{q}^{s,t}_{x,y}]^m+\mathcal{T}^{\vartheta,(t,1)}_{\ell,m}[\mathfrak{q}^{s,t}_{x,y}]^m.\end{aligned}$$

For the upper part on $(t,1)$, using a L^1-L^ℓ convolution inequality, we get

$$\mathcal{T}^{\vartheta,(t,1)}_{\ell,m}[\mathfrak{q}^{s,t}_{x,y}]^m\lesssim\int_t^1\frac{\mathrm{d}v}{v}v^{(1-\frac{\vartheta}{\alpha})m}\|\partial_v p_\alpha(v,\cdot)\|^m_{L^1}\|p(s,\cdot-x)\nabla^j p_\alpha(t-s,y-\cdot)\|^m_{L^\ell}$$

Next, taking $x=x'/2$ in (2.2.10) and letting $|x|\to+\infty$, we first obtain that $p(t,x)$ goes to zero, and, in turn, taking the limit in (2.2.10) as $|x'|\to+\infty$, we have, for a fixed $x\in\mathbb{R}^d$, for all $t>0$,

$$p(t,x)\le\kappa\bar p_{\alpha,c}(t,x).\tag{2.2.14}$$

In particular, using the L^p estimate (2.2.7), we have

$$\begin{aligned}\|p(s,\cdot-x)\nabla^j p_{\alpha,c}(t-s,y-\cdot)\|^m_{L^\ell}&\lesssim(t-s)^{-\frac{j}{\alpha}}\|p_{\alpha,c}(s,\cdot-x)p_{\alpha,c}(t-s,y-\cdot)\|^m_{L^\ell}\\&\lesssim\bar p_{\alpha,c_1}(t,y-x)(t-s)^{-\frac{j}{\alpha}}\left[s^{-\frac{dm}{\alpha p}}+(t-s)^{-\frac{dm}{\alpha p}}\right].\end{aligned}$$

This yields

$$\begin{aligned}\mathcal{T}^{\vartheta,(t,1)}_{\ell,m}[\mathfrak{q}^{s,t}_{x,y}]&\lesssim\bar p_\alpha(t,y-x)(t-s)^{-\frac{j}{\alpha}}\left[s^{-\frac{d}{\alpha p}}+(t-s)^{-\frac{d}{\alpha p}}\right]\left(\int_t^1 v^{-\frac{\vartheta m}{\alpha}-1}\,\mathrm{d}v\right)^{\frac{1}{m}}\\&\lesssim\bar p_\alpha(t,y-x)(t-s)^{-\frac{j}{\alpha}}\left[s^{-\frac{d}{\alpha p}}+(t-s)^{-\frac{d}{\alpha p}}\right]t^{-\frac{\vartheta}{\alpha}}.\end{aligned}\tag{2.2.15}$$

For the lower part, let us write

$$\|\partial_v p_\alpha(v,\cdot)\star \mathfrak{q}^{s,t}_{x,y}\|^\ell_{L^\ell} = \int \left|\int \partial_v p_\alpha(v,z-w)\mathfrak{q}^{s,t}_{x,y}(w)\,\mathrm{d}w\right|^\ell \mathrm{d}z$$
$$= \int \left|\int \partial_v p_\alpha(v,z-w)\big[\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)\big]\,\mathrm{d}w\right|^\ell \mathrm{d}z, \tag{2.2.16}$$

using a cancellation argument for the last equality. Next, let us distinguish whether this difference is in diagonal or off-diagonal regime.

- **Diagonal case:** $|z-w|\le s^{\frac{1}{\alpha}}$. Let us write

$$\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z) = p(s,w-x)\big[\nabla^j p_\alpha(t-s,y-w)-\nabla^j p_\alpha(t-s,y-z)\big]$$
$$-\big[p(s,z-x)-p(s,w-x)\big]\nabla^j p_\alpha(t-s,y-z).$$

  Using the spatial regularity of p_α, (2.2.5) and the Hölder regularity of p, (2.2.10), we get, for some $\eta\in(0,1]$ and for some constant $c\ge 1$,

$$|\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)| \lesssim \bar{p}_{\alpha,c}(s,w-x)(t-s)^{-\frac{j}{\alpha}}\frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}}\left[\bar{p}_{\alpha,c}(t-s,y-w)+\bar{p}_{\alpha,c}(t-s,y-z)\right]$$
$$+\left[\bar{p}_{\alpha,c}(s,z-x)+\bar{p}_{\alpha,c}(s,w-x)\right]\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}(t-s)^{-\frac{j}{\alpha}}\bar{p}_{\alpha,c}(t-s,y-z) \tag{2.2.17}$$

  In the previous, terms involving a cross-dependence in w and z are slightly more difficult to handle. In order to avoid them, we use the current diagonal regime in which, for any $c_1>c$, $\bar{p}_{\alpha,c}(s,w-x)\lesssim \bar{p}_{\alpha,c_1}(s,z-x)$ and $\bar{p}_{\alpha,c}(s,z-x)\lesssim \bar{p}_{\alpha,c_1}(s,w-x)$ to write

$$|\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)| \lesssim \bar{p}_{\alpha,c_1}(s,w-x)\bar{p}_{\alpha,c_1}(t-s,y-w)(t-s)^{-\frac{j}{\alpha}}\left[\frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}}+\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}\right]$$
$$+\bar{p}_{\alpha,c_1}(s,z-x)\bar{p}_{\alpha,c_1}(t-s,y-z)(t-s)^{-\frac{j}{\alpha}}\left[\frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}}+\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}\right]. \tag{2.2.18}$$

- **Off-diagonal case:** $|z-w|\ge s^{\frac{1}{\alpha}}$. Using a triangular inequality, the regularity of p_α, (**??**), (2.2.14) and the fact that $\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}\ge 1$, we trivially have the following:

$$|\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)| \lesssim \bar{p}_{\alpha,c_1}(s,w-x)\bar{p}_{\alpha,c_1}(t-s,y-w)(t-s)^{-\frac{j}{\alpha}}\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}$$
$$+\bar{p}_{\alpha}(s,z-x)\bar{p}_{\alpha,c_1}(t-s,y-z)(t-s)^{-\frac{j}{\alpha}}\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}. \tag{2.2.19}$$

Gathering (2.2.18) and (2.2.19), we have

$$|\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)| \lesssim \bar{p}_{\alpha,c_1}(s,w-x)\bar{p}_{\alpha,c_1}(t-s,y-w)(t-s)^{-\frac{j}{\alpha}}\left[\frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}}+\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}\right]$$
$$+\bar{p}_{\alpha,c_1}(s,z-x)\bar{p}_{\alpha,c_1}(t-s,y-z)(t-s)^{-\frac{j}{\alpha}}\left[\frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}}+\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}}\right] \tag{2.2.20}$$

Plugging this in (2.2.16) and using (2.2.2) (i.e. the fact that $\partial_v p_\alpha(v,\cdot) \lesssim v^{-1} p_{\alpha,c_1}(v,\cdot)$), we get

$$\begin{aligned}
&\|\partial_v p_\alpha(v,\cdot) \star \mathfrak{q}_{x,y}^{s,t}\|_{L^\ell}^\ell = \int \left| \int \partial_v p_\alpha(v, z-w) \mathfrak{q}_{x,y}^{s,t}(w)\,\mathrm{d}w \right|^\ell \mathrm{d}z \\
&\lesssim \int \Bigg(\int v^{-1} \bar{p}_{\alpha,c_1}(v, z-w) \bar{p}_{\alpha,c_1}(s, w-x) \bar{p}_{\alpha,c_1}(t-s, y-w) \\
&\qquad\qquad\qquad \times (t-s)^{-\frac{j}{\alpha}} \left[\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}} + \frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}} \right] \mathrm{d}w \Bigg)^\ell \mathrm{d}z \\
&\quad + \int \Bigg(\int v^{-1} \bar{p}_\alpha(v, z-w) \bar{p}_{\alpha,c_1}(s, z-x) \bar{p}_{\alpha,c_1}(t-s, y-z) \\
&\qquad\qquad\qquad \times (t-s)^{-\frac{j}{\alpha}} \left[\frac{|z-w|^\eta}{s^{\frac{\eta}{\alpha}}} + \frac{|z-w|^\eta}{(t-s)^{\frac{\eta}{\alpha}}} \right] \mathrm{d}w \Bigg)^\ell \mathrm{d}z.
\end{aligned}$$

From this point, we derive a smoothing effect in v by using the moments estimate (2.2.3). It is immediate for the second term, whereas for the first one, due to the order of integration, we need to use an $L^1 - L^\ell$ convolution inequality. This yields, using (2.2.7),

$$\begin{aligned}
&\|\partial_v p_\alpha(v,\cdot) \star \mathfrak{q}_{x,y}^{s,t}\|_{L^\ell}^\ell \\
&\lesssim \left(\frac{v^{-1+\frac{\eta}{\alpha}}}{(t-s)^{\frac{j}{\alpha}}} \left[s^{-\frac{\eta}{\alpha}} + (t-s)^{-\frac{\eta}{\alpha}} \right] \right)^\ell \Bigg[\int \left(\bar{p}_\alpha(s, w-x) \left(\bar{p}_{\alpha,c_1}(t-s, y-w) + \bar{p}_{\alpha,c_1}(t-s, y'-w) \right) \right)^\ell \mathrm{d}w \\
&\qquad\qquad + \int \left(\bar{p}_{\alpha,c_1}(s, z-x) \left(\bar{p}_{\alpha,c_1}(t-s, y-z) + \bar{p}_{\alpha,c_1}(t-s, y'-z) \right) \right)^\ell \mathrm{d}z \Bigg] \\
&\lesssim \left(\frac{v^{-1+\frac{\eta}{\alpha}}}{(t-s)^{\frac{j}{\alpha}}} \left[s^{-\frac{\eta}{\alpha}} + (t-s)^{-\frac{\eta}{\alpha}} \right] \right)^\ell \left[s^{-\frac{d}{\alpha p}} + (t-s)^{-\frac{d}{\alpha p}} \right]^\ell \bar{p}_{\alpha,c_1}(t, y-x)^\ell. \qquad (2.2.21)
\end{aligned}$$

Going back to the definition of $\mathcal{T}_{\ell,m}^{\vartheta,(0,t)}$, we thus obtain, recalling that $\eta > \vartheta$,

$$\begin{aligned}
\mathcal{T}_{\ell,m}^{\vartheta,(0,t)} &\lesssim (t-s)^{-\frac{j}{\alpha}} \left[\frac{1}{s^{\frac{\eta}{\alpha}}} + \frac{1}{(t-s)^{\frac{\eta}{\alpha}}} \right] \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right] \bar{p}_{\alpha,c_1}(t, y-x) \left(\int_0^t v^{m(\frac{\eta-\vartheta}{\alpha})-1} \mathrm{d}v \right)^{\frac{1}{m}} \\
&\lesssim (t-s)^{-\frac{j}{\alpha}} \left[\frac{1}{s^{\frac{\varsigma}{\alpha}}} + \frac{1}{(t-s)^{\frac{\varsigma}{\alpha}}} \right] \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right] \bar{p}_{\alpha,c_1}(t, y-x) t^{\frac{\eta-\vartheta}{\alpha}}.
\end{aligned}$$

This finally yields

$$\mathcal{T}_{\ell,m}^{\vartheta}[\mathfrak{q}_{x,y}^{s,t}] \lesssim \frac{t^{-\frac{\vartheta}{\alpha}}}{(t-s)^{\frac{j}{\alpha}}} \left[1 + \frac{t^{\frac{\eta}{\alpha}}}{s^{\frac{\eta}{\alpha}}} + \frac{t^{\frac{\eta}{\alpha}}}{(t-s)^{\frac{\eta}{\alpha}}} \right] \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right] \bar{p}_{\alpha,c_1}(t, y-x). \qquad (2.2.22)$$

Non-thermic part

Noticing that

$$\|\mathcal{F}(\phi) \star \mathfrak{q}_{x,y}^{s,t}\|_{L^\ell} \lesssim \|\mathcal{F}(\phi)\|_{L^1} \|\mathfrak{q}_{x,y}^{s,t}\|_{L^\ell},$$

we see that (2.2.22) is also a valid bound for the non-thermic part of $\|\mathfrak{q}_{x,y}^{s,t}\|_{\mathbb{B}_{\ell,m}^\vartheta}$, which concludes the proof of (2.2.11).

□

Part II

Heat kernel estimates

Chapter 3

Heat kernel estimates for stable-driven SDEs with distributional drift

This chapter is based on the article [Fit23], published in the Journal of Potential Analysis. Therein, we consider the *formal* SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d, \tag{E}$$

where $b \in L^r([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d))$ is a time-inhomogeneous Besov drift and Z_t is a symmetric d-dimensional α-stable process, $\alpha \in (1,2)$, whose spectral measure is absolutely continuous w.r.t. the Lebesgue measure on the sphere. Above, L^r and $\mathbb{B}^{\beta}_{p,q}$ respectively denote Lebesgue and Besov spaces. We show that, when $\beta > \frac{1-\alpha+\frac{\alpha}{r}+\frac{d}{p}}{2}$, the martingale solution associated with the formal generator of (E) admits a density which enjoys two-sided heat kernel bounds as well as gradient estimates w.r.t. the backward variable. Our proof relies on a suitable mollification of the singular drift aimed at using a Duhamel-type expansion. We then use a normalization method combined with Besov space properties (thermic characterization, duality and product rules) to derive estimates.

3.1 Introduction

For a fixed $T > 0$, we study the *formal* SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x, \qquad \forall t \in [0,T], \tag{3.1.1}$$

where $b \in L^r([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d)) = \left\{ f : [0,T] \times \mathbb{R}^d : \left\| t \mapsto \|f(t,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \right\|_{L^r([0,T])} < \infty \right\}$ and Z_t is a symmetric non-degenerate d-dimensional α-stable process, whose spectral measure is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{S}^{d-1}$.

We assume the following weak-wellposedness condition holds:

$$\gamma := 2\beta - \left(1 - \alpha + \frac{\alpha}{r} + \frac{d}{p}\right) > 0 \tag{WP}$$

For this section, as we work with $\alpha < 2$, we will drop the subscript c and denote p_α the density of the noise. The set of parameters on which the notations $\lesssim$ and $\asymp$ can depend is $\Theta := \{\alpha, d, \beta, r, p, q, \|b\|_{L^r - \mathbb{B}^{\beta}_{p,q}}\}$.

We call (3.1.1) *"formal"* equation because b can be a distribution when $\beta < 0$, in which case (3.1.1) is ill-defined as such. As there are multiple ways to define a solution to (3.1.1), each with its conditions on the parameters β, p, q, r and interpretation, we will go into details in Subsection 3.1.2.

The main idea behind the study of singular drift diffusions is that adding a noise regularizes ordinary differential equations, and helps restore existence and uniqueness in some appropriate sense. For example, in the case of a β-Hölder ($\beta \in (0,1)$) drift, the noise gives an "impulse" which permits to exit singular spots (see e.g. [DF14] in the Brownian case). Knowing that, one would expect that, the bigger the intensity of the noise, the stronger the regularizing effect, which we will see on the upcoming thresholds (see also [CdRM22b] and [MM21]). We will investigate cases in which the noise is strong enough to restore uniqueness even for distributional drifts.

Let us first review the probabilistic results and associated techniques used in the case $\beta \geq 0, \alpha \in (0,2)$ to derive weak or strong well-posedness, when the drift is a function. In order to establish well-posedness, a natural condition appeared in the seminal article by Tanaka *et al.* [TTW74] : $\beta + \alpha > 1$. The authors consider therein the scalar case, and proved that strong uniqueness holds for bounded β-Hölder drifts under this condition, while giving a counter example when $\beta + \alpha < 1$. The critical multidimensional case (i.e. $\alpha = 1$) in a time-inhomogeneous setting was investigated in [Kom84], in which weak uniqueness is derived for a continuous drift with, again, the driving noise having absolutely continuous spectral measure w.r.t. the Lebesgue measure on the sphere. Having in mind that weak (or strong) well-posedness is often investigated through the corresponding parabolic PDE, recalling that the associated expected parabolic gain is $\beta + \alpha$, the condition $\beta + \alpha > 1$ coincides with the regularity required to define the gradient of the solution. The aforementioned regularity gain is often obtained through Schauder-type estimates. We can mention [MP14] (bounded drift, stable-like generators), [CdRMP20a] (unbounded drift, general stable generators including e.g. the cylindrical one). These estimates naturally lead to weak uniqueness in the multidimensional setting for (3.1.1) through the martingale problem, which precisely requires a control of the gradient of the solution of the PDE.

Going towards strong solutions requires additional constraints on the parameters. It was e.g. shown by Priola in [Pri12] that pathwise uniqueness holds in the multidimensional case for general non-degenerate stable generators with $\alpha \geq 1$ for time-homogeneous bounded β-Hölder drifts under the assumption $\beta > 1 - \alpha/2$. Under the same assumption, [CZZ21] proved strong existence and uniqueness for any $\alpha \in (0,2)$, as well as weak uniqueness whenever $\beta + \alpha > 1$ for time-inhomogeneous drift with non-trivial diffusion coefficient. Those results are usually obtained using the Zvonkin transform (see [Zvo74], [Ver80]), which requires additional regularity on the underlying PDE, which again follow from Schauder-type estimates.

Once weak or strong well-posedness is established, a natural question concerns the behavior of the time marginal laws of the SDE. Such behavior is usually investigated through heat kernel estimates, which, in the stable setting, somehow forces to consider the stable-like case, i.e., the driving noise Z in (3.1.1) has a Lévy measure with *smooth* density w.r.t. to the isotropic α-stable measure (see Subsection 3.1.1 for detailed assumptions on the noise). In this setting, we can refer to the seminal work by Kolokoltsov [Kol00], who addressed the subcritical case $\alpha > 1$ for smooth bounded drifts. This work was extended in various directions, although mostly for non-negative β (see [Kul19], [CHZ20], [KK18]). In [MZ22], authors cover the whole range $\alpha \in (0,2)$ with Hölder unbounded drift. In those works, the authors establish that the time marginal laws of the process have a density which is "equivalent" (i.e. bounded from above and below) to the density of the noise, and that the spatial gradients exhibit the same time singularities and decay rates (see Theorem 3.1 below in the current setting).

Going towards negative β brings additional difficulties. The first challenge is to specify what is intended with "solution" to (3.1.1). To this end, a key tool is the following PDE:

$$(\partial_t + b \cdot D + \mathcal{L}^\alpha) u(t,x) = f(t,x) \text{ on } [0,T) \times \mathbb{R}^d, \qquad u(T,\cdot) = g \text{ on } \mathbb{R}^d \tag{3.1.2}$$

for suitable sources f and final conditions g, and where $\mathcal{L}^\alpha$ is the generator of the noise Z. When studying (3.1.2), defining the gradient of the solution still requires $\alpha + \beta > 1$, which now imposes $\alpha > 1$. This is anyhow not sufficient: we also need to be able to define $b \cdot Du$ as a distribution. Roughly, since b has spatial regularity β, this imposes $\beta + (\beta + \alpha - 1) > 0 \iff \beta > \frac{1-\alpha}{2}$ by usual paraproduct rules (note that this is the exact assumption we need if $p = r = +\infty$). This threshold already appears in [BC01] in the diffusive setting ($\alpha = 2$), where strong well-posedness is derived in the scalar case through Dirichlet forms techniques for specifically structured time-homogeneous drifts. The same threshold is exhibited in [FIR17], where the authors introduce the notion of *virtual solutions* to give a meaning to (3.1.1). Those solutions are defined through a Zvonkin-type transform formula, and, while not requiring any specific structure, do not yield a precise dynamics for the SDE. We can also refer to [ZZ17] and [ABM20], who specified the meaning to be given to (3.1.1), in the sense that the drift therein is defined through smooth approximating sequences of the singular b along the solution. Importantly, the limit drift is a Dirichlet process, highlighting once again that (3.1.1) is a *formal* equation. A thorough description of this Dirichlet process was done in the Brownian scalar case in [DD16] and extended in [CC18] for multidimensional SDEs. Assuming some additional structure on the drift, they manage to go beyond the above threshold and reach $\beta > -\frac{2}{3}$ (still with $p = r = \infty$). This work was extended in the multidimensional strictly stable case, still assuming a specific structure for the drift in [KP22], in which weak well-posedness is proved for $\beta > \frac{2-2\alpha}{3}$. Without any structure on the drift, a similar and consistent description of the dynamics for the weak solutions of (3.1.1) in the multidimensional setting is obtained in [CdRM22a] for $\beta > \frac{1-\alpha}{2}$. The case of a non-trivial diffusion coefficient was investigated in [LZ22] with the same thresholds. Note that, in the present work, we chose to work with a trivial diffusion coefficient as the most delicate issue is the handling of the singular drift (see Remark 15 in [CdRM22a] for the handling of a non trivial diffusion coefficient in a Duhamel expansion). We do believe our approach would be robust enough to treat this case. Let us mention that [ABM20] also obtained strong uniqueness with this threshold in the scalar case. We emphasize that most of the aforementioned results heavily rely on the Schauder-type regularization properties of the PDE (3.1.2).

In the scope of singular drift heat kernels estimates, the sole result we were able to gather is [PvZ22]. Using the Littlewood-Paley characterization of Besov spaces, Perkowski and van Zuijlen managed to derive explicit two-sided heat kernel estimates as well as gradient estimates w.r.t. the backward variable for the solution in the Brownian, time-inhomogeneous setting with time-continuous drift in $\mathbb{B}^{\beta}_{\infty,1}, \beta > -\frac{1}{2}$.

The goal of the current paper is to establish heat kernel and gradient estimates for stable driven SDEs with drifts in $L^r - \mathbb{B}^{\beta}_{p,q}$ and symmetric non-degenerate d-dimensional α-stable noise with absolutely continuous Lévy measure for $\beta \in \left(\frac{1-\alpha+\frac{d}{p}+\frac{\alpha}{r}}{2}, 0\right)$. As compared to the previous results, this represents a slight modification of the threshold, due to integrability concerns. For $p = r = +\infty$, we work under the usual $\beta > \frac{1-\alpha}{2}$ assumption.

This paper is organized as follows. We first discuss the properties of the noise in Subsection 3.1.1. We then define the notions of martingale solutions for (3.1.1) and mild solutions for (3.1.2) along with required assumptions in Subsection 3.1.2. We state our main results in Subsection and detail the dynamics of (3.1.1) in 3.2.1. Section 3.3 is dedicated to obtaining estimates on a mollified equation with smooth drift and Section 3.4 links those estimates back to the main SDE (3.1.1) through compactness arguments.

3.1.1 Driving noise and related density properties

Let us denote by $\mathcal{L}^{\alpha}$ the generator of the driving noise Z. In the case $\alpha = 2$, $\mathcal{L}^{\alpha}$ is the usual Laplacian $\frac{1}{2}\Delta$. When $\alpha \in (1, 2)$, in whole generality, the generator of a symmetric stable process writes, $\forall \phi \in C_0^{\infty}(\mathbb{R}^d, \mathbb{R})$ (smooth compactly supported functions),

$$\begin{aligned}\mathcal{L}^{\alpha}\phi(x) &= \text{p.v.} \int_{\mathbb{R}^d} [\phi(x+z) - \phi(x)]\, \nu(\, \mathrm{d}z) \\ &= \text{p.v.} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} [\phi(x+\rho\xi) - \phi(x)]\, \mu(\, \mathrm{d}\xi) \frac{\mathrm{d}\rho}{\rho^{1+\alpha}}\end{aligned}$$

(see [Sat99] for the polar decomposition of the spectral measure) where μ is a non-degenerate measure on the unit sphere $\mathbb{S}^{d-1}$, i.e. μ is symmetric and $\exists \kappa \geq 1 : \forall \lambda \in \mathbb{R}^d$,

$$\kappa^{-1}|\lambda|^{\alpha} \leq \int_{\mathbb{S}^{d-1}} |\lambda \cdot \xi|^{\alpha} \mu(\, \mathrm{d}\xi) \leq \kappa|\lambda|^{\alpha},$$

where "·" stands for the usual scalar product in $\mathbb{R}^d$.
This general setting will not allow us to derive heat kernel estimates, because it does not lead to global estimates of the noise density. In [Wat07], Watanabe investigates the behavior of the density of an α-stable process in terms of properties fulfilled by the support of its spectral measure. From this work, we know that whenever the measure μ is not equivalent to the Lebesgue measure on the unit sphere, accurate estimates on the density of the stable process are delicate to obtain. However, Watanabe (see [Wat07], Theorem 1.5) and Kolokoltsov ([Kol00], Propositions 2.1–2.5) showed that if $C^{-1}m(\,\mathrm{d}\xi) \le \mu(\,\mathrm{d}\xi) \le Cm(\,\mathrm{d}\xi)$ (where m is the uniform density on $\mathbb{S}^{d-1}$), the following estimates hold: there exists a constant C depending only on α, d, s.t. $\forall u \in \mathbb{R}_+^*, z \in \mathbb{R}^d$,

$$\frac{C^{-1}}{u^{\frac{d}{\alpha}}} \frac{1}{\left(1+\frac{|z|}{u^{\frac{1}{\alpha}}}\right)^{d+\alpha}} \le p_\alpha(u,z) \le \frac{C}{u^{\frac{d}{\alpha}}} \frac{1}{\left(1+\frac{|z|}{u^{\frac{1}{\alpha}}}\right)^{d+\alpha}}.$$

As our approach heavily relies on these global bounds, we have to assume that μ is equivalent to the Lebesgue measure on the sphere and that $\alpha \in (1,2)$.

3.1.2 Defining solutions to the distributional drift SDE

We will use the following notations :

- The set of all parameters will be denoted $\Theta := \{\alpha, d, \beta, r, p, q, \|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\}$
- $a \lesssim b$ if there exists a constant C, which depends only on parameters from Θ, such that $a \le Cb$.
- $a \asymp b$ if there exists a constant C, which depends only on parameters from Θ, such that $C^{-1}b \le a \le Cb$.
- $\star$ denotes the spatial convolution.
- $C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R})$ is the space of continuous in time and differentiable in space functions, $C_b^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}) = C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}) \cap L^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$ and $C_b^0([0,T]\times\mathbb{R}^d,\mathbb{R})$ is the space of bounded continuous space-time functions.
- For $f \in \mathcal{S}'(\mathbb{R}^d)$ (the dual of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$) and $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\phi(0) \ne 0$, we set $\phi(D)f = \mathcal{F}^{-1}(\phi \times \mathcal{F}(f)) = \mathcal{F}^{-1}(\phi) \star f$, where $\mathcal{F}$ denotes the Fourier transform.
- For $p \in [1,+\infty]$, we always denote by $p' \in [1,+\infty]$ s.t. $\frac{1}{p}+\frac{1}{p'} = 1$ its conjugate.

As we work with a distributional drift, we need to specify what we call a "solution" to (3.1.1). There are two ways to define a solution to (3.1.1) which we will investigate. We will first introduce the usual *martingale solutions*. Those are defined through the mild solutions of the underlying PDE and are the ones that require the least regularity. Importantly, they are sufficient to state Theorem **??**. In Subsection 3.2.1, we will then give details about *weak solutions*, as defined in [CdRM22a] in order to give a concrete dynamics for the solution.

Although our results are proved for martingale solutions (in which case they can be understood as a *formal* discussion on the density of the process), they are mainly useful in the scope of weak solutions, as those introduce a dynamics and could be a starting point to establish numerical schemes for those equations.

Let us now fix $p,q,r \ge 1$. For the definition of a **martingale** solution to (3.1.1), we need the following conditions on α, β, which we call a **good relation (GR)** :

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2\right) \qquad \beta \in \left(\frac{1-\alpha+\frac{d}{p}+\frac{\alpha}{r}}{2}, 0\right) \tag{GR}$$

and we will denote

$$\theta := \beta + \alpha - \frac{d}{p} - \frac{\alpha}{r}, \tag{3.1.3}$$

which corresponds to the parabolic bootstrap induced by the drift. As explained in [CdRM22a], this choice of θ implies that

$$(t,x) \mapsto \int_t^T P^\alpha_{s-t}[G \cdot v](s,x)\,\mathrm{d}s$$

is well defined and belongs to $\mathcal{C}^{0,1}_b([0,T]\times\mathbb{R}^d,\mathbb{R})$ as soon as $G \in L^r([0,T],\mathbb{B}^\beta_{p,q}(\mathbb{R}^d,\mathbb{R}^d))$ and $v \in L^\infty([0,T],\mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ for some $0 \le \varepsilon \ll 1$.

Remark 3.1. *Note that, here, we are only trying to give a meaning to the distributional product $G \cdot v$. Roughly speaking, for $p = r = +\infty$, by Bony's paraproduct rule, the total regularity of $G \cdot v$ is $\beta + \theta - 1 - \varepsilon$, which we need to be positive. This is only possible if α and β satisfy **(GR)**, hence the definition of the latter. The additional $\frac{d}{p} + \frac{\alpha}{r}$ corresponds to the lack of global boundedness of the drift b.*

This allows us to give the definition of mild solution to a PDE:

Definition 3.1. *Mild solution of the underlying PDE.*
*Let $\alpha \in (1,2)$, $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$. For a given $T > 0$, we say that $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a **mild** solution of the formal Cauchy problem $\mathcal{C}(b,\mathcal{L}^\alpha,f,g,T)$*

$$(\partial_t + b \cdot D + \mathcal{L}^\alpha)\, u(t,x) = f(t,x) \text{ on } [0,T) \times \mathbb{R}^d, \qquad u(T,\cdot) = g \text{ on } \mathbb{R}^d,$$

if it belongs to $\mathcal{C}^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R})$ with $Du \in \mathcal{C}^0_b([0,T],\mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty})$ for any $0 < \varepsilon \ll 1$ and $\theta = \beta + \alpha - \frac{d}{p} - \frac{\alpha}{r}$, and if it satisfies

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, u(t,x) = P^\alpha_{T-t}[g](x) - \int_t^T P^\alpha_{s-t}[f - b \cdot Du](s,x)\,\mathrm{d}s. \tag{3.1.4}$$

In [CdRM22a], Chaudru de Raynal and Menozzi proved existence and uniqueness of such solutions under (GR), and also give information on their time regularity. Let us now introduce the notion of martingale problem (introduced in [SV97] and then generalized in [EK86]).

Definition 3.2. *Solution of the martingale problem*
Let $\Omega = \mathcal{D}([0,T],\mathbb{R}^d)$ (the Skorokhod space of càdlàg functions). We say that a probability measure $\mathbb{P}$ on Ω equipped with its canonical filtration is a solution of the martingale problem associated with $(b,\mathcal{L}^\alpha,x)$ for $x \in \mathbb{R}^d$ if, denoting by $(x_t)_{t\in[0,T]}$ the associated canonical process,

(i) $\mathbb{P}(x_0 = x) = 1$,

(ii) $\forall f \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d,\mathbb{R}))$, $g \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R})$ *with* $Dg \in \mathbb{B}^{\theta-1}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d)$,

$$\left(u(t,x_t) - \int_0^t f(s,x_s)ds - u(0,x) \right)_{0\le t\le T}$$

*is a martingale under $\mathbb{P}$ where u is the **mild** solution of the Cauchy problem $\mathcal{C}(b,\mathcal{L}^\alpha,f,g,T)$.*

Remark 3.2. *The choice of the class of f (here, $\mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d,\mathbb{R}))$) is not critical. We only need it to be rich enough to characterize marginal laws, i.e. a class of functions Φ is sufficient if whenever two probability measures μ_1 and μ_2 satisfy*

$$\int \phi\,\mathrm{d}\mu_1 = \int \phi\,\mathrm{d}\mu_2, \qquad \forall \phi \in \Phi,$$

then $\mu_1 = \mu_2$.

Again, in [CdRM22a], it is proved that there exists a unique solution to the martingale problem in the sense of the previous definition. We will call *"martingale solution to* (3.1.1)*"* the associated canonical process.

3.2 Main results

Theorem 3.1. *Fix the parameters $T > 0$ and $\Theta = \{\alpha, d, \beta, r, p, q, \|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\}$. Take $b \in L^r([0,T], \mathbb{B}^\beta_{p,q}(\mathbb{R}^d))$ and assume* (WP) *holds. Consider the solution $\mathbb{P}$ to the martingale problem associated with $(b, \mathcal{L}^\alpha, x)$ starting at time s and denote $(x_t)_{t\in[s,T]}$ the associated canonical process. For all $t \in (s,T]$, x_t admits a density $\Gamma(s,x,t,\cdot)$ such that there exists $C := C(T,\Theta,\rho) \geq 1$ such that for all $y \in \mathbb{R}^d$,*

$$C^{-1} p_\alpha(t-s, y-x) \leq \Gamma(s,x,t,y) \leq C p_\alpha(t-s,y-x), \tag{3.2.1}$$

$$|\nabla_x \Gamma(s,x,t,y)| \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}} p_\alpha(t-s,y-x), \tag{3.2.2}$$

$$\forall (y,y') \in \mathbb{R}^d, \qquad |\Gamma(s,x,t,y) - \Gamma(s,x,t,y')| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}} \left(p_\alpha(t-s,y-x) + p_\alpha(t-s,y'-x)\right), \tag{3.2.3}$$

$$\forall (y,y') \in \mathbb{R}^d, \qquad |\nabla_x \Gamma(s,x,t,y) - \nabla_x \Gamma(s,x,t,y)| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \left(p_\alpha(t-s,x-y) + p_\alpha(t-s,x-y')\right), \tag{3.2.4}$$

for any $\rho \in (-\beta, -\beta+\gamma/2)$, where $\gamma := 2\beta + \alpha - 1 - \frac{\alpha}{r} - \frac{d}{p} > 0$ is the "gap to singularity".

Moreover, for any $\varepsilon \in (0,-\beta)$ and $\rho \in (-\beta, -\beta+\varepsilon/2)$, for $t' > t$ such that $(t'-t) < t/2$,

$$\left\| \frac{\Gamma(s,x,t,\cdot) - \Gamma(s,x,t',\cdot)}{p_\alpha(t-s,\cdot - x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \leq C \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon-\beta}{\alpha}}} \tag{3.2.5}$$

Remark 3.3 (Logarithmic gradient estimates.)**.** *Note that, in the current strictly stable regime ($\alpha \in (1,2)$) and given the previous theorem, one can easily compute global logarithmic gradient estimates for Γ:*

$$|\nabla_x \log \Gamma(s,x,t,y)| = \frac{|\nabla_x \Gamma(s,x,t,y)|}{\Gamma(s,x,t,y)} \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}}.$$

The sketch of the proof of Theorem 3.1 is as follows:

- Take a smooth $b^m \in C_b^\infty$ to approach b and consider the *mollified* equation
$$\mathrm{d}X_t^m = b^m(t, X_t^m)\,\mathrm{d}t + \mathrm{d}Z_t. \tag{3.2.6}$$
- Compute estimates on the density of (X_t^m) which are uniform in m, using a Duhamel expansion and a normalization method first introduced by [MPZ21] (Brownian setting with unbounded Hölder drift) and then exploited in [JM21] (Brownian setting with $L^q - L^p$ drift) and [MZ22] (unbounded drift, stable driven with multiplicative isotropic noise).
- Conclude with a compactness argument.

3.2.1 Weak solutions

Although mild solutions allow for a formal discussion on the density of the underlying process in the SDE (3.1.1), they do not exhibit anything about its dynamics nor about its SDE interpretation. In order to build the dynamics of the equation, [CdRM22a] introduced a weak formulation of the problem. To such end, they used the notion of L^ℓ stochastic Young integral, in the sense of the definition first introduced by [CG16] and [DD16].

In order to define the upcoming notion of solution, we need slightly stronger assumptions on α, β. We say that α, β satisfy a **good relation for the dynamics (GRD)** if the following holds:

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2\right) \qquad \beta \in \left(\frac{1-\alpha+\frac{2d}{p}+\frac{2\alpha}{r}}{2}, 0\right). \tag{GRD}$$

This stronger condition is required in order for the following definition to make sense:

Definition 3.3. *We call **weak** solution of the formal SDE (3.1.1) a pair (Y, Z) of adapted processes on a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_T\}_{t\geq 0}, \mathbb{P})$ such that Z is an $\{\mathcal{F}_T\}_{t\geq 0}$ α-stable process and (Y, Z) satisfies*

$$Y_t = x + \int_0^t \mathfrak{B}(s, Y_s, \mathrm{d}s) + Z_t, \qquad \mathbb{P}-\text{a.s.}, \qquad \mathbb{E}\left|\int_0^t \mathfrak{B}(s, Y_s, \mathrm{d}s)\right| < \infty \tag{3.2.7}$$

for any $t \in [0, T]$, where

$$\mathfrak{B} : (v, x, h) \mapsto \int_0^h \mathrm{d}r \int_{\mathbb{R}^d} p_\alpha(h - r, x - y) b(v + r, y)\, \mathrm{d}y \tag{3.2.8}$$

*and where the integral in (3.2.7) is understood as an L^1 stochastic Young integral and imposes the stronger **(GRD)** condition.*

With this explicit definition, it becomes fathomable to develop numerical schemes for the SDE. In particular, as Theorem 3.1 is proved under **(GR)**, it is valid under the stronger **(GRD)** conditions and thus holds for the density of weak solutions.

3.3 Estimates on the mollified SDE

In this section, we only consider the *mollified* SDE

$$\mathrm{d}X_t^m = b^m(t, X_t^m)\, \mathrm{d}t + \mathrm{d}Z_t, \tag{3.3.1}$$

where $(b^m)_{m\in\mathbb{N}} \in C_b^\infty$ is an approximating sequence of the drift, as given by Proposition 2.1. As thus, this SDE is a classical one, and we have strong well-posedness and uniqueness. In this setting, it is known that the density of $(X_t^m)_{t\geq s}$ exists for $t > s$ (see e.g. [Kol00] or [Lea85] for a more general additive noise). We will prove the following theorem:

Theorem 3.2. *Fix the parameters $T > 0$ and $\Theta = \{\alpha, d, \beta, r, p, q, \|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\}$. Assume* (WP) *holds. For any m, consider the solution $\mathbb{P}^m$ to the martingale problem associated with $(b^m, \mathcal{L}^\alpha, x)$ starting at time s and denote $(x_t^m)_{t\in[s,T]}$ the associated canonical process. For all $t \in (s, T]$, x_t^m admits a density $\Gamma^m(s, x, t, \cdot)$ such that there exists $C := C(T, \Theta, \rho) \geq 1$ such that for all $(x, y) \in \mathbb{R}^d$,*

$$C^{-1}\bar{p}_\alpha(t - s, y - x) \leq \Gamma^m(s, x, t, y) \leq C p_\alpha(t - s, y - x), \tag{3.3.2}$$

$$|\nabla_x \Gamma^m(s, x, t, y)| \leq \frac{C}{(t - s)^{\frac{1}{\alpha}}} p_\alpha(t - s, y - x), \tag{3.3.3}$$

$$\forall (y, y') \in \mathbb{R}^d, \qquad |\Gamma^m(s, x, t, y) - \Gamma^m(s, x, t, y')| \leq \frac{C|y - y'|^\rho}{(t - s)^{\frac{\rho}{\alpha}}} \left(p_\alpha(t - s, y - x) + p_\alpha(t - s, y' - x)\right), \tag{3.3.4}$$

$$\forall (y, y') \in \mathbb{R}^d, \qquad |\nabla_x \Gamma^m(s, x, t, y) - \nabla_x \Gamma^m(s, x, t, y')| \leq \frac{C|y - y'|^\rho}{(t - s)^{\frac{\rho+1}{\alpha}}} \left(p_\alpha(t - s, x - y) + p_\alpha(t - s, x - y')\right), \tag{3.3.5}$$

for any $\rho \in (-\beta, -\beta + \gamma/2)$, where $\gamma := 2\beta + \alpha - 1 - \frac{\alpha}{r} - \frac{d}{p} > 0$ is the "gap to singularity".

Moreover, for any $\varepsilon \in (0, -\beta)$ and $\rho \in (-\beta, -\beta + \varepsilon/2)$, for $t' > t$ such that $(t' - t) < t/2$,

$$\left\|\frac{\Gamma^m(s, x, t, \cdot) - \Gamma^m(s, x, t', \cdot)}{p_\alpha(t - s, \cdot - x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}} \leq C \frac{(t' - t)^{\frac{\gamma - \varepsilon}{\alpha}}}{(t - s)^{\frac{\gamma - \varepsilon - \beta}{\alpha}}} \tag{3.3.6}$$

We insist that this statement is uniform in m as C does not depend on m. We will see in the proof that this is made possible by (2.1.6). We could in fact already obtain those bounds from [Kol00], but they would not be uniform in m.

Proof. We will prove Theorem 3.2 for $T \in (0,1)$. To extend this proof to any $T > 0$, it suffices to use the Chapman-Kolmogorov property of p_α.

As equation (3.3.1) can be understood in a classical way, we can perform a Duhamel expansion on the density of the solution (see e.g. [MZ22]). Namely, $\forall 0 \le s < t \le T, \forall (x,y) \in \mathbb{R}^d$,

$$\Gamma^m(s,x,t,y) = p_\alpha(t-s, y-x) + \int_s^t \int \Gamma^m(s,x,u,z) b^m(u,z) \nabla_z p_\alpha(t-u, y-z)\, \mathrm{d}z\, \mathrm{d}u. \tag{3.3.7}$$

Let us now denote, for fixed $(s,x) \in [0,1] \times \mathbb{R}^d$,

$$g^m_{s,x}(t,y) := \frac{\Gamma^m(s,x,t,y)}{p_\alpha(t-s,y-x)}.$$

The aforementioned normalization method then consists in writing the following:

$$g^m_{s,x}(t,y) \lesssim 1 + \frac{1}{p_\alpha(t-s,y-x)} \int_s^t \left| \int g_{s,x}(u,z) b^m(u,z) p_\alpha(s,x,u,z) \nabla_z p_\alpha(t-u,y-z)\, \mathrm{d}z \right| \mathrm{d}u.$$

From this point, our goal is to use a Gronwall-Volterra lemma on this expansion. This will give us bounds on $g^m_{s,x}$, which we need to be uniform in m. In our case, we do not know much about b^m, and the most we might be able to rely on is that $\|b^m - b\|_{L^{\tilde r} - \mathbb{B}^{\tilde\beta}_{p,q}} \to 0$ (with $\tilde r$ and $\tilde\beta < \beta$ as defined in Propostion 2.1). On the flipside, we know a lot about the stable kernel p_α and its derivatives. In particular, it is very smooth, and we should be able to control its Besov norm rather well. Hence we will use the duality inequality (2.1.3) and the product rule (2.1.2) with any $\rho > \max\left\{\beta, d\left(\frac{1}{p} - 1\right)_+ - \beta\right\} = -\beta$ to derive:

$$\begin{aligned} g^m_{s,x}(t,y) &\lesssim 1 + \frac{1}{p_\alpha(t-s,y-x)} \int_s^t \|g^m_{s,x}(u,\cdot) b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|p_\alpha(u-s,\cdot - x) \nabla p_\alpha(t-u, y - \cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\, \mathrm{d}u \\ &\lesssim 1 + \frac{1}{p_\alpha(t-s,y-x)} \int_s^t \|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|p_\alpha(u-s,\cdot - x) \nabla p_\alpha(t-u, y - \cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\, \mathrm{d}u, \end{aligned}$$

Using (2.2.9), we get

$$\begin{aligned} g^m_{s,x}(t,y) \lesssim 1 + \int_s^t & \|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \\ & \times \frac{(t-s)^{\frac{\beta+\rho}{\alpha}}}{(t-u)^{\frac{1}{\alpha}}} \left[(t-u)^{-\frac{d}{\alpha p}} + (u-s)^{-\frac{d}{\alpha p}}\right] \left[(t-u)^{-\frac{\rho}{\alpha}} + (u-s)^{-\frac{\rho}{\alpha}}\right] \mathrm{d}u. \end{aligned} \tag{3.3.8}$$

We now need to retrieve $\|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}$ on the l.h.s. to use a Gronwall-Volterra lemma.

$$\begin{aligned} \|g^m_{s,x}(t,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}} &= \|g^m_{s,x}(t,\cdot)\|_{L^\infty} + \sup_{v \in (0,1]} v^{1-\frac{\rho}{\alpha}} \|\partial_v p_\alpha(v,\cdot) \star g^m_{s,x}(t,\cdot)\|_{L^\infty} \\ &= \|g^m_{s,x}(t,\cdot)\|_{L^\infty} + \mathcal{T}^\rho_{\infty,\infty}[g^m_{s,x}(t,\cdot)]. \end{aligned}$$

The non-thermic part can already be estimated from (3.3.8). For the thermic part, let us mention that since the space $\mathbb{B}^\rho_{\infty,\infty}$ is continuously embedded in the classical Hölder space $\mathcal{C}^\rho$, the thermic part is in fact a Hölder modulus, which we control in the following way:

For $(y,y') \in (\mathbb{R}^d)^2$ such that $|y-y'| \ge (t-s)^{\frac{1}{\alpha}}$, we trivially have

$$|g^m_{s,x}(t,y) - g^m_{s,x}(t,y')| \lesssim \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}} \left\| g^m_{s,x}(t,\cdot) \right\|_{L^\infty}.$$

For $(y,y')\in(\mathbb{R}^d)^2$ such that $|y-y'|\le (t-s)^{\frac{1}{\alpha}}$, using (2.2.5), we have

$$\begin{aligned}
|g^m_{s,x}(t,y)-g^m_{s,x}(t,y')| &= \left|\frac{\Gamma^m(s,x,t,y)}{p_\alpha(t-s,y-x)}-\frac{\Gamma^m(s,x,t,y')}{p_\alpha(t-s,y'-x)}\right|\\
&\lesssim \left|\frac{\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t,y')}{p_\alpha(t-s,y-x)}\right| + \Gamma^m(s,x,t,y')\left|\frac{1}{p_\alpha(t-s,y-x)}-\frac{1}{p_\alpha(t-s,y'-x)}\right|\\
&\lesssim \left|\frac{\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t,y')}{p_\alpha(t-s,y-x)}\right| + \Gamma^m(s,x,t,y')\left|\frac{p_\alpha(t-s,y'-x)-p_\alpha(t-s,y-x)}{p_\alpha(t-s,y'-x)p_\alpha(t-s,y-x)}\right|\\
&\lesssim \left|\frac{\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t,y')}{p_\alpha(t-s,y-x)}\right| + \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}}\left\|g^m_{s,x}(t,\cdot)\right\|_{L^\infty}.
\end{aligned}$$

It thus remains to control $|\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t,y')|$. Using (2.2.5), we have

$$\begin{aligned}
&|\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t,y')| \le |p_\alpha(t-s,y-x)-p_\alpha(t-s,y'-x)|\\
&\quad+\int_s^t\int \Gamma^m(s,x,u,z)b^m(u,z)\left[\nabla p_\alpha(t-u,y-z)-\nabla p_\alpha(t-u,y'-z)\right]\mathrm{d}z\,\mathrm{d}u\\
&\lesssim \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}}\left(p_\alpha(t-s,y-x)+p_\alpha(t-s,y'-x)\right)\\
&\quad+\int_s^t \|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}\|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\|p_\alpha(u-s,\cdot-x)\left[\nabla p_\alpha(t-u,y-\cdot)-\nabla p_\alpha(t-u,y'-\cdot)\right]\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}u,
\end{aligned}$$

Using now (2.2.12) and the fact that $|y-y'|\le(t-s)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
\frac{|\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t,y')|}{p_\alpha(t-s,y-x)} &\lesssim \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}}\Bigg(1+\int_s^t\|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}\|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\\
&\qquad\frac{(t-s)^{\frac{\beta+2\rho}{\alpha}}}{(t-u)^{\frac{1+\rho}{\alpha}}}\left[(t-u)^{-\frac{d}{\alpha p}}+(u-s)^{-\frac{d}{\alpha p}}\right]\left[(t-u)^{-\frac{\rho}{\alpha}}+(u-s)^{-\frac{\rho}{\alpha}}\right]\mathrm{d}u\Bigg).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\mathcal{T}^\rho_{\infty,\infty}[g^m_{s,x}(t,\cdot)] &\lesssim \frac{1}{(t-s)^{\frac{\rho}{\alpha}}}\Bigg(1+\int_s^t\|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}\|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\\
&\qquad\times\frac{(t-s)^{\frac{\beta+\rho}{\alpha}}}{(t-u)^{\frac{1}{\alpha}}}\left[(t-u)^{-\frac{d}{\alpha p}}+(u-s)^{-\frac{d}{\alpha p}}\right]\left[(t-u)^{-\frac{\rho}{\alpha}}+(u-s)^{-\frac{\rho}{\alpha}}\right]\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}}\,\mathrm{d}u\Bigg).
\end{aligned}$$

This indicates that the thermic part of $\|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}$ is not homogeneous to its non-thermic part. We therefore introduce a normalized version of $\|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}$ on which to perform a Gronwall-Volterra lemma, accounting for the right time singularity. Denote

$$\tilde g^m_{s,x}(t):=\|g^m_{s,x}(t,\cdot)\|_{L^\infty}+(t-s)^{\frac{\rho}{\alpha}}\mathcal{T}^\rho_{\infty,\infty}[g^m_{s,x}(t,\cdot)]. \tag{3.3.9}$$

With the previous lemma, we can write

$$\begin{aligned}
\tilde g^m_{s,x}(t) &\lesssim 1+\int_s^t\|g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}\|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \qquad\qquad (3.3.10)\\
&\qquad\times\frac{(t-s)^{\frac{\beta+\rho}{\alpha}}}{(t-u)^{\frac{1}{\alpha}}}\left[(t-u)^{-\frac{d}{\alpha p}}+(u-s)^{-\frac{d}{\alpha p}}\right]\left[(t-u)^{-\frac{\rho}{\alpha}}+(u-s)^{-\frac{\rho}{\alpha}}\right]\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}}\,\mathrm{d}u.
\end{aligned}$$

Notice that, because $(u-s)\le(t-s)\le 1$,

$$\|(u-s)^{\frac{\rho}{\alpha}}g^m_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}}=\|(u-s)^{\frac{\rho}{\alpha}}g^m_{s,x}(u,\cdot)\|_{L^\infty}+(u-s)^{\frac{\rho}{\alpha}}\mathcal{T}^\rho_{\infty,\infty}[g^m_{s,x}(u,\cdot)]\le\tilde g^m_{s,x}(u).$$

Because of this, (3.3.10) yields

$$\begin{aligned}
\tilde g^m_{s,x}(t) &\lesssim 1 + \int_s^t \frac{\tilde g^m_{s,x}(u)}{(u-s)^{\frac{\rho}{\alpha}}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \\
&\qquad\qquad \times \frac{(t-s)^{\frac{\beta+\rho}{\alpha}}}{(t-u)^{\frac{1}{\alpha}}} \left[(t-u)^{-\frac{d}{\alpha p}} + (u-s)^{-\frac{d}{\alpha p}}\right] \left[(t-u)^{-\frac{\rho}{\alpha}} + (u-s)^{-\frac{\rho}{\alpha}}\right] \frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}} \, \mathrm{d}u \\
&\lesssim 1 + \|b^m\|_{L^r - \mathbb{B}^\beta_{p,q}} \qquad (3.3.11)\\
&\quad \times \left(\int_s^t \frac{\tilde g^m_{s,x}(u)^{r'} (t-s)^{r'\left(\frac{\beta+2\rho}{\alpha}\right)}}{(u-s)^{\frac{r'\rho}{\alpha}} (t-u)^{r'\left(\frac{1+\rho}{\alpha}\right)}} \left[(t-u)^{-\frac{dr'}{\alpha p}} + (u-s)^{-\frac{dr'}{\alpha p}}\right] \left[(t-u)^{-\frac{\rho r'}{\alpha}} + (u-s)^{-\frac{\rho r'}{\alpha}}\right] \mathrm{d}u \right)^{\frac{1}{r'}}.
\end{aligned}$$

In the previous, the most singular term is the one involving powers of $t-u$. Those are integrable if and only if

$$1 - r'\left(\frac{1+2\rho+\frac{d}{p}}{\alpha}\right) > 0 \iff 1 - \frac{1}{r} - \frac{1}{\alpha} - \frac{d}{\alpha p} - \frac{2\rho}{\alpha} > 0 \iff \frac{\gamma}{\alpha} + \frac{2(\rho-\beta)}{\alpha} > 0 \iff \rho < -\beta + \frac{\gamma}{2},$$

which we assumed to hold. We see here that the threshold $\rho < \gamma/2 - \beta$ is due to integrability of the previous singularity, while the constraint $\rho > -\beta$ comes from the above use of a duality inequality.

Using a Gronwall-Volterra lemma, we get

$$\tilde g^m_{s,x}(t) \lesssim 1 + \|b^m\|_{L^r - \mathbb{B}^\beta_{p,q}} \lesssim 1,$$

using as well the bound $\|b^m\|_{L^r - \mathbb{B}^\beta_{p,q}} \lesssim \|b\|_{L^r - \mathbb{B}^\beta_{p,q}}$ from Proposition 2.1 and we thus obtain the uniform boundedness of $\tilde g^m_{s,x}(t)$.

From the definition of $g^m_{s,x}(t,\cdot)$ we obtain the upper bound of (3.3.2) and (3.3.4). To obtain the lower bound of (3.3.2), it suffices to write

$$g^m_{s,x}(t,y) \geq C - \frac{1}{p_\alpha(t-s,y-x)} \left| \int_s^t \int \frac{\Gamma^m(u-s,z-x)}{p_\alpha(u-s,z-x)} b^m(u,z) p_\alpha(u-s,z-x) \nabla p_\alpha(t-u,y-z) \, \mathrm{d}z \, \mathrm{d}u \right| \tag{3.3.12}$$

and to follow the same steps.

For items (3.3.3) and (3.3.5), it suffices to notice that the whole proof remains the same if we add a derivative w.r.t. the initial value x, and using (2.2.2) to account for the gradient at the end. Namely, we would have the following Duhamel-type expansion:

$$\nabla_x \Gamma^m(s,x,t,y) = \nabla_x p_\alpha(t-s,y-x) + \int_s^t \int \nabla_x \Gamma^m(s,x,u,z) b^m(u,z) \nabla_z p_\alpha(t-u,y-z) \, \mathrm{d}z \, \mathrm{d}u.$$

In turn, this means we have to study

$$G^m_{s,x}(t,y) := (t-s)^{\frac{1}{\alpha}} \frac{\nabla_x \Gamma^m(s,x,t,y)}{p_\alpha(t-s,y-x)}.$$

Computations then remain the same as in this section, up to a factor $\frac{(t-s)^{1/\alpha}}{(u-s)^{1/\alpha}}$ that will disappear through time integration when using the Gronwall-Volterra lemma. To be precise, it exactly adds $-r'/\alpha$ to the exponent of $(u-s)$ in (3.3.11). Importantly, the condition allowing the integral to converge remains the same. Denoting $\tilde G^m_{s,x}(t) := \|G^m_{s,x}(t,\cdot)\|_{L^\infty} + (t-s)^{\frac{\rho}{\alpha}} \mathcal{T}^\rho_{\infty,\infty}[G^m_{s,x}(t,\cdot)]$, this means we obtain the uniform in m boundedness of $\tilde G^m_{s,x}(t)$, hence (3.3.3) and (3.3.5).

Let us turn to the proof of (3.3.6). For $t' > t$ such that $t' - t \le t/2$, using the duality inequality (2.1.3) and the product rule (2.1.2), we have

$$
\begin{aligned}
&|\Gamma^m(s,x,t,y) - \Gamma^m(s,x,t',y)| \le |p_\alpha(t-s,y-x) - p_\alpha(t'-s,y-x)| \\
&\quad + \int_s^t \left| \int \Gamma^m(s,x,u,z) b^m(u,z) \left[\nabla_z p_\alpha(t-u,y-z) - \nabla_z p_\alpha(t'-u,y-z)\right] \mathrm{d}z \right| \mathrm{d}u \\
&\quad + \int_t^{t'} \left| \int \Gamma^m(s,x,u,z) b^m(u,z) \nabla_z p_\alpha(t'-u,y-z) \, \mathrm{d}z \right| \mathrm{d}u \\
&\lesssim |p_\alpha(t-s,y-x) - p_\alpha(t'-s,y-x)| \\
&\quad + \int_s^t \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \left\| g^m_{s,x}(u,\cdot) \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \|p_\alpha(u-s,\cdot-x) \left[\nabla_z p_\alpha(t-u,y-\cdot) - \nabla_z p_\alpha(t'-u,y-\cdot)\right]\|_{\mathbb{B}^{-\beta}_{p',q'}} \, \mathrm{d}u \\
&\quad + \int_t^{t'} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \left\| g^m_{s,x}(u,\cdot) \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \|p_\alpha(u-s,\cdot-x) \nabla_z p_\alpha(t'-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \, \mathrm{d}u.
\end{aligned}
$$

Using the Hölder modulus of the stable kernel, (2.2.4), for the first term and the Besov estimates (2.2.13) and (2.2.11) for the second and third terms respectively

$$
\begin{aligned}
&|\Gamma^m(s,x,t,y) - \Gamma^m(s,x,t',y)| \lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}} (p_\alpha(t-s,y-x) + p_\alpha(t'-s,y-x)) \\
&\quad + \int_s^t \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \frac{\tilde{g}^m_{s,x}(u)}{(u-s)^{\frac{\rho}{\alpha}}} \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}} (t'-s)^{\frac{\beta+\rho}{\alpha}}}{(t-u)^{\frac{1+\gamma-\varepsilon}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t-u)^{-\frac{d}{\alpha p}}\right] \\
&\qquad \times (p_\alpha(t-s,y-x) + p_\alpha(t'-s,y-x)) \, \mathrm{d}u \\
&\quad + \int_t^{t'} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \frac{\tilde{g}^m_{s,x}(u)}{(u-s)^{\frac{\rho}{\alpha}}} \frac{(t'-s)^{\frac{\beta+\rho}{\alpha}}}{(t'-u)^{\frac{1}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t'-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t'-u)^{-\frac{d}{\alpha p}}\right] \\
&\qquad \times p_\alpha(t'-s,y-x) \, \mathrm{d}u.
\end{aligned}
$$

Recalling that we just proved the (uniform in m) boundedness of $g^m_{s,x}(u)$, we get, for any $\varepsilon \in (0,-\beta)$ designed to be small,

$$
\begin{aligned}
&|\Gamma^m(s,x,t,y) - \Gamma^m(s,x,t',y)| \lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}} (p_\alpha(t-s,y-x) + p_\alpha(t'-s,y-x)) \Bigg(1 \\
&\quad + (t-s)^{\frac{\gamma-\varepsilon}{\alpha}} \left(\int_s^t \left(\frac{(t'-s)^{\frac{\beta+\rho}{\alpha}}}{(u-s)^{\frac{\rho}{\alpha}} (t-u)^{\frac{1+\gamma-\varepsilon}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t-u)^{-\frac{d}{\alpha p}}\right] \right)^{r'} \mathrm{d}u \right)^{\frac{1}{r'}} \\
&\quad + \frac{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}} \left(\int_t^{t'} \left(\frac{(t'-s)^{\frac{\beta+\rho}{\alpha}}}{(u-s)^{\frac{\rho}{\alpha}} (t'-u)^{\frac{1}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t'-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t'-u)^{-\frac{d}{\alpha p}}\right] \right)^{r'} \mathrm{d}u \right)^{\frac{1}{r'}} \Bigg)
\end{aligned}
$$

The first integral converges if and only if

$$
1 - \frac{r'}{\alpha}\left(1 + \gamma - \varepsilon + \rho + \frac{d}{p}\right) > 0 \qquad \text{and} \qquad 1 - \frac{r'}{\alpha}\left(2\rho + \frac{d}{p}\right) > 0, \tag{3.3.13}
$$

where

$$
\gamma = \alpha - 1 - \frac{\alpha}{r} - \frac{d}{p} + 2\beta,
$$

in which case

$$\left(\int_s^t \left(\frac{(t'-s)^{\frac{\beta+\rho}{\alpha}}}{(u-s)^{\frac{\rho}{\alpha}}(t-u)^{\frac{1+\gamma-\varepsilon}{\alpha}}}\left[(u-s)^{-\frac{\rho}{\alpha}}+(t-u)^{-\frac{\rho}{\alpha}}\right]\left[(u-s)^{-\frac{d}{\alpha p}}+(t-u)^{-\frac{d}{\alpha p}}\right]\right)^{r'} \mathrm{d}u\right)^{\frac{1}{r'}}$$
$$\lesssim (t'-s)^{1-\frac{1}{r}+\frac{\beta-\rho-1-\gamma+\varepsilon-\frac{d}{p}}{\alpha}}$$
$$\lesssim (t'-s)^{\frac{\varepsilon-(\beta+\rho)}{\alpha}}.$$

The second integral converges if and only if

$$1-\frac{r'}{\alpha}\left(1+\rho+\frac{d}{p}\right)>0 \qquad \text{and} \qquad 1-\frac{r'}{\alpha}\left(2\rho+\frac{d}{p}\right)>0, \tag{3.3.14}$$

in which case

$$\left(\int_t^{t'} \left(\frac{(t'-s)^{\frac{\beta+\rho}{\alpha}}}{(u-s)^{\frac{\rho}{\alpha}}(t'-u)^{\frac{1}{\alpha}}}\left[(u-s)^{-\frac{\rho}{\alpha}}+(t'-u)^{-\frac{\rho}{\alpha}}\right]\left[(u-s)^{-\frac{d}{\alpha p}}+(t'-u)^{-\frac{d}{\alpha p}}\right]\right)^{r'} \mathrm{d}u\right)^{\frac{1}{r'}}$$
$$\lesssim (t'-s)^{\frac{\beta+\rho}{\alpha}}(t'-t)^{1-\frac{1}{r}-\frac{1}{\alpha}-\frac{d}{\alpha p}-2\frac{\rho}{\alpha}}$$
$$\lesssim (t'-s)^{\frac{\beta+\rho}{\alpha}}(t'-t)^{\frac{\gamma-2(\beta+\rho)}{\alpha}}.$$

Notice that choosing $\rho \in (-\beta, -\beta+\varepsilon/2)$, conditions (3.3.13) and (3.3.14) are fullfilled. Bounding non-negative powers of $t-s$, $t'-t$ and $t'-s$ by powers of T that then get absorbed in the underlying inequality constant, we get

$$|\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t',y)| \lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}\left(p_\alpha(t-s,y-x)+p_\alpha(t'-s,y-x)\right)$$
$$\times\left(1+(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}(t'-s)^{\frac{\varepsilon-(\beta+\rho)}{\alpha}}+\frac{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}(t'-s)^{\frac{\beta+\rho}{\alpha}}(t'-t)^{\frac{\gamma-2(\beta+\rho)}{\alpha}}\right)$$
$$\lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}\left(p_\alpha(t-s,y-x)+p_\alpha(t'-s,y-x)\right)\left(1+(t'-t)^{\frac{\varepsilon-2\beta-\rho}{\alpha}}\right)$$
$$\lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}\left(p_\alpha(t-s,y-x)+p_\alpha(t'-s,y-x)\right).$$

Recalling that we are in the regime $t'-t \le t/2$, we obtain

$$\left\|\frac{\Gamma^m(s,x,t,\cdot)-\Gamma^m(s,x,t',\cdot)}{p_\alpha(t-s,\cdot-x)}\right\|_{L^\infty} \lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}. \tag{3.3.15}$$

Let us now turn to the thermic part of $\left\|\frac{\Gamma^m(s,x,t,\cdot)-\Gamma^m(s,x,t',\cdot)}{p_\alpha(t-s,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}$. Again, it can be estimated from the

ρ-Hölder modulus of the same function. Let us write, for $(y,y')\in(\mathbb{R}^d)^2$ such that $|y-y'|\le(t-s)^{\frac{1}{\alpha}}$,

$$\begin{aligned}
&|\Gamma^m(s,x,t,y)-\Gamma^m(s,x,t',y)-\Gamma^m(s,x,t,y')+\Gamma^m(s,x,t',y')|\\
&\quad\le\left|\int_0^1\left(\nabla p_\alpha(t-s,y'+\lambda(y-y')-x)-\nabla p_\alpha(t'-s,y'+\lambda(y-y')-x)\right)(y'-y)\,\mathrm{d}\lambda\right|\\
&\qquad+\int_s^t\mathbb{1}_{|y-y'|\le(t-u)^{\frac{1}{\alpha}}}\left|\int\Gamma^m(s,x,u,z)b^m(u,z)\int_0^1\big[\nabla^2p_\alpha(t-u,y'+\lambda(y-y')-z)\right.\\
&\qquad\qquad\left.-\nabla^2p_\alpha(t'-u,y'+\lambda(y-y')-z)\big](y'-y)\,\mathrm{d}\lambda\,\mathrm{d}z\right|\mathrm{d}u\\
&\qquad+\int_s^t\mathbb{1}_{|y-y'|\ge(t-u)^{\frac{1}{\alpha}}}\left|\int\Gamma^m(s,x,u,z)b^m(u,z)[\nabla p_\alpha(t-u,y-z)-\nabla p_\alpha(t-u,y'-z)\right.\\
&\qquad\qquad\left.-\nabla p_\alpha(t'-u,y-z)+\nabla p_\alpha(t'-u,y'-z)]\,\mathrm{d}z\right|\mathrm{d}u\\
&\qquad+\int_t^{t'}\left|\int\Gamma^m(s,x,u,z)b^m(u,z)\left[\nabla_zp_\alpha(t'-u,y-z)-\nabla_zp_\alpha(t'-u,y'-z)\right]\mathrm{d}z\right|\mathrm{d}u\\
&=:T_1+T_2+T_3+T_4.
\end{aligned}$$

Using the Hölder modulus of the stable kernel, (2.2.4), we have

$$T_1\lesssim\frac{|y-y'|}{(t-s)^{\frac{1}{\alpha}}}\frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}p_\alpha(t-s,y-x)\lesssim\frac{|y-y'|^\rho(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon}{\alpha}}}p_\alpha(t-s,y-x).\tag{3.3.16}$$

Similarly, for T_2, using the Besov estimate on the convolution of stable kernels (2.2.13) and using the current regime to raise $|y-y'|/(t-u)^{1/\alpha}$ to the power ρ, we have

$$\begin{aligned}
T_2&\lesssim\int_s^t\mathbb{1}_{|y-y'|\le(t-u)^{\frac{1}{\alpha}}}\int_0^1\left|\int\Gamma^m(s,x,u,z)b^m(u,z)\big(\nabla^2p_\alpha(t-u,y'+\lambda(y-y')-z)\right.\\
&\qquad\left.-\nabla^2p_\alpha(t'-u,y'+\lambda(y-y')-z)\big)(y'-y)\,\mathrm{d}z\,\mathrm{d}\lambda\right|\mathrm{d}u\\
&\lesssim|y-y'|\int_s^t\mathbb{1}_{|y-y'|\le(t-u)^{\frac{1}{\alpha}}}\left\|\frac{\Gamma^m(s,x,u,\cdot)}{p_\alpha(u-s,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}\|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\int_0^1\\
&\quad\times\|p_\alpha(u-s,\cdot-x)\big(\nabla^2p_\alpha(t-u,y'+\lambda(y-y')-\cdot)-\nabla^2p_\alpha(t'-u,y'+\lambda(y-y')-\cdot)\big)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}\lambda\,\mathrm{d}u\\
&\lesssim|y-y'|^\rho\int_0^1p_\alpha(t-s,y'+\lambda(y-y')-x)\,\mathrm{d}\lambda\\
&\quad\times\int_s^t\|b(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{(t-s)^{\frac{\beta+\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(u-s)^{\frac{\rho}{\alpha}}(t-u)^{\frac{\gamma-\varepsilon+1+\rho}{\alpha}}}\left[(u-s)^{-\frac{\rho}{\alpha}}+(t-u)^{-\frac{\rho}{\alpha}}\right]\left[(u-s)^{-\frac{d}{\alpha p}}+(t-u)^{-\frac{d}{\alpha p}}\right]\mathrm{d}u\\
&\lesssim|y-y'|^\rho p_\alpha(t-s,y-x)(t-s)^{\frac{\beta+\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}\\
&\quad\times\left(\int_s^t\left(\frac{1}{(u-s)^{\frac{\rho}{\alpha}}(t-u)^{\frac{\gamma-\varepsilon+1+\rho}{\alpha}}}\left[(u-s)^{-\frac{\rho}{\alpha}}+(t-u)^{-\frac{\rho}{\alpha}}\right]\left[(u-s)^{-\frac{d}{\alpha p}}+(t-u)^{-\frac{d}{\alpha p}}\right]\right)^{r'}\mathrm{d}u\right)^{\frac{1}{r'}}.
\end{aligned}$$

This integral converges if and only if

$$1-\frac{r'}{\alpha}\left(\gamma-\varepsilon+1+2\rho+\frac{d}{p}\right)>0\qquad\text{and}\qquad1-\frac{r'}{\alpha}\left(2\rho+\frac{d}{p}\right)>0,$$

which imposes $\rho\in(-\beta,-\beta+\varepsilon/2)$. This yields

$$\begin{aligned}
T_2&\lesssim|y-y'|^\rho p_\alpha(t-s,y-x)(t-s)^{\frac{\beta+\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}(t-s)^{\frac{\varepsilon-2(\rho+\beta)}{\alpha}-\frac{\rho}{\alpha}}\\
&\lesssim|y-y'|^\rho p_\alpha(t-s,y-x)(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}(t-s)^{\frac{\beta}{\alpha}}.
\end{aligned}\tag{3.3.17}$$

For T_3, let us write, using (**??**) twice along with the fact that $|y-y'|^\rho/(t-u)^{\rho/\alpha} > 1$,

$$
\begin{aligned}
T_3 \lesssim \int_s^t \frac{|y-y'|^\rho}{(t-u)^{\frac{\rho}{\alpha}}} \left\| \frac{\Gamma^m(s,x,u,\cdot)}{p_\alpha(u-s,\cdot-x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \Big(\|p_\alpha(u-s,\cdot-x)\big[\nabla p_\alpha(t-u,y-z) - p_\alpha(t'-u,y-z)\big]\|_{\mathbb{B}^{-\beta}_{p',q'}} \\
+ \|p_\alpha(u-s,\cdot-x)\big[\nabla p_\alpha(t-u,y'-z) - p_\alpha(t'-u,y'-z)\big]\|_{\mathbb{B}^{-\beta}_{p',q'}} \Big) \mathrm{d}u \\
\lesssim \int_s^t \frac{|y-y'|^\rho}{(t-u)^{\frac{\rho}{\alpha}}} \frac{\tilde{g}^m_{s,x}(u)}{(u-s)^{\frac{\rho}{\alpha}}} \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}(t'-s)^{\frac{\beta+\rho}{\alpha}}}{(t-u)^{\frac{1+\gamma-\varepsilon}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t-u)^{-\frac{d}{\alpha p}}\right] \\
\times (p_\alpha(t-s,y-x) + p_\alpha(t'-s,y-x) + p_\alpha(t-s,y'-x) + p_\alpha(t'-s,y'-x)).
\end{aligned}
$$

This integral is exactly the same which appeared in the previous computation, and we get

$$
T_3 \lesssim |y-y'|^\rho p_\alpha(t-s,y-x)(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}(t-s)^{\frac{\beta}{\alpha}}. \tag{3.3.18}
$$

For T_4, using (2.2.9), we have

$$
\begin{aligned}
T_4 &\lesssim \int_t^{t'} \left\| \frac{\Gamma^m(s,x,u,\cdot)}{p_\alpha(u-s,\cdot-x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \\
&\qquad \times \|p_\alpha(u-s,\cdot-x)\left[\nabla_z p_\alpha(t'-u,y-z) - \nabla_z p_\alpha(t'-u,y'-z)\right]\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u \\
&\lesssim \int_t^{t'} \|b(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} (p_\alpha(t'-s,y-x) + p_\alpha(t'-s,y'-x)) \\
&\qquad \times \frac{|y-y'|^\rho (t'-s)^{\frac{\beta+\rho}{\alpha}}}{(u-s)^{\frac{\rho}{\alpha}}(t'-u)^{\frac{1+\rho}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t-u)^{-\frac{d}{\alpha p}}\right] \mathrm{d}u.
\end{aligned}
$$

This integral converges if and only if

$$
1 - \frac{r'}{\alpha}\left(1 + 2\rho + \frac{d}{p}\right) > 0 \qquad \text{and} \qquad 1 - \frac{r'}{\alpha}\left(2\rho + \frac{d}{p}\right) > 0,
$$

which holds true for $\rho \in (-\beta, -\beta+\varepsilon/2)$. Recalling that $t'-t < t/2$ so that $(t'-s)^{\frac{\beta+\rho}{\alpha}} \lesssim (t-s)^{\frac{\beta+\rho}{\alpha}}$, we get

$$
\begin{aligned}
T_4 &\lesssim \int_t^{t'} \|b(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} (p_\alpha(t'-s,y-x) + p_\alpha(t'-s,y'-x)) \\
&\qquad \times \frac{|y-y'|^\rho (t'-s)^{\frac{\beta+\rho}{\alpha}}}{(t-s)^{\frac{\rho}{\alpha}}(t'-u)^{\frac{1+\rho}{\alpha}}} \left[(u-s)^{-\frac{\rho}{\alpha}} + (t-u)^{-\frac{\rho}{\alpha}}\right] \left[(u-s)^{-\frac{d}{\alpha p}} + (t-u)^{-\frac{d}{\alpha p}}\right] \mathrm{d}u \\
&\lesssim |y-y'|^\rho p_\alpha(t'-s,y-x)(t'-s)^{\frac{\beta+\rho}{\alpha}}(t-s)^{-\frac{\rho}{\alpha}}(t'-t)^{\frac{\gamma-2(\beta+\rho)}{\alpha}} \\
&\lesssim |y-y'|^\rho p_\alpha(t-s,y-x)(t-s)^{\frac{\beta}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}(t'-t)^{\frac{\varepsilon-2(\beta+\rho)}{\alpha}} \\
&\lesssim |y-y'|^\rho p_\alpha(t-s,y-x)(t-s)^{\frac{\beta}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}.
\end{aligned} \tag{3.3.19}
$$

Gathering (3.3.16), (3.3.17), (3.3.18) and (3.3.19), we obtain

$$
\mathcal{T}^\rho_{\infty,\infty}\left[\frac{\Gamma^m(s,x,t,\cdot) - \Gamma^m(s,x,t',\cdot)}{p_\alpha(t-s,\cdot-x)}\right] \lesssim \frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon-\beta}{\alpha}}}. \tag{3.3.20}
$$

Along with (3.3.15), we obtain (3.3.6) for any $\varepsilon \in (0,-\beta)$ and for any $\rho \in (-\beta, -\beta+\varepsilon/2)$.

3.4 From the smooth approximation to the actual SDE

By Proposition 2.1, let $(b^m)_{m\in\mathbb{N}}$ be a sequence of smooth bounded functions s.t.

$$
\|b - b^m\|_{L^{\tilde{r}}-\mathbb{B}^{\tilde{\beta}}_{p,q}} \underset{m\to\infty}{\longrightarrow} 0, \qquad \forall \tilde{\beta} < \beta,
$$

with $\tilde r = r$ if $r < \infty$ and for any $\tilde r < \infty$ otherwise and let $\kappa \geq 1$:

$$\sup_{m\in\mathbb{N}} \|b^m\|_{L^{\tilde r}-\mathbb{B}^{\beta}_{p,q}} \leq \kappa \|b\|_{L^{\tilde r}-\mathbb{B}^{\beta}_{p,q}}.$$

The following was already discussed in [CdRM22a], but we reproduce it here for the sake of completeness.

Tightness of the sequence of probability measures $(\mathbb{P}^m)_{m\in\mathbb{N}}$

Notice that when considering the *mollified* equation (3.3.1), for every m, the martingale problem associated with $(b^m, \mathcal{L}^\alpha, x)$ is well posed (see [CdRM22a]). Let us denote by $\mathbb{P}^m$ its solution and by $(x_t^m)_{t\geq 0}$ the associated canonical process. Let $u_m = (u_m^1, ..., u_m^d)$ where, $\forall i, u_m^i$ is a mild solution of the classical Cauchy problem $\mathcal{C}(b^m, \mathcal{L}^\alpha, -b_i^m, 0, T)$ (i.e. with terminal condition $u_m^i(T,\cdot) = 0$ and source term $-b_i^m$, the i^{th} component of b^m), so that

$$\left(u_m(t, x_t^m) + \int_0^t b^m(s, x_s^m)\,\mathrm{d}s - u(0,x)\right)_{0\leq t\leq T}$$

is a $\mathbb{P}^m$-martingale, which we can express, through Itô's formula, as

$$M_{v,s}(u_m, x^m) := \int_v^s \int_{\mathbb{R}^d\setminus\{0\}} [u_m(r, x_{r^-}^m + x) - u_m(r, x_{r^-}^m)]\,\tilde N(\,\mathrm{d}r,\,\mathrm{d}x), \qquad \forall s \geq v, \tag{3.4.1}$$

where $\tilde N$ is the compensated Poisson measure. Itô's formula now writes

$$x_s^m - x_v^m = M_{v,s}(u_m, x^m) + Z_s - Z_v - [u_m(s, x_s^m) - u_m(v, x_v^m)]\,. \tag{3.4.2}$$

We will use an Aldous criterion to prove the tightness of $(\mathbb{P}^m)_{m\in\mathbb{N}}$, which means we need a control of the form $\mathbb{E}[|X_s^m - X_v^m|^p] \leq c(s-v)^\zeta$ for some $p > 0$ and some $\zeta > 0$ (see Proposition 34.9 from [Bas11]). Since $\forall i, u_m^i$ is the mild solution of the Cauchy problem $\mathcal{C}(b^m, \mathcal{L}^\alpha, -b_m^i, 0, T)$, we can write

$$|u_m(v, x_v^m) - u_m(s, x_s^m)| \leq |u_m(v, x_v^m) - u_m(v, x_s^m)| + |u_m(v, x_s^m) - u_m(s, x_s^m)|, \tag{3.4.3}$$

and use Proposition 9 from [CdRM22a] to get the required space and time controls. Namely, for the spatial part, $\exists C_T$ s.t. $C_T \to 0$ as $T \to 0$ and $|u_m(v, x_v^m) - u_m(v, x_s^m)| < C_T|x_v^m - x_s^m|$. For the time part, we use the Hölder continuity in time of u_m. For $M_{v,s}(u_m, x^m)$, the control follows from the Burkholder–Davis–Gundy inequality and, finally, for $Z_s - Z_v$, it follows from (2.2.3) and the stationarity of Z.

Limit probability measure

We will now prove that any limit probability measure $\mathbb{P}$ is a martingale solution to (**??**) in the sense of Definition 3.2. Let $f \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d, \mathbb{R}))$, $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ with $Dg \in \mathbb{B}^{\theta-1}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Let $u_m \in \mathcal{C}^{0,1}([0,T]\times\mathbb{R}^d)$ be the classical solution of the *mollified* Cauchy problem $\mathcal{C}(b^m, \mathcal{L}^\alpha, f, g, T)$, with $Du_m \in \mathcal{C}_b^0([0,T], \mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty})$ for some $0 < \varepsilon \ll 1$. By Theorem 3.2, we have a uniform control of the modulus of continuity of u_m and Du_m. By the Arzelà-Ascoli Theorem, we can extract a subsequence $(u_{m_k}, Du_{m_k})_k$ s.t. $(u_{m_k})_k$ and $(Du_{m_k})_k$ converge uniformly on every compact subsets of $[0,T]\times\mathbb{R}^d$ to some functions $u \in \mathcal{C}^{0,1}([0,T]\times\mathbb{R}^d)$ and $Du \in \mathcal{C}_b^0([0,T], \mathbb{B}^{\theta-1-\epsilon}_{\infty,\infty}), \forall\epsilon \in (0,\varepsilon)$ respectively (Du being the space-derivative of u). Because of this uniform convergence, (3.1.4) holds for the limit, i.e.

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d, u(t,x) = P^\alpha_{T-t}[g](x) - \int_t^T P^\alpha_{s-t}[f - b\cdot Du](s,x)\,\mathrm{d}s, \tag{3.4.4}$$

hence u is a mild solution to $\mathcal{C}(b, \mathcal{L}^\alpha, f, g, T)$. Together with a control of the moments of X^m (which we already obtained in the last paragraph), we deduce that

$$\left(u(t, x_t) + \int_0^t f(s, x_s)\,\mathrm{d}s - u(0, x_0)\right)_{0\leq t\leq T}$$

is a $\mathbb{P}$-martingale.

Uniqueness of the limit probability measure

Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two solution of the martingale problem associated with $(b, \mathcal{L}^\alpha, x_0)$ for some $x_0 \in \mathbb{R}^d$. Thus, $\forall f \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d, \mathbb{R}))$, taking $g = 0$,

$$u(0, x_0) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(s, x_s)\,\mathrm{d}s\right] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^T f(s, x_s)\,\mathrm{d}s\right],$$

which is sufficient to prove uniqueness in law (see e.g. [EK86]).

Since $X_t^m = x_t^m$, Γ^m is the density of the canonical process under $\mathbb{P}^m$. From the Arzelà-Ascoli theorem which can be applied from the estimates derived in Theorem 3.2, we can extract a subsequence $(\Gamma^{m_k}, \nabla_x \Gamma^{m_k})_k$ s.t. $(\Gamma^{m_k})_k$ and $(\nabla_x \Gamma^{m_k})_k$ converge uniformly on every compact subset to some functions Γ and $\nabla_x \Gamma$ ($\nabla_x \Gamma$ being indeed the gradient of Γ). By the uniqueness results from [CdRM22a], Γ is the time marginal of $\mathbb{P}$, and enjoys the estimates of Theorem 3.1.

□

Part III

Discretization schemes for singular drift SDEs

Chapter 4

Weak Error on the densities for the Euler scheme of stable additive SDEs with Holder drift

This chapter is based on the article [FM24], written with Stéphane Menozzi[1] and published in Stochastic Processes and their Applications. Therein, we are interested in the Euler-Maruyama dicretization of the SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d,$$

where Z_t is a symmetric isotropic d-dimensional α-stable process, $\alpha \in (1, 2]$ and the drift $b \in L^\infty\left([0, T], \mathcal{C}^\beta(\mathbb{R}^d, \mathbb{R}^d)\right)$, $\beta \in (0, 1)$, is bounded and Hölder regular in space. Using an Euler scheme with a randomization of the time variable, we show that, denoting $\gamma := \alpha + \beta - 1$, the weak error on densities related to this discretization converges at the rate γ/α.

4.1 Introduction

For a fixed finite time horizon $T > 0$, we are interested in the Euler-Maruyama dicretization of the SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x, \qquad \forall t \in [0, T], \tag{4.1.1}$$

where Z_t is a symmetric isotropic d-dimensional α-stable process, $\alpha \in (1, 2]$ and $b \in L^\infty\left([0, T], \mathcal{C}^\beta(\mathbb{R}^d, \mathbb{R}^d)\right)$, $\beta \in (0, 1)$, i.e. it is bounded and Hölder regular in space. In this setting, weak well-posedness holds for (4.1.1) since the *natural* condition

$$\gamma := \beta + \alpha - 1 > 0 \iff \beta + \alpha > 1, \tag{4.1.2}$$

is always satisfied. The condition (4.1.2) actually ensures weak well-posedness for the SDE (4.1.1), even in the super-critical case $\alpha \in (0, 1]$, provided the drift is time homogeneous or bounded in time (see [TTW74], [MP14], [CZZ21], see also [Pri12], [CZZ21] for strong well-posedness established under the more stringent condition $\beta + \alpha/2 > 1$).

The goal of this paper is to prove a convergence rate for the weak error on densities associated with an *appropriate* Euler scheme for (4.1.1).

4.1.1 Definition of the scheme

We will use a discretization scheme with n time steps over $[0, T]$, with constant step size $h := T/n$. For the rest of this paper, we denote, $\forall k \in \{0, ..., n\}, t_k := kh$ and $\forall s > 0, \tau_s^h := h\lfloor \frac{s}{h} \rfloor \in (s - h, s]$, which is the last

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grid point before time s. Namely, if $s \in [t_k, t_{k+1}), \tau_s^h = t_k$.

We define a step of the Euler scheme, starting from $X_0 = x$, as

$$X^h_{t_{k+1}} = X^h_{t_k} + hb(U_k, X^h_{t_k}) + (Z_{t_{k+1}} - Z_{t_k}),\ k \in \mathbb{N}, \tag{4.1.3}$$

where the $(U_k)_{k\in\mathbb{N}}$ are independent random variables, independent as well from the driving noise, s.t. $U_k \overset{\text{(law)}}{=} \mathcal{U}([t_k, t_{k+1}])$, i.e. U_k is uniform on the time interval $[t_k, t_{k+1}]$. We consider the corresponding time interpolation defined as the solution to

$$\mathrm{d}X^h_t = b(U_{\tau^h_t/h}, X^h_{\tau^h_t})\,\mathrm{d}t + \,\mathrm{d}Z_t. \tag{4.1.4}$$

As b is bounded, equation (4.1.4) is well-posed and X^h_t admits a density for $t > 0$. We can refer to this end to [FJM24] for related estimates. We will denote by $\Gamma^h(0, x, t, \cdot)$ this density at time $t \in (0, T]$ when starting from x at time 0.

4.1.2 Euler scheme - state of the art

For all $0 \le s < t \le T$, it is known that the unique weak solution to (4.1.1), starting in x at time s admits a density, which we will denote $\Gamma(s, x, t, \cdot)$. It has as well been established in [MZ22] that in the current setting Γ enjoys two-sided stable heat kernel estimates for $\alpha \in (1, 2)$ whereas this property can already be derived from Friedman [Fri64] (under some additional smoothness in time for b) or [MPZ21] in the Brownian case. In this paper, we are interested in the weak error on densities, which is defined as the quantity

$$|\Gamma(s, x, t, y) - \Gamma^h(s, x, t, y)|. \tag{4.1.5}$$

In particular we want to bound it, up to a multiplicative constant, by the product of an appropriate power of the time step h and a density which provides an upper bound for the one of the driving noise. This would then in particular allow to integrate against possibly irregular test functions having the corresponding convergence rate.

The general definition of the weak error is

$$\mathcal{E}(f, t, x, h) := \mathbb{E}_{0,x}\left[f(X^h_t) - f(X_t)\right], \tag{4.1.6}$$

for f belonging to a suitable class of test functions, and where the meaning of the expectation subscript for the rest of the paper is $\mathbb{E}_{0,x}[\cdot] := \mathbb{E}[\cdot|X^h_0 = X_0 = x]$.

Deriving convergence results for the weak error involves studying the PDE

$$(\partial_s + b(s, x) \cdot \nabla_x + \mathcal{L}^\alpha)\, u(s, x) = 0 \text{ on } [0, T) \times \mathbb{R}^d, \qquad u(T, \cdot) = f \text{ on } \mathbb{R}^d, \tag{4.1.7}$$

where $\mathcal{L}^\alpha$ is the generator of the noise. When the coefficients of (4.1.1) and the test function f are smooth, the seminal paper of Talay and Tubaro ([TT90]) gives a convergence rate of order 1 in h in the Brownian case. Similar results were obtained for the densities in [KM02] and [KM10] respectively in the Brownian and pure-jump settings. With β-Hölder coefficients and again a smooth f in (4.1.6), the work of Mikulevicius and Platen ([MP91]) proves a convergence in $h^{\frac{\beta}{2}}$ in the Brownian case. This result was extended to densities in [KM17]. In these works, when applying Itô's formula, authors use the regularity of the drift to treat terms of the form $b(r, X^h_r) - b(U_{\tau^h_r/h}, X^h_{\tau^h_r})$ but do not exploit the full parabolic bootstrap associated with the PDE (4.1.7). Note that this approach intrinsically leads to a *strong* convergence order and that all the previously quoted results are for an SDE with multiplicative noise which is as well β-Hölder continuous in space. In that setting, we believe that the rate is sharp. However, for an additive noise as in (4.1.1) (or a multiplicative noise with *smooth* diffusion coefficient), the rate can be significantly improved.

In the Brownian setting, one way to proceed is to use the stochastic sewing lemma introduced in [Lê20], which allows to quantify the discretization error along rough functionals of the Brownian path. In the specific case of a β-Hölder continuous drift and terminal condition f, in the work [Hol24], the author improves the convergence rate from [MP91] to $h^{\frac{\beta+1}{2}-\varepsilon}$, $\varepsilon > 0$. We aim to extend this result to the pure-jump setting $\alpha \in (1, 2]$ and to a more general class of test functions by working on densities achieving as well $\varepsilon = 0$. Let us also mention the work [LL21], which proves a *strong* (i.e. on trajectories) rate of convergence of order 1/2 (up to a logarithmic factor) in the Brownian setting for $L^q_t - L^p_x$ drifts under the Krylov-Röckner type condition $d/p + 2/q < 1$. However, the use of stochastic sewing techniques still does not allow to take full advantage of the parabolic bootstrap associated with the fundamental solution of (4.1.7) when the test function is rough, e.g. Dirac masses leading to the weak error on densities.

In the current work, we precisely focus on these types of errors of the form $\mathcal{E}(\delta_y, t, x, h)$ (where δ_y is the Dirac mass at point y). From Itô's formula, (4.1.6) and (4.1.7), this formally writes

$$\mathcal{E}(\delta_y, t, x, h) = \mathbb{E}_{0,x}\left[\int_0^t \left(b(r, X^h_r) - b(U_{\tau^h_r/h}, X^h_{\tau^h_r}))\right) \cdot \nabla_z \Gamma(r, z, t, y)|_{z=X^h_r} \, \mathrm{d}r\right]. \tag{4.1.8}$$

To analyze the corresponding error, a new idea was introduced in [BJ22]. The drift was therein assumed to be merely measurable and bounded so that no rate could be *a priori* derived from the difference in (4.1.8). The point then consists in using the regularity of the solution to (4.1.7) instead of that of b. Namely, writing

$$\begin{aligned}\mathbb{E}_{0,x}[b(r, X^h_r) \cdot \nabla\Gamma(r, X^h_r, t, y) - b(r, X^h_{\tau^h_r}) \cdot \nabla\Gamma(r, X^h_{\tau^h_r}, t, y)]\\ = \int [\Gamma^h(0, x, r, z) - \Gamma^h(0, x, \tau^h_r, z)] b(r, z) \cdot \nabla\Gamma(r, z, t, y) \, \mathrm{d}z\end{aligned} \tag{4.1.9}$$

one can exploit some additional (or-bootstrapped regularity) of Γ^h in its forward time variable. Namely, it was proved in the Brownian setting of [BJ22] that for a bounded drift, this regularity was of order 1/2, which actually formally corresponds to the exponent γ/α (with γ defined in (4.1.2)) when taking $\beta = 0$. This result still holds in the current setting with $\beta \in (0, 1)$ and provides a significant improvement, corresponding to the expected regularity deriving from the parabolic bootstrap in the forward variable, when compared to the β-Hölder regularity of b in space. To handle the error from (4.1.8) one would need as well to investigate a space sensitivity of the gradient of the density Γ in its backward variable. This could as well be done by exploiting the parabolic bootstrap.

On the other hand, we also have to account for terms involving $b(r, X_r) - b(U_{\tau^h_r/h}, X_r)$. One way to achieve the expected convergence rate is to make strong assumptions on the time regularity of b: we would need $b(\cdot, z)$ to be γ/α-Hölder. Importantly, without making any assumption on the time regularity of the drift, those terms can be handled thanks to the randomization of the time argument introduced in (4.1.3), which allows for a convenient use of the Fubini theorem in the error analysis (see (4.2.10) below). This averaging procedure can somehow be seen as well as a regularization by noise phenomenon.

Let us mention that for the proofs below we will not rely on the previous expansion of the error, which we presented here in order to give an idea of the main crucial steps and tools for the error analysis, but on the Duhamel representations of the densities expanded using the density of the driving noise as proxy (see Proposition 4.3 below).

From the above techniques (forward time regularity of Γ^h and time randomization), a rate of order $\frac{\alpha-1-\left(\frac{d}{p}+\frac{\alpha}{q}\right)}{\alpha} > 0$ is derived in [JM24a] and [FJM24], respectively in the Brownian and pure-jump settings, for a Lebesgue drift in $L^q_t - L^p_x$ for the difference of the densities (4.1.5). Comparing this rate to that of [LL21], although $1/\alpha$ is lost due to the gradient in (4.1.9) (time singularity induced by the gradient of the density of the noise), one sees that the convergence rate displays explicitly the "gap to singularity" $\alpha - 1 - (d/p + \alpha/q)$ or Serrin condition in that setting (critical stable parabolic scaling in Lebesgue spaces).

In Theorem 4.1, we derive a weak error rate in $h^{\frac{\gamma}{\alpha}}$, where $\gamma := \beta + \alpha - 1$ is the corresponding "gap to singularity" in the Hölder case. Importantly, if we interpret $-\left(\frac{d}{p} + \frac{\alpha}{q}\right)$ as the regularity in the former works[2], there is continuity of the rate of the convergence w.r.t. the regularity of the drift. Continuity w.r.t. the stability index α also holds when comparing Theorem 4.1 to the results in [Hol24] (and getting rid of the ε in the rate therein), thus extending the former to a more general class of test functions and noises.

The restriction to the sub-critical case $\alpha \in (1, 2)$, for which the intensity of the driving noise somehow dominates the drift in *small time*, is here required mainly for approximation purposes. Again, in the [CZZ21] paper, weak well-posedness is obtained under (4.1.2) without the restriction $\alpha \in (1, 2)$. On the other hand, as we are interested in heat kernel estimates, it is also well known, see e.g. Kulik *et al.* [KK18], [Kul19], [MZ22], that in the super-critical regime $\alpha \in (0, 1)$ for which the Hölder regularity needs to be large enough to compensate the lower regularizing effects of the noise, considerations on some flows related to the drift in (4.1.1) are needed. These aspects become a rather difficult issue when considering associated discretization schemes (see Konakov *et al.* [KM23] in connection with stochastic algorithms of Robbins Monro type).

The paper is organized as follows: in Section 4.1.3 we specify some properties of the driving noise in (4.1.1). Section 4.1.4 is then dedicated to the statement of the main results (we give some controls on the densities of the SDE and the Euler scheme in Proposition 4.1 and the convergence rate for the weak error in Theorem 4.1). Section 4.2 is devoted to the proof of the main theorem. The proof is achieved via exploiting some additional quantitative properties of the density of the driving noise, the Duhamel representation of the densities (see Proposition 4.3) and the regularity results of Proposition 4.1 which are in turn proved in Section 4.3.

4.1.3 Driving noise and related density properties

Let us denote by $\mathcal{L}^\alpha$ the generator of the driving noise Z and $p_\alpha : \mathbb{R}_+\backslash\{0\} \times \mathbb{R}^d \to \mathbb{R}_+$ its density. In the case $\alpha = 2$, $\mathcal{L}^\alpha$ is the usual normalized Laplacian $\frac{1}{2}\Delta$. The noise is a Brownian Motion and its gaussian marginal densities are explicit.

When $\alpha \in (1, 2)$, in whole generality, the generator of a symmetric stable process writes, $\forall \phi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ (smooth compactly supported functions),

$$\begin{aligned}\mathcal{L}^\alpha \phi(x) &= \text{p.v.} \int_{\mathbb{R}^d} [\phi(x+z) - \phi(x)]\, \nu(\,\mathrm{d}z)\\ &= \text{p.v.} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} [\phi(x+\rho\xi) - \phi(x)]\, \mu(\,\mathrm{d}\xi) \frac{\mathrm{d}\rho}{\rho^{1+\alpha}}\end{aligned}$$

(see [Sat99] for the polar decomposition of the stable Lévy measure) where μ is a symmetric measure on the unit sphere $\mathbb{S}^{d-1}$. We will here restrict to the case where $\mu = m$ the Lebesgue measure on the sphere but it is very likely that the analysis below can be extended to the case where μ is symmetric and $\exists \kappa \geq 1 : \forall \lambda \in \mathbb{R}^d$,

$$C^{-1} m(\,\mathrm{d}\xi) \leq \mu(\,\mathrm{d}\xi) \leq C m(\,\mathrm{d}\xi),$$

i.e. it is equivalent to the Lebesgue measure on the sphere. Indeed, in that setting Watanabe (see [Wat07], Theorem 1.5) and Kolokoltsov ([Kol00], Propositions 2.1–2.5) showed that if $C^{-1}m(\,\mathrm{d}\xi) \leq \mu(\,\mathrm{d}\xi) \leq Cm(\,\mathrm{d}\xi)$, the following estimates hold: there exists a constant C depending only on α, d, s.t. $\forall v \in \mathbb{R}_+\backslash\{0\}, z \in \mathbb{R}^d$,

$$C^{-1} v^{-\frac{d}{\alpha}} \left(1 + \frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \leq p_\alpha(v, z) \leq C v^{-\frac{d}{\alpha}} \left(1 + \frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}. \tag{4.1.10}$$

On the other hand let us mention that the sole non-degeneracy condition

$$\kappa^{-1}|\lambda|^\alpha \leq \int_{\mathbb{S}^{d-1}} |\lambda \cdot \xi|^\alpha \mu(\,\mathrm{d}\xi) \leq \kappa|\lambda|^\alpha,$$

[2] actually this exponent naturally appears as the negative regularity parameter when embedding the time-space Lebesgue space in a Besov space with infinite integrability indexes (which can be identified with a usual Hölder space when the regularity index is positive), see e.g. [Saw18].

does not allow to derive *global* heat kernel estimates for the noise density. In [Wat07], Watanabe investigates the behavior of the density of an α-stable process in terms of properties fulfilled by the support of its spectral measure μ. From this work, we know that whenever the measure μ is not equivalent to the Lebesgue measure m on the unit sphere, accurate estimates on the density of the stable process are delicate to obtain.

From now on, and in particular in Section 4.3 which is dedicated to the proof of technical lemmas, we will be using the *proxy* notation

$$\bar{p}_\alpha(v,z) := \begin{cases} C_\alpha v^{-\frac{d}{\alpha}}\left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} & \text{if } \alpha\in(1,2) \\ (2\pi c v)^{-\frac{d}{2}}\exp\left(-c^{-1}\frac{|z|^2}{2v}\right),\ c\geq 1 & \text{if } \alpha=2, \end{cases}, \qquad v>0, z\in\mathbb{R}^d, \tag{4.1.11}$$

where, for $\alpha\in(1,2)$, C_α is chosen so that $\forall v>0, \int \bar{p}_\alpha(v,y)\,\mathrm{d}y=1$, and $c:=c(d)$ is a global given constant for $\alpha=2$. We will explicitly rely on the global bounds provided by $\bar{p}_\alpha$. Observe importantly that, keeping in mind that (4.1.10), in the pure jump case, there exists $C\geq 1$ s.t. for all $(v,z)\in\mathbb{R}_+\backslash\{0\}\times\mathbb{R}^d$,

$$C^{-1}\bar{p}_\alpha(v,z)\leq p_\alpha(v,z)\leq C\bar{p}_\alpha(v,z), \tag{4.1.12}$$

and the results could be stated with either the proxy density $\bar{p}_\alpha$ or the density p_α of the noise itself. However, the equivalence in (4.1.12) fails in the Gaussian case, due to the exponential tails. This is why the results will be stated in terms of $\bar{p}_\alpha$. Observe as well that from the definition in (4.1.11) we readily have the following important properties:

- (Approximate) convolution property: there exists a constant $\mathfrak{c}\geq 1$ s.t. for all $u,v\in\mathbb{R}_+\backslash\{0\},\ x,y\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\bar{p}_\alpha(u,z-x)\bar{p}_\alpha(v,y-z)\,\mathrm{d}z\leq \mathfrak{c}\bar{p}_\alpha(u+v,y-x). \tag{4.1.13}$$

In particular, for $\alpha=2$ the convolution is *exact* and $\mathfrak{c}=1$.

- Time-scale for the spatial moments: for all $0\leq\delta<\alpha, \alpha\in(1,2)$ and for all $\delta\geq 0$ if $\alpha=2$, there exists $C_{\alpha,\delta}$ s.t.

$$\int_{\mathbb{R}^d}|z|^\delta\bar{p}_\alpha(v,z)\,\mathrm{d}z\leq C_{\alpha,\delta}v^{\frac{\delta}{\alpha}}. \tag{4.1.14}$$

Further properties related to the density of the driving noise, notably concerning its time-space derivatives, are stated in Lemma 4.2 below.

4.1.4 Main results

We first give some important estimates concerning the densities of the SDE (4.1.1) and its associated Euler scheme (4.1.4).

Proposition 4.1 (Density estimates for the diffusion and its Euler scheme)**.** *The unique weak solution to Equation* (4.1.1) *starting from x at time $s\in[0,T]$ admits for all $t\in(s,T]$ a density $\Gamma(s,x,t,\cdot)$. Furthermore there exists a constant $C:=C(d,b,\alpha,T)$ s.t. for all $y\in\mathbb{R}^d$ the following upper-bound holds:*

$$\Gamma(s,x,t,y)\leq C\bar{p}_\alpha(t-s,y-x), \tag{4.1.15}$$

with $\bar{p}_\alpha$ defined in (4.1.11)*, as well as the following control for the Hölder regularity in the forward time variable:*

$$\forall 0\leq s<t<t'\leq T,\ |t-t'|\leq(t-s),\ |\Gamma(s,x,t,y)-\Gamma(s,x,t',y)|\leq C\frac{(t'-t)^{\frac{\gamma}{\alpha}}}{(t-s)^{\frac{\gamma}{\alpha}}}\bar{p}_\alpha(t'-s,y-x). \tag{4.1.16}$$

Also, for $\varepsilon\in(0,\gamma\wedge 1]$, there exists $C_\varepsilon:=C_\varepsilon(d,b,\alpha,T)$ s.t. forall $0\leq s<t\leq T,\ x,y,w\in\mathbb{R}^d$ s.t. $|y-w|\leq(t-s)^{\frac{1}{\alpha}}$,

$$|\Gamma(s,x,t,y)-\Gamma(s,x,t,w)|\leq C_\varepsilon\left(\frac{|y-w|}{(t-s)^{\frac{1}{\alpha}}}\right)^{\gamma_\varepsilon}\bar{p}_\alpha(t-s,w-x),\ \gamma_\varepsilon:=(\gamma\wedge 1)-\varepsilon. \tag{4.1.17}$$

Similarly, for any positive integer n and $h = \frac{T}{n}$, the corresponding Euler scheme X^h defined in (4.1.4) *starting from $x \in \mathbb{R}^d$ at time $t_k := kh, k \in [\![0, n-1]\!]$ admits for $t \in (t_k, T]$ a transition density $\Gamma^h(t_k, x, t, \cdot)$, for which, for all $y \in \mathbb{R}^d$:*

$$\Gamma^h(t_k, x, t, y) \le C\bar{p}_\alpha(t - t_k, y - x). \tag{4.1.18}$$

Also, for all $0 < t_j < t_k \le T$, $x, y, w \in \mathbb{R}^d$, $|y - w| \le (t_k - t_j)^{\frac{1}{\alpha}}$,

$$|\Gamma^h(t_j, x, t_k, y) - \Gamma^h(t_j, x, t_k, w)| \le C_\varepsilon \left(\frac{|y-w|+h^{\frac{1}{\alpha}}}{(t_k - t_j)^{\frac{1}{\alpha}}} \right)^{\gamma_\varepsilon} \bar{p}_\alpha(t_k - t_j, w - x). \tag{4.1.19}$$

Existence of the densities and the related Aronson type bounds (4.1.15) and (4.1.18) readily follow from [JM24a] and [FJM24]. The sensitivity controls (4.1.16), (4.1.17) and (4.1.19) are proven in Section 4.3.

Remark 4.1 (About additional controls on the density of the SDE and the Euler scheme)**.** *Let us point out that the densities Γ, Γ^h also satisfy additional controls. Namely, some gradient controls in the backward spatial variable could be established. Anyhow, for the error analysis, these controls are not needed. They could be derived following the approach of [FJM24].*

The main result of the paper is then the following theorem.

Theorem 4.1 (Convergence Rate for the stable-driven Euler scheme with $L^\infty_t \mathcal{C}^\beta_x$ drift)**.** *Denoting by Γ and Γ^h the respective densities of the SDE* (4.1.1) *and its Euler scheme defined in* (4.1.3)*, there exists a constant $C := C(d, b, \alpha, T) < \infty$ s.t. for all $h = T/n$ with $n \in \mathbb{N}^*$, and all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,*

$$|\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)| \le C\big(1 + t^{-\frac{\beta}{\alpha}}\big) h^{\frac{\gamma}{\alpha}} \bar{p}_\alpha(t, y - x), \tag{4.1.20}$$

where $\gamma = \beta + \alpha - 1 > 0$ is again the "gap to singularity" defined in (4.1.2)*.*

Remark 4.2 (Weak error involving an additional test function)**.** *Let us mention that if one is interested in the weak error for some test function f, $\mathcal{E}(f, x, t, h) := \mathbb{E}_{0,x}[f(X^h_t) - f(X_t)]$, as soon as f is $\delta \in [\beta, 1]$-Hölder (not necessarily bounded) then, a rate can be derived as a consequence of the convergence of $|\Gamma(s, x, t, y) - \Gamma^h(s, x, t, y)|$ using a simple cancellation argument:*

$$\begin{aligned}
\mathcal{E}(f, x, t, h) &= \int_{\mathbb{R}^d} (\Gamma^h - \Gamma)(0, x, t, y) f(y)\, \mathrm{d}y = \int_{\mathbb{R}^d} (\Gamma^h - \Gamma)(0, x, t, y)\big(f(y) - f(x)\big)\, \mathrm{d}y, \\
|\mathcal{E}(f, x, t, h)| &\le C h^{\frac{\gamma}{\alpha}} \big(1 + t^{-\frac{\beta}{\alpha}}\big) \int_{\mathbb{R}^d} \bar{p}_\alpha(t, y - x)|x - y|^\delta\, \mathrm{d}y \underset{(4.1.11)}{\le} \tilde{C}\big(1 + t^{-\frac{\beta}{\alpha}}\big) t^{\frac{\delta}{\alpha}} h^{\frac{\gamma}{\alpha}}.
\end{aligned}$$

Precisely, the smoothness of f allows to absorb the time-singularity from (4.1.20) *in small time.*

4.2 Proof of the main results

We begin this section with recalling some quantitative properties of the density of the driving noise as well as the Duhamel representations of the densities which will be the starting point to analyze the corresponding error.

4.2.1 Representation and Estimates on the densities of the diffusion and its Euler scheme

As in the papers [JM24a], [FJM24] in which the weak error was investigated for Lebesgue drifts, we will expand the densities of the SDE and its Euler scheme along the underlying heat equation. In particular, since the drift we consider is here bounded, existence of the density and related Aronson type upper-bounds readily follow from these works (see e.g. Propositions 2.1. and 2.3 in [JM24a] for the Brownian case and Theorem 1 and Proposition 1 in [FJM24] for the pure jump one).

In the current Hölder setting in space, in order to explicitly take advantage of the additional spatial regularity we will rely on appropriate cancellation techniques. We start recalling some useful controls for the underlying heat kernel p_α, the density of the driving noise Z.

Controls on the density of the stable noise

Proposition 4.2 (Density estimates for the heat equation)**.** *There exists $C \geq 1, \bar{c} \geq 1$ s.t. for all $0 \leq s < t \leq T$, $(x, w) \in (\mathbb{R}^d)^2$, and any muti-index $\zeta \in \mathbb{N}^d$, $|\zeta| \in \{1, 2\}$, $\theta \in \{0, 1\}$,*

$$|\partial_t^\theta \nabla_x^\zeta p_\alpha(t-s, w-x)| \leq C(t-s)^{-(\theta + \frac{|\zeta|+d}{\alpha})} \left(1 + \frac{|w-x|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha-|\zeta|}, \ \alpha \in (1,2),$$

$$|\partial_t^\theta \nabla_x^\zeta p_\alpha(t-s, w-x)| \leq C(t-s)^{-(\theta + \frac{|\zeta|+d}{\alpha})} g_{\bar{c}}(t-s, w-x), \ \alpha = 2, \tag{4.2.1}$$

where $g_{\bar{c}}(u, z) := \frac{1}{(2\pi \bar{c})^{\frac{d}{2}}} \exp(-\frac{|z|^2}{2\bar{c}u})$ stands for the Gaussian density of the Gaussian vector with variance $\bar{c} I_d$. In particular, under (4.1.2),

$$|w-x|^\beta |\partial_t^\theta \nabla_x^\zeta p_\alpha(t-s, w-x)| \leq C(t-s)^{-\theta + \frac{\beta - |\zeta|}{\alpha}} \bar{p}_\alpha(t-s, w-x), \tag{4.2.2}$$

and, for $w' \in \mathbb{R}^d$ s.t. $|w-w'| \lesssim (t-s)^{\frac{1}{\alpha}}$,

$$|w-x|^\beta |\partial_t^\theta \nabla_x^\zeta p_\alpha\left(t-s, w-x+(w-w')\right)| \leq C(t-s)^{-\theta + \frac{\beta-|\zeta|}{\alpha}} \bar{p}_\alpha(t-s, w-x). \tag{4.2.3}$$

Proof. The estimates in (4.2.1) are plain to prove directly if $\alpha = 2$. Turning now to $\alpha \in (1, 2)$, since we have assumed Z to be isotropic, it is well known that the spatial derivatives of the density of the driving noise enjoy better concentration properties, see e.g. Lemma 2.8 in [MZ22], which proves (4.2.1) for $\theta = 0$. For $\theta = 1$, let us proceed as follows: for $|w-x| \leq (t-s)^{\frac{1}{\alpha}}$, the bound follows from the Fourier representation of the density:

$$p_\alpha(t, x) = (2\pi)^{-d} \int \exp\left[ix \cdot \xi\right] \exp\left[-c_\alpha t |\xi|^\alpha\right] \mathrm{d}\xi, \qquad c_\alpha > 0. \tag{4.2.4}$$

For $|w-x| \geq (t-s)^{\frac{1}{\alpha}}$, let us recall that the density of the isotropic stable process can be expressed as:

$$\begin{aligned} p_\alpha(t-s, w-x) &= \int_0^\infty g(r, w-x) p_{S^{\frac{\alpha}{2}}}(t-s, r)\, \mathrm{d}r \\ &= \int_0^\infty g(r, w-x) \frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\frac{\alpha}{2}}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right) \mathrm{d}r, \end{aligned}$$

where $g : (r, x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto (2\pi r)^{-d/2} \exp\left[-|x|^2/(2r)\right]$ denotes the standard gaussian density and $p_{S^{\alpha/2}}$ stands for the density of the $\alpha/2$ stable subordinator. Hence, integrating by parts,

$$\begin{aligned} \partial_t p_\alpha(t-s, w-x) &= \int_0^\infty g(r, w-x) \partial_t \left(\frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\alpha/2}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right)\right) \mathrm{d}r \\ &= -\frac{2}{\alpha}\Bigg[\frac{1}{t-s} \int_0^\infty g(r, w-x) \frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\frac{\alpha}{2}}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right) \mathrm{d}r \\ &\qquad + \int_0^\infty g(r, w-x) \frac{1}{(t-s)^{\frac{2}{\alpha}}} \partial_r p_{S^{\alpha/2}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right) \frac{r}{t-s}\, \mathrm{d}r\Bigg] \\ &= -\frac{2}{\alpha(t-s)}\left[p(t-s, w-x) - \int_0^\infty \partial_r(r g(r, w-x)) \frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\alpha/2}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right) \mathrm{d}r\right] \\ &= -\frac{2}{\alpha(t-s)} \int_0^\infty r \partial_r g(r, w-x) \frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\alpha/2}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right) \mathrm{d}r. \end{aligned}$$

Recalling that $\partial_r g(r, x) = \Big(-\frac{d}{2r} + \frac{|x|^2}{2r^2}\Big) g(r, x)$, we have, for ζ s.t. $|\zeta| \in \{1, 2\}$,

$$|r \nabla_x^\zeta \partial_r g(r, x)| \lesssim \left(\frac{1}{r^{\frac{|\zeta|}{2}}} + \left(\frac{|x|}{r}\right)^{|\zeta|}\right) r^{-\frac{d}{2}} \exp\left[-\frac{|x|^2}{2r}\right] \lesssim \frac{1}{r^{\frac{|\zeta|+d}{2}}} \exp\left[-\lambda \frac{|x|^2}{2r}\right], \tag{4.2.5}$$

for some $\lambda > 1$. Plugging this into the previous equation and setting $u = |w-x|^2/r$, we get

$$|\nabla_x^\zeta \partial_t p_\alpha(t-s, w-x)| \lesssim \frac{1}{t-s}\int_0^\infty \frac{1}{r^{\frac{|\zeta|+d}{2}}} \exp\left[-\lambda \frac{|w-x|^2}{2r}\right] \frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\alpha/2}}\left(1, \frac{r}{(t-s)^{\frac{2}{\alpha}}}\right) \mathrm{d}r$$
$$\lesssim \frac{1}{t-s}\int_0^\infty \left(\frac{|w-x|^2}{u}\right)^{-\frac{|\zeta|+d}{2}} \exp\left[-\frac{\lambda u}{2}\right] \frac{1}{(t-s)^{\frac{2}{\alpha}}} p_{S^{\alpha/2}}\left(1, \frac{|w-x|^2}{u(t-s)^{\frac{2}{\alpha}}}\right) \frac{|w-x|^2}{u^2}\,\mathrm{d}u.$$

Using now the global bound on the law of the stable subordinator (see e.g. the proof of Lemma 4.8 in [?], in particular how the global bound is derived from Equation (4.8)):

$$\forall s > 0, \qquad p_{S^{\alpha/2}}(1, s) \lesssim s^{-1-\frac{\alpha}{2}},$$

we have,

$$|\nabla_x^\zeta \partial_t p_\alpha(t-s, w-x)| \lesssim \frac{1}{t-s}\int_0^\infty \left(\frac{|w-x|^2}{u}\right)^{-\frac{|\zeta|+d}{2}} \exp\left[-\frac{\lambda u}{2}\right] \frac{1}{(t-s)^{\frac{2}{\alpha}}} \left(\frac{|w-x|^2}{u(t-s)^{\frac{2}{\alpha}}}\right)^{-1-\frac{\alpha}{2}} \frac{|w-x|^2}{u^2}\,\mathrm{d}u$$
$$\lesssim |w-x|^{-|\zeta|-d-\alpha} \int_0^\infty u^{\frac{|\zeta|+d+\alpha}{2}-1} \exp\left[-\frac{\lambda u}{2}\right] \mathrm{d}u \lesssim |w-x|^{-|\zeta|-d-\alpha}$$
$$\lesssim ((t-s)^{\frac{1}{\alpha}} + |w-x|)^{-|\zeta|-d-\alpha} \lesssim (t-s)^{-\left(1+\frac{|\zeta|+d}{\alpha}\right)} \left(1 + \frac{|w-x|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha-|\zeta|},$$

recalling for the last line that $|w-x| > (t-s)^{\frac{1}{\alpha}}$. This concludes the proof of (4.2.1).

An important consequence of the above estimates is precisely (4.2.2). Namely, for $\alpha \in (1,2)$ we get

$$|w-x|^\beta |\partial_t^\theta \nabla_x^\zeta p_\alpha(t-s, w-x)| \le C(t-s)^{-\theta+\frac{\beta-|\zeta|}{\alpha}} \left(\frac{|w-x|}{(t-s)^{\frac{1}{\alpha}}}\right)^\beta \left(1 + \frac{|w-x|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-|\zeta|} \bar{p}_\alpha(t-s, w-x)$$
$$\le C(t-s)^{-\theta+\frac{\beta-|\zeta|}{\alpha}} \bar{p}_\alpha(t-s, w-x).$$

Also, for $\alpha = 2$,

$$|w-x|^\beta |\partial_t^\theta \nabla_x^\zeta p_\alpha(t-s, w-x)| \le C(t-s)^{-\theta+\frac{\beta-|\zeta|}{2}} \left(\frac{|w-x|}{(t-s)^{\frac{1}{2}}}\right)^\beta g_{\bar{c}}(t-s, w-x)$$
$$\le C(t-s)^{-\theta+\frac{\beta-|\zeta|}{2}} \bar{p}_2(t-s, w-x).$$

In that case the concentration constant is slightly deteriorated whereas in the pure jump case we took advantage of the concentration improvement for the derivatives. In any case (4.2.2) is proven. Equation (4.2.3) follows from the same proof as (4.2.2) noting that in the diagonal regime $|w-w'| \le (t-s)^{\frac{1}{\alpha}}$,

$$\bar{p}_\alpha(t-s, w-x+(w-w')) \le C\bar{p}_\alpha(t-s, w-x), \ \alpha \in (1,2),$$
$$g_{\bar{c}}(t-s, w-x+(w-w')) \le C g_{\tilde{c}}(t-s, w-x), \tilde{c} < c, \ \alpha = 2.$$

Hence, $|w-w'|$ can be seen as a negligible perturbation. □

Duhamel representation for the densities

To compute the error rate, we will start from the following Duhamel representations, which are proved respectively in [JM24a] and [FJM24] for $\alpha = 2$ and $\alpha \in (1,2)$:

Proposition 4.3 (Duhamel representations for the densities of the SDE and the Euler scheme)**.** *The density $\Gamma(s,x,t,\cdot)$ of the unique weak solution to Equation (4.1.1) starting from x at time $s \in [0,T)$ admits the following Duhamel representation: for all $t \in (s,T]$, $y \in \mathbb{R}^d$,*

$$\Gamma(s,x,t,y) = p_\alpha(t-s, y-x) - \int_s^t \mathbb{E}_{s,x}\left[b(r, X_r)\cdot \nabla_y p_\alpha(t-r, y-X_r)\right] \mathrm{d}r, \tag{4.2.6}$$

where the expectation subscript means that $X_s = x$.
Similarly, for $k \in [\![0, n-1]\!]$, $t \in (t_k, T]$, the density of X_t^h admits, conditionally to $X_{t_k}^h = x$, a transition density $\Gamma^h(t_k, x, t, \cdot)$, which again enjoys a Duhamel type representation: for all $y \in \mathbb{R}^d$,

$$\Gamma^h(t_k, x, t, y) = p_\alpha(t - t_k, y - x) - \int_{t_k}^t \mathbb{E}_{t_k,x}\left[b(U_{\tau_r^h/h}, X_{\tau_r^h}^h) \cdot \nabla_y p_\alpha(t - r, y - X_r^h)\right] \mathrm{d}r, \tag{4.2.7}$$

where the expectation subscript means that $X_{t_k}^h = x$.

4.2.2 Proof of Theorem 4.1

In this section we will use for two quantities A and B the symbol $A \lesssim B$ whenever there exists a constant $C := C(d, b, \alpha, T)$ s.t. $A \leq CB$. Namely,

$$A \lesssim B \Longleftrightarrow \exists C := C(d, b, \alpha, T),\ A \leq CB. \tag{4.2.8}$$

Starting from Proposition 4.3 and comparing the Duhamel formula of the scheme, (4.2.7), to that of the diffusion, (4.2.6), we get

$$\begin{aligned}
&\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y) \\
&\quad = \mathbb{E}_{0,x}\left[\int_0^t \Big(b(s, X_s) \cdot \nabla_y p_\alpha(t - s, y - X_s) - b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \nabla_y p_\alpha(t - s, y - X_s^h)\Big)\, \mathrm{d}s\right].
\end{aligned}$$

In the previous equation and the rest of this paper, we denote $\mathbb{E}_{0,x}[\cdot] := \mathbb{E}[\cdot | X_0 = X_0^h = x]$. We will split the error in the following way:

$$\begin{aligned}
&\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y) \\
&\quad = \int_0^h \mathbb{E}_{0,x}\left[b(s, X_s) \cdot \nabla_y p_\alpha(t - s, y - X_s) - b(U_0, x) \cdot \nabla_y p_\alpha(t - s, y - X_s^h)\right] \mathrm{d}s \\
&\qquad + \int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(s, X_s) \cdot \nabla_y p_\alpha(t - s, y - X_s) - b(s, X_{\tau_s^h}) \cdot \nabla_y p_\alpha(t - s, y - X_{\tau_s^h})\right] \mathrm{d}s \\
&\qquad + \int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(s, X_{\tau_s^h}) \cdot \nabla_y p_\alpha(t - s, y - X_{\tau_s^h}) - b(s, X_{\tau_s^h}^h) \cdot \nabla_y p_\alpha(t - s, y - X_{\tau_s^h}^h)\right] \mathrm{d}s \\
&\qquad + \int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \Big(\nabla_y p_\alpha(t - s, y - X_{\tau_s^h}^h) - \nabla_y p_\alpha(t - s, y - X_s^h)\Big)\right] \mathrm{d}s \\
&\qquad + \int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \Big(\nabla_y p_\alpha(t - U_{\tau_s^h/h}, y - X_{\tau_s^h}^h) - \nabla_y p_\alpha(t - s, y - X_{\tau_s^h}^h)\Big)\right] \mathrm{d}s \\
&\qquad + \int_{\tau_t^h - h}^t \mathbb{E}_{0,x}\left[b(s, X_s) \cdot \nabla_y p_\alpha(t - s, y - X_s) - b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \nabla_y p_\alpha(t - s, y - X_s^h)\right] \mathrm{d}s \\
&\quad =: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6,
\end{aligned} \tag{4.2.9}$$

where we exploited that:

$$\begin{aligned}
&\int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(U_{[\tau_s^h/h]}, X_{\tau_s^h}^h) \cdot \nabla_y p_\alpha(t - U_{[\tau_s^h/h]}, y - X_{\tau_s^h}^h)\right] \mathrm{d}s \\
&\qquad = \sum_{i=1}^{\tau_t^h/h - 1} \frac{1}{h} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathbb{E}_{0,x}\left[b(r, X_{t_i}^h) \cdot \nabla_y p_\alpha(t - r, y - X_{t_i}^h)\right] \mathrm{d}s\, \mathrm{d}r \\
&\qquad = \int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(s, X_{\tau_s^h}^h) \cdot \nabla_y p_\alpha(t - s, y - X_{\tau_s^h}^h)\right] \mathrm{d}s,
\end{aligned} \tag{4.2.10}$$

for the correspondence between the last term in Δ_3 and the first one in Δ_5.

For Δ_1, we rely on the fact that we work on the first time step, and thus we do not even need the smoothing effect in $(t-s)^\beta$ provided by the drift in (4.2.2). Let us first expand the expectation:

$$\Delta_1 = \int_0^h \int \Big(\Gamma(0,x,s,z)b(s,z)\cdot\nabla_y p_\alpha(t-s,y-z) - \Gamma^h(0,x,s,z)b(U_0,x)\cdot\nabla_y p_\alpha(t-s,y-z)\Big)\,\mathrm{d}z\,\mathrm{d}s.$$

Assuming w.l.o.g. that $t > 2h$ so that for $s \in (0,h)$, $(t-s)^{-\frac{1}{\alpha}} < (t/2)^{-\frac{1}{\alpha}}$, then using (4.1.18), (4.1.15), (4.2.1) and the boundedness of b, we have

$$\begin{aligned}
|\Delta_1| &\lesssim \int_0^h \int \frac{1}{(t-s)^{\frac{1}{\alpha}}}\bar{p}_\alpha(s,z-x)\bar{p}_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)\int_0^h \frac{1}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}s \lesssim \bar{p}_\alpha(t,y-x)ht^{-\frac{1}{\alpha}} \\
&\lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}h^{\frac{1-\beta}{\alpha}}t^{-\frac{1}{\alpha}} \lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{-\frac{\beta}{\alpha}},
\end{aligned} \tag{4.2.11}$$

exploiting the *convolution property* (4.1.13) of the density $\bar{p}_\alpha$[3] for the second inequality and recalling that $h \le t$ for the last inequality.

Let us turn to Δ_2. Expanding the inner expectation and using the time regularity of Γ in the forward time variable (see (4.1.16))

$$\begin{aligned}
|\Delta_2| &= \left|\int_h^{\tau_t^h - h}\int \left[\Gamma(0,x,s,z) - \Gamma(0,x,\tau_s^h,z)\right] b(s,z)\cdot\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\right| \\
&\lesssim \int_h^t \int \frac{(s-\tau_s^h)^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}}\bar{p}_\alpha(s,z-x)(t-s)^{-\frac{1}{\alpha}}\bar{p}_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s,
\end{aligned}$$

using as well (4.2.1) for the last inequality. Again, from (4.1.13), along with the fact that $s - \tau_s^h \le h$ and that for $s \ge h$, $(\tau_s^h)^{-1} \le 2s^{-1}$, we can write

$$|\Delta_2| \lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}\int_h^t s^{-\frac{\gamma}{\alpha}}(t-s)^{-\frac{1}{\alpha}}\,\mathrm{d}s \lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}. \tag{4.2.12}$$

The term Δ_3, which is the one that will allow to apply a Gronwall type argument, will be treated at the end of the current error analysis.

Let us turn to the term Δ_4 in (4.2.9), which is by far the more delicate. Let us then introduce the following lemma:

Lemma 4.1 (Smoothing effect of the drift)**.** *Let ζ be a multi-index with length $1 \le |\zeta| \le 2$ and $\delta \in \{0,1\}$. Then, for all $(x,\mathfrak{y}) \in (\mathbb{R}^d)^2$, $0 \le h \le t/2 \le s < \tau_t^h - h \le T$, $r > 0$,*

$$\left|\int \Gamma^h(0,x,\tau_s^h,z)b(r,z)\partial_t^\delta\nabla_{\mathfrak{y}}^\zeta p_\alpha(t-\tau_s^h,\mathfrak{y}-z)\,\mathrm{d}z\right| \lesssim \bar{p}_\alpha(t,\mathfrak{y}-x)(t-\tau_s^h)^{-\delta+\frac{\beta-|\zeta|}{\alpha}}\left(1+(\tau_s^h)^{-\frac{\beta}{\alpha}}\right). \tag{4.2.13}$$

Proof. Let use the following cancellation argument:

$$\begin{aligned}
I &:= \int \Gamma^h(0,x,\tau_s^h,z)b(r,z)\partial_t^\delta\nabla_{\mathfrak{y}}^\zeta p_\alpha(t-\tau_s^h,\mathfrak{y}-z)\,\mathrm{d}z \\
&= \int [\Gamma^h(0,x,\tau_s^h,z)b(r,z) - \Gamma^h(0,x,\tau_s^h,\mathfrak{y})b(r,\mathfrak{y})]\partial_t^\delta\nabla_{\mathfrak{y}}^\zeta p_\alpha(t-\tau_s^h,\mathfrak{y}-z)\,\mathrm{d}z.
\end{aligned}$$

[3]which is actually just an *approximate* convolution property in the pure jump case.

Then, using the regularity of b and (4.1.19), taking therein ε such that $\gamma_\varepsilon \geq \beta$, we have

$$
\begin{aligned}
&|\Gamma^h(0,x,\tau_s^h,z)b(r,z) - \Gamma^h(0,x,\tau_s^h,\mathfrak{y})b(r,\mathfrak{y})| \\
&\quad \lesssim \bar{p}_\alpha(\tau_s^h, z-x)|\mathfrak{y}-z|^\beta + \|b\|_{L^\infty} \frac{(|\mathfrak{y}-z| + h^{\frac{1}{\alpha}})^\beta}{(\tau_s^h)^{\frac{\beta}{\alpha}}} \left[\bar{p}_\alpha(\tau_s^h, z-x) + \bar{p}_\alpha(\tau_s^h, \mathfrak{y}-x)\mathbb{1}_{|\mathfrak{y}-z|^\alpha \geq \tau_s^h}\right] \qquad (4.2.14) \\
&\quad \lesssim \bar{p}_\alpha(\tau_s^h, z-x)|\mathfrak{y}-z|^\beta \left(1 + (\tau_s^h)^{-\frac{\beta}{\alpha}}\right) + \frac{(|\mathfrak{y}-z|^\beta + (t-\tau_s^h)^{\frac{\beta}{\alpha}})}{(\tau_s^h)^{\frac{\beta}{\alpha}}} \bar{p}_\alpha(t, \mathfrak{y}-x),
\end{aligned}
$$

where we used the fact that we consider times $s \geq t/2$ to write $\bar{p}_\alpha(\tau_s^h, \mathfrak{y}-x) \lesssim \bar{p}_\alpha(t, \mathfrak{y}-x)$ and the fact that for $s \leq \tau_t^h - h$, $h \leq t - \tau_s^h$ for the last inequality. Plugging this into I and using (4.2.2), we get (4.2.13). $\square$

Going back to the bound for Δ_4, conditioning w.r.t. $\sigma(X_{\tau_s}^h, U_{\tau_s^h/h})$ (the sigma-algebra generated by the random variables $X_{\tau_s^h}$ and $U_{\tau_s^h}$) and using the harmonicity of the (gradient of the) stable heat kernel (or Itô's formula between τ_s^h and s) in order to get rid of the noise increment, we can write that for any bounded and measurable $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$,

$$
\mathbb{E}\left[\nabla_y p_\alpha\left(t-s, y - \varphi(X_{\tau_s^h}^h, U_{\tau_s^h}) + (Z_s - Z_{\tau_s^h})\right) |\sigma(X_{\tau_s}^h, U_{\tau_s^h/h})\right] = \nabla_y p_\alpha\left(t - \tau_s^h, y - \varphi(X_{\tau_s^h}^h, U_{\tau_s^h})\right). \qquad (4.2.15)
$$

Using this, we get

$$
\begin{aligned}
\Delta_4 &= \int_h^{t/2} \mathbb{E}_{0,x}\left[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \left(\nabla_y p_\alpha(t-s, y - X_{\tau_s^h}^h) - \nabla_y p_\alpha(t-s, y - X_s^h)\right)\right] \mathrm{d}s \\
&\quad + \int_{t/2}^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \left(\nabla_y p_\alpha(t-s, y - X_{\tau_s^h}^h) - \nabla_y p_\alpha(t-s, y - (X_{\tau_s^h}^h + Z_s - Z_{\tau_s^h}))\right)\right] \mathrm{d}s \\
&\quad + \int_{t/2}^{\tau_t^h - h} \mathbb{E}_{0,x}\Big[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \Big(\nabla_y p_\alpha(t-s, y - (X_{\tau_s^h}^h + Z_s - Z_{\tau_s^h})) \\
&\qquad\qquad - \nabla_y p_\alpha(t-s, y - (X_{\tau_s^h}^h + b(U_{\tau_s^h/h}, X_{\tau_s^h}^h)(s - \tau_s^h) + Z_s - Z_{\tau_s^h}))\Big)\Big] \mathrm{d}s \\
&= \int_h^{t/2} \mathbb{E}_{0,x}\left[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \left(\nabla_y p_\alpha(t-s, y - X_{\tau_s^h}^h) - \nabla_y p_\alpha(t - \tau_s^h, y - (X_{\tau_s^h}^h + b(U_{\tau_s^h/h}, X_{\tau_s^h}^h)(s - \tau_s^h)))\right)\right] \mathrm{d}s \\
&\quad + \int_{t/2}^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \left(\nabla_y p_\alpha(t-s, y - X_{\tau_s^h}^h) - \nabla_y p_\alpha(t - \tau_s^h, y - X_{\tau_s^h}^h)\right)\right] \mathrm{d}s \\
&\quad + \int_{t/2}^{\tau_t^h - h} \mathbb{E}_{0,x}\Big[b(U_{\tau_s^h/h}, X_{\tau_s^h}^h) \cdot \Big(\nabla_y p_\alpha(t - \tau_s^h, y - X_{\tau_s^h}^h) \\
&\qquad\qquad - \nabla_y p_\alpha(t - \tau_s^h, y - (X_{\tau_s^h}^h + b(U_{\tau_s^h/h}, X_{\tau_s^h}^h)(s - \tau_s^h)))\Big)\Big] \mathrm{d}s \\
&=: \Delta_{41} + \Delta_{42} + \Delta_{43}. \qquad (4.2.16)
\end{aligned}
$$

For Δ_{41}, there is no need to compensate for singularities in $(t-s)$ on the considered time interval:

$$
\begin{aligned}
|\Delta_{41}| &= \Big|\int_h^{t/2} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int \Gamma^h(0,x,\tau_s^h,z)b(r,z)\big[\nabla_y p_\alpha(t-s, y-z) - \nabla_y p_\alpha(t - \tau_s^h, y-z) \\
&\qquad + \nabla_y p_\alpha(t - \tau_s^h, y-z) - \nabla_y p_\alpha(t - \tau_s^h, y - z - b(r,z)(s - \tau_s^h))\big] \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}s\Big| \\
&\lesssim \int_h^{t/2} \int \bar{p}_\alpha(\tau_s^h, z-x)\|b\|_{L^\infty} \left[\frac{(s - \tau_s^h)}{(t-s)^{1+\frac{1}{\alpha}}} + \frac{(s - \tau_s^h)\|b\|_{L^\infty}}{(t - \tau_s^h)^{\frac{2}{\alpha}}}\right] \bar{p}_\alpha(t - \tau_s^h, y-z)\, \mathrm{d}z\, \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t, y-x) h^{\frac{\gamma}{\alpha}} \int_h^{t/2} \left[(t-s)^{-\frac{\gamma+1}{\alpha}} + (t - \tau_s^h)^{1 - \frac{\gamma+2}{\alpha}}\right] \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t, y-x) h^{\frac{\gamma}{\alpha}} t^{1 - \frac{\gamma+1}{\alpha}} = \bar{p}_\alpha(t, y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}. \qquad (4.2.17)
\end{aligned}
$$

For Δ_{42}, we will make the same time sensitivity appear and then expand it with a Taylor formula:

$$\Delta_{42} = \int_{t/2}^{\tau_t^h - h} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int \Gamma^h(0,x,\tau_s^h,z) b(r,z) \cdot \left[\nabla_y p_\alpha(t-s, y-z) - \nabla_y p_\alpha(t-\tau_s^h, y-z)\right] \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}s$$

$$= -\int_{t/2}^{\tau_t^h - h} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int \Gamma^h(0,x,\tau_s^h,z) b(r,z) \cdot \int_0^1 \partial_t \nabla_y p_\alpha(t-s+\mu(s-\tau_s^h), y-z)(s-\tau_s^h)\, \mathrm{d}\mu\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}s.$$

Then, using the same cancellation techniques as in the proof of (4.2.13) with $\mathfrak{y} = y$, we get

$$\begin{aligned}
|\Delta_{42}| &\lesssim (1 + t^{-\frac{\beta}{\alpha}}) \int_{t/2}^{\tau_t^h - h} \int_0^1 \frac{(s-\tau_s^h)}{(t-s+\mu(s-\tau_s^h))^{1+\frac{1-\beta}{\alpha}}} \bar{p}_\alpha(t+\tau_s^h - s + \mu(s-\tau_s^h), y-x)\, \mathrm{d}\mu\, \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)(1+t^{-\frac{\beta}{\alpha}}) \int_{t/2}^{\tau_t^h - h} \frac{(s-\tau_s^h)}{(t-\tau_s^h)^{1+\frac{1-\beta}{\alpha}}}\, \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)(1+t^{-\frac{\beta}{\alpha}}) h \int_{t/2}^{\tau_t^h - h} (t-s)^{-1+\frac{\beta-1}{\alpha}}\, \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)(1+t^{-\frac{\beta}{\alpha}}) h (t-\tau_t^h + h)^{\frac{\beta-1}{\alpha}} \lesssim \bar{p}_\alpha(t,y-z)(1+t^{-\frac{\beta}{\alpha}}) h^{\frac{\gamma}{\alpha}}.
\end{aligned} \tag{4.2.18}$$

Let us turn to Δ_{43}, let us write

$$\begin{aligned}
\Delta_{43} = \int_{t/2}^{\tau_t^h - h} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int \Gamma^h(0,x,\tau_s^h,z) b(r,z) \cdot \big[&\nabla_y p_\alpha(t-\tau_s^h, y-z) - \nabla_w p_\alpha(t-\tau_s^h, w)|_{w=y-z-b(r,y)(s-\tau_s^h)} \\
&+ \nabla_w p_\alpha(t-\tau_s^h, w)|_{w=y-z-b(r,y)(s-\tau_s^h)} - \nabla_y p_\alpha(t-\tau_s^h, y-(z+b(r,z)(s-\tau_s^h)))\big]\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}s \\
:= \Delta_{431} + \Delta_{432}.&
\end{aligned} \tag{4.2.19}$$

We carefully mention that this additional pivot is needed in order to use cancellation arguments for the first term (in order to have a drift which does not depend on the spatial integration variable) and to take full force of the regularity of the drift for the second one. For Δ_{431}, we use a Taylor expansion and then (4.2.13):

$$\begin{aligned}
|\Delta_{431}| &= \Bigg| \int_{t/2}^{\tau_t^h - h} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int \Gamma^h(0,x,\tau_s^h,z) b(r,z) \\
&\qquad \cdot \int_0^1 \nabla_w^2 p_\alpha(t-\tau_s^h, w)|_{w=y-z-\mu b(r,y)(s-\tau_s^h)} \cdot b(r,y)(s-\tau_s^h)\, \mathrm{d}\mu\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}s \Bigg| \\
&\lesssim (1+t^{-\frac{\beta}{\alpha}}) \int_{t/2}^{\tau_t^h - h} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int_0^1 \bar{p}_\alpha(t, y-\mu b(r,y)(s-\tau_s^h) - x) \|b\|_{L^\infty} \frac{(s-\tau_s^h)}{(t-\tau_s^h)^{\frac{2-\beta}{\alpha}}}\, \mathrm{d}\mu\, \mathrm{d}r\, \mathrm{d}s \\
&\lesssim (1+t^{-\frac{\beta}{\alpha}}) \bar{p}_\alpha(t,y-x) h \int_{t/2}^{\tau_t^h - h} (t-\tau_s^h)^{-\frac{2-\beta}{\alpha}}\, \mathrm{d}s.
\end{aligned}$$

Recalling that $\bar{p}_\alpha(t, y-\mu b(r,y)(s-\tau_s^h) - x) \lesssim \bar{p}_\alpha(t,y-x)$ for the second inequality (with a slight notational abuse in the gaussian case since the variance is then modified). Now, if $(2-\beta)/\alpha < 1 \iff \alpha+\beta > 2$ (which is e.g. always the case in the Brownian setting), the time integral is convergent and uniformly bounded in h. The term Δ_{431} then has order h regarding the time step. If now $(2-\beta)/\alpha = 1$, it has order $h|\ln(h)| \lesssim h^{\frac{\gamma}{\alpha}}$. We can thus assume w.l.o.g. that $(2-\beta)/\alpha > 1$. Then,

$$\begin{aligned}
|\Delta_{431}| &\lesssim (1+t^{-\frac{\beta}{\alpha}}) \bar{p}_\alpha(t,y-x) h (t-\tau_t^h + h)^{1-\frac{2-\beta}{\alpha}} \\
&\lesssim (1+t^{-\frac{\beta}{\alpha}}) \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}+(1-\frac{1}{\alpha})} \lesssim \bar{p}_\alpha(t,y-x)(1+t^{-\frac{\beta}{\alpha}}) h^{\frac{\gamma}{\alpha}},
\end{aligned} \tag{4.2.20}$$

Let us turn to Δ_{432}, which we first expand with a Taylor formula:

$$\begin{aligned}
\Delta_{432} = \int_{t/2}^{\tau_t^h - h} \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h + h} \int \Gamma^h(0,x,\tau_s^h,z) b(r,z) \\
\cdot \int_0^1 \nabla_w^2 p_\alpha\left(t-\tau_s^h, w\right)|_{w=y-z-[b(r,z)-\mu[b(r,y)-b(r,z)]](s-\tau_s^h)} \cdot [b(r,y)-b(r,z)](s-\tau_s^h)\, \mathrm{d}\mu\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}s.
\end{aligned}$$

Since $t-\tau_s^h \ge s-\tau_s^h$, using the boundedness and the Hölder regularity of b and then (4.2.3), we have

$$\begin{aligned}&\left|\nabla_w^2 p_\alpha\left(t-\tau_s^h, w\right)|_{w=y-z-[b(r,z)-\mu[b(r,y)-b(r,z)]](s-\tau_s^h)} \cdot [b(r,y)-b(r,z)]\right|\\ &\lesssim (t-\tau_s^h)^{\frac{\beta-2}{\alpha}} \bar p_\alpha(t-\tau_s^h, y-z), \end{aligned}\tag{4.2.21}$$

thus yielding, with the same computations of the time integrals as for Δ_{431},

$$\begin{aligned}|\Delta_{432}| &\lesssim (1+t^{-\frac{\beta}{\alpha}})\bar p_\alpha(t,y-x)\int_{t/2}^{\tau_t^h-h}(s-\tau_s^h)(t-\tau_s^h)^{\frac{\beta-2}{\alpha}}\,\mathrm{d}s\\ &\lesssim (1+t^{-\frac{\beta}{\alpha}})\bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}.\end{aligned}\tag{4.2.22}$$

Plugging (4.2.20) and (4.2.22) into (4.2.19) and from (4.2.17), (4.2.18) and (4.2.16) , we obtain

$$|\Delta_4| \lesssim (1+t^{-\frac{\beta}{\alpha}})\bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}.\tag{4.2.23}$$

Observe now that the term Δ_5 in (4.2.9) could be handled just as Δ_{41} and Δ_{42} above. Indeed, the time variable is there randomized, but once expanded through the density, the quantity is really similar. Importantly, it makes a pure time sensitivity of the stable kernel appear. This therefore yields

$$|\Delta_5| \lesssim (1+t^{-\frac{\beta}{\alpha}})\bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}.\tag{4.2.24}$$

It thus remains to handle the contribution Δ_6 associated with the last time steps. The quantities involved can actually again be estimated using cancellation arguments. Write:

$$\begin{aligned}|\Delta_6| &\le \left|\int_{\tau_t^h-h}^t \mathbb{E}_{0,x}\left[b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s) - b(U_{\tau_s^h/h}, X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X_s^h)\right]\mathrm{d}s\right|\\ &\le \left|\int_{\tau_t^h-h}^t\int\Big(\Gamma(0,s,x,z)b(s,z)-\Gamma(0,s,x,y)b(s,y)\Big)\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\right|\\ &\quad+\left|\int_{\tau_t^h-h}^t\frac{1}{h}\int_{\tau_s^h}^{\tau_s^h+h}\int\Big(\Gamma^h(0,\tau_s^h,x,z)b(r,z)-\Gamma^h(0,\tau_s^h,x,y)b(r,y)\Big)\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\right|\\ &=: \Delta_{61}+\Delta_{62}.\end{aligned}$$

Using (4.2.14) for Δ_{62} and the corresponding inequality based on the Hölder estimate (4.1.17) for the density of the diffusion (still taking therein ε s.t. $\gamma_\varepsilon \ge \beta$), and exploiting (4.2.2) as in the proof of Lemma (4.2.13), we then get:

$$|\Delta_6| \lesssim \int_{\tau_t^h-h}^t\Big((t-s)^{-\frac{1}{\alpha}+\frac{\beta}{\alpha}}+(t-s)^{-\frac{1}{\alpha}}h^{\frac{\beta}{\alpha}}\Big)(1+(\tau_s^h)^{-\frac{\beta}{\alpha}})\bar p_\alpha(t,y-x)\,\mathrm{d}s \lesssim (1+t^{-\frac{\beta}{\alpha}})\bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}.\tag{4.2.25}$$

Gathering estimates (4.2.11), (4.2.12), (4.2.23), (4.2.24), (4.2.25), we have

$$\begin{aligned}&|\Gamma^h(0,x,t,y)-\Gamma(0,x,t,y)| \lesssim \bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}(1+t^{-\frac{\beta}{\alpha}})\\ &\quad+\left|\int_h^t \mathbb{E}_{0,x}\left[b(s,X_{\tau_s^h})\cdot\nabla_y p_\alpha(t-\tau_s^h,y-X_{\tau_s^h})-b(s,X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-\tau_s^h,y-X^h_{\tau_s^h})\right]\mathrm{d}s\right|.\end{aligned}\tag{4.2.26}$$

Set

$$f_h(u)=\sup_{(x,z)\in(\mathbb{R}^d)^2}\frac{|\Gamma^h(0,x,u,z)-\Gamma(0,x,u,z)|}{\bar p_\alpha(u,z-x)}.\tag{4.2.27}$$

Observe from (4.1.18) and (4.1.15) that f_h is bounded uniformly in h and the time variable. We then have, using (4.2.2), the boundedness of b and the convolution property of the stable kernel,

$$\begin{aligned}\frac{|\Gamma^h(0,x,t,y)-\Gamma(0,x,t,y)|}{p_\alpha(t,y-x)} &\lesssim h^{\frac{\gamma}{\alpha}}(1+t^{-\frac{\beta}{\alpha}})+\frac{1}{\bar p_\alpha(t,y-x)}\int_h^t\frac{f_h(\tau_s^h)}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\int\bar p_\alpha(\tau_s^h,z-x)\bar p_\alpha(t-\tau_s^h,y-z)\,\mathrm{d}z\,\mathrm{d}s\\ &\lesssim h^{\frac{\gamma}{\alpha}}(1+t^{-\frac{\beta}{\alpha}})+\int_h^t\frac{f_h(\tau_s^h)}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\,\mathrm{d}s.\end{aligned}$$

The previous bound being uniform in x and y, we get

$$f_h(t) \lesssim h^{\frac{\gamma}{\alpha}}(1+t^{-\frac{\beta}{\alpha}}) + \int_h^t \frac{f_h(\tau_s^h)}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\,\mathrm{d}s.$$

Using a *discrete* Grönwall-Volterra lemma, we obtain Theorem 4.1.

4.3 Proof of the regularity results from Proposition 4.1

This section is devoted to the proof of the controls (4.1.17), (4.1.19) concerning the Hölder continuity in space in the forward variable for the density of the diffusion and the scheme respectively as well as to (4.1.16), Hölder regularity in time in the forward time variable. These estimates were crucial in order to prove the main theorem in the previous section. Importantly, we achieve respectively the order $\gamma_\varepsilon = (\gamma \wedge 1) - \varepsilon,\ \varepsilon > 0$ in space and γ/α in time.

Note that the lower exponent attained in space is sufficient for the previous proof of the main result to work. We actually used the spatial regularity of the densities of the diffusion and the scheme in a cancellation argument involving a product with the drift (see former (4.2.14)). In this context, we just need in practice the lower order β-regularity corresponding to the spatial smoothness of the drift. Indeed, for $\beta \in (0,1)$ one can find ε s.t. $(\gamma \wedge 1) - \varepsilon \geq \beta$. We mention as well that this first spatial estimate actually allows in a second time to derive the expected exponent γ when $\gamma < 1$ for the Hölder regularity in the forward variable. The proof is provided for completeness (since we actually insisted on the parabolic bootstrap phenomenon) for the diffusion in Appendix 4.4.

We start this section recalling some usual yet important controls on the density of the driving noise that we will profusely use in order to prove (4.1.16). The proof is somehow standard and can be e.g. found in [FJM24] for $\alpha \in (1,2)$ and [JM24a] in the gaussian case.

Lemma 4.2 (Stable sensitivities - Estimates on the α-stable kernel)**.** *For each multi-index ζ with length $|\zeta| \leq 2$, and for all $0 < u \leq u' \leq T$, $(x,x') \in (\mathbb{R}^d)^2$, $\theta \in (0,1]$,*

- *Time Hölder regularity:*

$$\left|\nabla_x^\zeta p_\alpha(u,x) - \nabla_x^\zeta p_\alpha(u',x)\right| \lesssim \frac{|u-u'|^\theta}{u^{\theta+\frac{|\zeta|}{\alpha}}}\left(\bar{p}_\alpha(u,x) + \bar{p}_\alpha(u',x)\right). \tag{4.3.1}$$

- *Spatial Hölder regularity:*

$$\left|\nabla_x^\zeta p_\alpha(u,x) - \nabla_{x'}^\zeta p_\alpha(u,x')\right| \lesssim \left(\frac{|x-x'|^\theta}{u^{\frac{\theta}{\alpha}}} \wedge 1\right)\frac{1}{u^{\frac{|\zeta|}{\alpha}}}\left(\bar{p}_\alpha(u,x) + \bar{p}_\alpha(u,x')\right). \tag{4.3.2}$$

4.3.1 Proof of the spatial regularity

We start this section providing the estimate (4.1.17) for the diffusion. We could actually have established (4.1.19) only and then derive (4.1.17) passing to the limit exploiting the convergence in law of the Euler scheme to the diffusion, which would have allowed to transfer the estimates on densities. However, we provide a complete proof on the diffusion first since it is actually simpler than the one for the scheme and already emphasizes the key ideas, which will as well appear in the proof of (4.1.16) (time regularity).

Proof of (4.1.17): forward spatial Hölder regularity for the diffusion

Define, for $\eta > 0$ meant to be small (viewed as a spatial viscosity parameter),

$$h_{s,x}^{\varepsilon,\eta}(t) := \sup_{(z,z')\in(\mathbb{R}^d)^2}\left\{\frac{|\Gamma(s,x,t,z) - \Gamma(s,x,t,z')|(t-s)^{\frac{\gamma_\varepsilon}{\alpha}}}{\left(\bar{p}_\alpha(t-s,z-x) + \bar{p}_\alpha(t-s,z'-x)\right)(|z-z'| \vee \eta)^{\gamma_\varepsilon}}\right\}.$$

Since we already know from (4.1.15) that $\frac{|\Gamma(s,x,t,z)-\Gamma(s,x,t,z')|}{\bar p_\alpha(t-s,z-x)+\bar p_\alpha(t-s,z'-x)} < \infty$, we immediately have $h^{\varepsilon,\eta}_{s,x}(t) \lesssim \eta^{-\gamma_\varepsilon} < +\infty$. W.l.o.g., we take $s=0$ for simplicity and assume T is *small*, in particular $T \le 1$. Let us write for $0 < t \le T, x, y, y' \in \mathbb{R}^d$ the following:

$$\begin{aligned}
\Gamma(0,x,t,y') - \Gamma(0,x,t,y) &= p_\alpha(t, y'-x) - p_\alpha(t,y-x) \\
&\quad + \int_0^{t/2}\int \Gamma(0,x,s,z) b(s,z)\cdot (\nabla_y p_\alpha(t-s,y-z) - \nabla_y p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s \\
&\quad + \int_{t/2}^{t-(|y'-y|\vee\eta)^\alpha}\int \Gamma(0,x,s,z) b(s,z)\cdot (\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s \\
&\quad + \int_{t-(|y'-y|\vee\eta)^\alpha}^{t}\int \Gamma(0,x,s,z) b(s,z)\cdot (\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s \\
&=: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4.
\end{aligned}$$

We tacitly assume as well that $(|y'-y|\vee\eta)^\alpha \le t/2$ since otherwise, i.e. in the off-diagonal case, the expected control $[(\Gamma(0,x,t,y)-\Gamma(0,x,t,y'))t^{\frac{\gamma_\varepsilon}{\alpha}}]/[(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x))(|y-y'|\vee\eta)^{\gamma_\varepsilon}] \le C$ readily follows from the Aronson type bounds (4.1.15).

For Δ_1, we use the regularity of the stable kernel, (4.3.2) to write

$$|\Delta_1| \lesssim \frac{|y-y'|^{\gamma_\varepsilon}}{t^{\frac{\gamma_\varepsilon}{\alpha}}}(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x)). \tag{4.3.3}$$

For Δ_2, using again (4.3.2), we write

$$|\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z)| \lesssim \frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{\gamma_\varepsilon+1}{\alpha}}}(\bar p_\alpha(t-s,y-z)+\bar p_\alpha(t-s,y'-z))$$

which yields, along with (4.1.18) and the convolution property (4.1.13) of $\bar p_\alpha$,

$$\begin{aligned}
|\Delta_2| &\lesssim \int_0^{t/2}\int \bar p_\alpha(s,z-x)\|b\|_{L^\infty}\frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{\gamma_\varepsilon+1}{\alpha}}}(\bar p_\alpha(t-s,y-z)+\bar p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s \\
&\lesssim \frac{|y-y'|^{\gamma_\varepsilon}}{t^{\frac{\beta_\varepsilon}{\alpha}}}(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x)),
\end{aligned} \tag{4.3.4}$$

where $\beta_\varepsilon := \beta-\varepsilon$, noting that for $\gamma = \alpha+\beta-1 \in [1,2)$, $\gamma_\varepsilon + 1 - \alpha = 2-(\alpha+\varepsilon) \le \beta - \varepsilon$ and recalling that we have assumed T to be *small*.

For Δ_3, using a Taylor expansion and then a cancellation argument, we have

$$\begin{aligned}
\Delta_3 &= \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\int\int_0^1 \Gamma(0,x,s,z)b(s,z)\cdot\nabla_y^2 p_\alpha(t-s,y+\lambda(y'-y)-z)(y'-y)\,\mathrm{d}\lambda\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\int_0^1\int [\Gamma(0,x,s,z)b(s,z) - \Gamma(0,x,s,y+\lambda(y'-y))b(s,y+\lambda(y'-y))] \\
&\qquad\qquad \cdot\nabla_y^2 p_\alpha(t-s,y+\lambda(y'-y)-z)(y'-y)\,\mathrm{d}z\,\mathrm{d}\lambda\,\mathrm{d}s.
\end{aligned} \tag{4.3.5}$$

We then write

$$
\begin{aligned}
&|\Gamma(0,x,s,z)b(s,z)-\Gamma(0,x,s,y+\lambda(y'-y))b(s,y+\lambda(y'-y))|\\
&\quad\lesssim |y+\lambda(y'-y)-z|^{\beta}\bar p_\alpha(s,z-x)+\|b\|_{L^\infty}\Big(h_{0,x}^{\varepsilon,\eta}(s)\frac{|y+\lambda(y'-y)-z|^{\gamma_\varepsilon}}{s^{\frac{\gamma_\varepsilon}{\alpha}}}\mathbb{1}_{|y+\lambda(y'-y)-z|\le s^{\frac1\alpha}}\\
&\qquad+\frac{|y+\lambda(y'-y)-z|^{\beta}}{s^{\frac{\beta}{\alpha}}}\mathbb{1}_{|y+\lambda(y'-y)-z|\ge s^{\frac1\alpha}}\Big)\left(\bar p_\alpha(s,z-x)+\bar p_\alpha(s,y+\lambda(y'-y)-x)\right)\\
&\quad\lesssim |y+\lambda(y'-y)-z|^{\beta}\left(1+s^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right)\left(\bar p_\alpha(s,z-x)+\bar p_\alpha(s,y+\lambda(y'-y)-x)\right),
\end{aligned}
\tag{4.3.6}
$$

recalling that $\beta\le\gamma_\varepsilon$ for the last inequality. Plugging this into (4.3.5) and using (4.2.2), recalling that on the considered time integration, $(t-s)\ge|y'-y|^\alpha$ (local diagonal regime), we get,

$$
\begin{aligned}
|\Delta_3|&\lesssim\int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\int_0^1\int\left(\bar p_\alpha(s,z-x)+\bar p_\alpha(s,y+\lambda(y'-y)-x)\right)\left(1+s^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right)\\
&\quad\times\frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{1+\gamma_\varepsilon-\beta}{\alpha}}}\bar p_\alpha(t-s,y+\lambda(y'-y)-z)\,\mathrm{d}z\,\mathrm{d}\lambda\,\mathrm{d}s\\
&\lesssim\int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\int_0^1\bar p_\alpha(t,y+\lambda(y'-y)-x)\left(1+s^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right)\frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{1+\gamma_\varepsilon-\beta}{\alpha}}}\,\mathrm{d}\lambda\,\mathrm{d}s\\
&\lesssim\left(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x)\right)\int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\left(1+s^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right)\frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{1+\gamma_\varepsilon-\beta}{\alpha}}}\,\mathrm{d}s,
\end{aligned}
$$

where, for the two last inequalities, we use the fact that for $s\ge t/2$, up to a modification of the underlying variance in the Brownian case, $\bar p_\alpha(s,y+\lambda(y'-y)-x)\lesssim\bar p_\alpha(t,y+\lambda(y'-y)-x)$ and since $|y-y'|\le t^{\frac1\alpha}$, $\bar p_\alpha(t,y+\lambda(y'-y)-x)\le\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x)$ with the same previous abuse of notation if $\alpha=2$. Finally, noting from the above definition of $\gamma_\varepsilon=(1\wedge\gamma)-\varepsilon$ that $(1+\gamma_\varepsilon-\beta)/\alpha<1\iff\gamma>\gamma_\varepsilon$, this yields

$$
|\Delta_3|\lesssim\left(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x)\right)|y-y'|^{\gamma_\varepsilon}t^{\frac{\gamma-\gamma_\varepsilon}{\alpha}}\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right).\tag{4.3.7}
$$

For Δ_4, write:

$$
\begin{aligned}
|\Delta_4|\le&\left|\int_{t-(|y'-y|\vee\eta)^\alpha}^{t}\int\left[\Gamma(0,x,s,z)b(s,z)-\Gamma(0,x,s,y)b(s,y)\right]\cdot\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\right|\\
&+\left|\int_{t-(|y'-y|\vee\eta)^\alpha}^{t}\int\left[\Gamma(0,x,s,z)b(s,z)-\Gamma(0,x,s,y')b(s,y')\right]\cdot\nabla_{y'} p_\alpha(t-s,y'-z)\,\mathrm{d}z\,\mathrm{d}s\right|\\
\lesssim&\int_{t-(|y'-y|\vee\eta)^\alpha}^{t}\int\bar p_\alpha(s,z-x)\left(1+s^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right)\frac{\bar p_\alpha(t-s,y-z)+\bar p_\alpha(t-s,y'-z)}{(t-s)^{\frac1\alpha-\frac{\beta}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}s,
\end{aligned}
$$

where we used (4.3.6) (with respectively $\lambda=0$ and $\lambda=1$ therein) and (4.2.2) for the second inequality. We get:

$$
\begin{aligned}
|\Delta_4|&\lesssim(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x))(|y'-y|\vee\eta)^{\gamma}\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right)\\
&\lesssim(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x))(|y'-y|\vee\eta)^{\gamma_\varepsilon}t^{\frac{\gamma-\gamma_\varepsilon}{\alpha}}\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}h_{0,x}^{\varepsilon,\eta}(r)\right)\right),
\end{aligned}
\tag{4.3.8}
$$

where we also used the fact that $(|y-y'|\vee\eta)^\alpha \le t/2$ as previously mentioned for the last inequality. Gathering estimates (4.3.3), (4.3.4), (4.3.7) and (4.3.8) and considering the fact that $|y-y'|^{\gamma_\varepsilon} \le (|y'-y|\vee\eta)^{\gamma_\varepsilon}$, we obtain

$$|\Gamma(0,x,t,y)-\Gamma(0,x,t,y')| \lesssim (\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x))\,\frac{(|y'-y|\vee\eta)^{\gamma_\varepsilon}}{t^{\frac{\gamma_\varepsilon}{\alpha}}}$$
$$\times\left[1+t^{\frac{\gamma_\varepsilon-\beta_\varepsilon}{\alpha}}+t^{\frac{\gamma}{\alpha}}(1+t^{-\frac{\beta}{\alpha}})+t^{\frac{\gamma-\beta}{\alpha}}\sup_{s\in(0,T]} h^{\varepsilon,\eta}_{0,x}(s)\right].$$

Noting that all the exponents of t appearing in brackets in the above equation are positive, we get, in turn taking the supremum for $t\in(0,T]$ on the l.h.s.

$$h^{\varepsilon,\eta}_{0,x}(t)\lesssim\left[1+T^{\frac{\gamma-\beta}{\alpha}}\sup_{r\in(0,T]} h^{\varepsilon,\eta}_{0,x}(r)\right].$$

Provided T is small enough, we obtain the (uniform in η) boundedness of $h^{\varepsilon,\eta}_{0,x}$. Taking the limit $\eta\to 0$ concludes the proof of (4.1.17). □

Proof of (4.1.19)**: forward spatial Hölder regularity for the scheme**

Let ε such that $\gamma_\varepsilon=(\gamma\wedge 1)-\varepsilon\ge\beta$ and set

$$g^{h,\varepsilon}_{s,x}(t):=\sup_{(z,z')\in(\mathbb{R}^d)^2}\left\{\frac{|\Gamma^h(s,x,t,z)-\Gamma^h(s,x,t,z')|(t-s)^{\frac{\gamma_\varepsilon}{\alpha}}}{(p_\alpha(t-s,z-x)+p_\alpha(t-s,z'-x))\,(|z-z'|+h^{\frac{1}{\alpha}})^{\gamma_\varepsilon}}\right\}.$$

We emphasize that the time shift in the Duhamel representation of the scheme (4.2.7), associated with the term $b(U_{\tau^h_s/h},X^h_{\tau_s})\nabla_y p_\alpha(t-s,y-X^h_s)$, induces the additional term in $h^{1/\alpha}$ in the normalization. Intuitively, this can be explained since if $|y'-y|^\alpha\le h$, then, close to the time-boundary, i.e. for s close to t, the local drift transition of the scheme of order $s-\tau^h_s$ is not negligible w.r.t $t-s$. When looking at the diffusion, this is usually dealt with by introducing a cut-off level at $t-|y'-y|^\alpha$. But on the remaining time interval, one can still have $s-\tau^h_s\ge|y'-y|^\alpha$ and the drift somehow prevails for the scheme.

Let us importantly point out that, from the Aronson type inequality (4.1.18) for the scheme, it readily follows that $g^{h,\varepsilon}_{s,x}(t)\lesssim h^{-\frac{\gamma_\varepsilon}{\alpha}}<+\infty$. In particular, this means that this quantity can be used in a Gronwall or circular type procedure as we actually do below.

Let us then introduce the following lemma:

Lemma 4.3 (Smoothing effect of the drift)**.** *Let ζ be a multi-index with length $1\le|\zeta|\le 2$. Then, for all $(x,\mathfrak{y})\in(\mathbb{R}^d)^2$, $0\le t/2\le s<t\le T$, $r>0$,*

$$\left|\int\Gamma^h(0,x,\tau^h_s,z)b(r,z)\nabla^\zeta_{\mathfrak{y}}p_\alpha(t-\tau^h_s,\mathfrak{y}-z)\,\mathrm{d}z\right|$$
$$\lesssim\left((t-\tau^h_s)^{\frac{\beta-|\zeta|}{\alpha}}+h^{\frac{\beta}{\alpha}}(t-\tau^h_s)^{-\frac{|\zeta|}{\alpha}}\right)\left(1+\frac{1}{(\tau^h_s)^{\frac{\beta}{\alpha}}}+\frac{g^{h,\varepsilon}_{0,x}(\tau^h_s)}{(\tau^h_s)^{\frac{\beta}{\alpha}}}\right)\bar p_\alpha(t,\mathfrak{y}-x). \tag{4.3.9}$$

Proof of Lemma 4.3. To prove this, let us use the following cancellation argument

$$\begin{aligned}I&:=\int\Gamma^h(0,x,\tau^h_s,z)b(r,z)\nabla_{\mathfrak{y}}p_\alpha(t-\tau^h_s,\mathfrak{y}-z)\,\mathrm{d}z\\&=\int[\Gamma^h(0,x,\tau^h_s,z)b(r,z)-\Gamma^h(0,x,\tau^h_s,\mathfrak{y})b(r,\mathfrak{y})]\nabla_{\mathfrak{y}}p_\alpha(t-s,\mathfrak{y}-z)\,\mathrm{d}z.\end{aligned} \tag{4.3.10}$$

Then, we write similarly to (4.3.6),

$$
\begin{aligned}
&|\Gamma^h(0,x,\tau_s^h,z)b(s,z)-\Gamma^h(0,x,\tau_s^h,\mathfrak{y})b(s,\mathfrak{y})|\\
&\quad\lesssim |\mathfrak{y}-z|^\beta\bar{p}_\alpha(\tau_s^h,z-x)+\|b\|_{L^\infty}\Big(1+g_{0,x}^{h,\varepsilon}(\tau_s^h)\Big)\frac{(|\mathfrak{y}-z|+h^{\frac{1}{\alpha}})^\beta}{(\tau_s^h)^{\frac{\beta}{\alpha}}}\left(\bar{p}_\alpha(\tau_s^h,z-x)+\bar{p}_\alpha(\tau_s^h,\mathfrak{y}-x)\right)\\
&\quad\lesssim (|\mathfrak{y}-z|+h^{\frac{1}{\alpha}})^\beta\left(1+\frac{1}{(\tau_s^h)^{\frac{\beta}{\alpha}}}+\frac{g_{0,x}^{h,\varepsilon}(\tau_s^h)}{(\tau_s^h)^{\frac{\beta}{\alpha}}}\right)\left(\bar{p}_\alpha(\tau_s^h,z-x)+\bar{p}_\alpha(\tau_s^h,\mathfrak{y}-x)\right). \qquad (4.3.11)
\end{aligned}
$$

Plugging this into (4.3.10) and using (4.2.2), we get

$$
\begin{aligned}
&\left|\int\Gamma^h(0,x,\tau_s^h,z)b(r,z)\nabla_{\mathfrak{y}}^\zeta p_\alpha(t-\tau_s^h,\mathfrak{y}-z)\,\mathrm{d}z\right|\\
&\lesssim\Big((t-\tau_s^h)^{\frac{\beta-|\zeta|}{\alpha}}+h^{\frac{\beta}{\alpha}}(t-\tau_s^h)^{-\frac{|\zeta|}{\alpha}}\Big)\left(1+\frac{1}{(\tau_s^h)^{\frac{\beta}{\alpha}}}+\frac{g_{0,x}^{h,\varepsilon}(\tau_s^h)}{(\tau_s^h)^{\frac{\beta}{\alpha}}}\right)\left[\bar{p}_\alpha(t,\mathfrak{y}-x)+\bar{p}_\alpha(\tau_s^h,\mathfrak{y}-x)\right].
\end{aligned}
$$

Using the fact that $s\geq t/2$, up to a modification of the underlying variance in the Brownian case, $\bar{p}_\alpha(\tau_s^h,\mathfrak{y}-x)\lesssim\bar{p}_\alpha(t,\mathfrak{y}-x)$, we obtain (4.3.9). □

Let us first write

$$
\begin{aligned}
\Gamma^h&(0,x,t,y')-\Gamma^h(0,x,t,y)=p_\alpha(t,y'-x)-p_\alpha(t,y-x)\\
&+\int_0^{t/2}\frac{1}{h}\int_{\tau_s^h}^{\tau_s^h+h}\int\Gamma^h(0,x,\tau_s^h,z)b(r,z)\\
&\qquad\times\mathbb{E}_{\tau_s^h,z,r}\left[\nabla_y p_\alpha(t-s,y-X_s^h)-\nabla_{y'}p_\alpha(t-s,y'-X_s^h)\right]\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\\
&+\int_{t/2}^{t}\frac{1}{h}\int_{\tau_s^h}^{\tau_s^h+h}\int\Gamma^h(0,x,\tau_s^h,z)b(r,z)\\
&\qquad\times\mathbb{E}_{\tau_s^h,z,r}\left[\nabla_y p_\alpha(t-s,y-X_s^h)-\nabla_{y'}p_\alpha(t-s,y'-X_s^h)\right]\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\\
=:&\ \Delta_1+\Delta_2+\Delta_3, \qquad (4.3.12)
\end{aligned}
$$

where we denoted $\mathbb{E}_{\tau_s^h,z,r}[\cdot]:=\mathbb{E}[\cdot|X_{\tau_s^h}^h=z,U_{\tau_s^h/h}=r]$. For Δ_1, we use (4.3.2) to write

$$
\begin{aligned}
|\Delta_1|=|p_\alpha(t,y'-x)-p_\alpha(t,y-x)|&\lesssim\frac{|y-y'|^{\gamma_\varepsilon}}{t^{\frac{\gamma_\varepsilon}{\alpha}}}(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x))\\
&\lesssim\frac{(|y-y'|^\alpha\vee h)^{\frac{\gamma_\varepsilon}{\alpha}}}{t^{\frac{\gamma_\varepsilon}{\alpha}}}(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)). \qquad (4.3.13)
\end{aligned}
$$

For Δ_2, we use (4.3.2) to write

$$
|\nabla_y p_\alpha(t-s,y-w)-\nabla_{y'}p_\alpha(t-s,y'-w)|\lesssim\frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{\gamma_\varepsilon+1}{\alpha}}}(\bar{p}_\alpha(t-s,y-w)+\bar{p}_\alpha(t-s,y'-w))
$$

which yields, similarly to (4.3.4),

$$
\begin{aligned}
|\Delta_2|\lesssim&\int_0^{t/2}\int\bar{p}_\alpha(\tau_s^h,z-x)\|b\|_{L^\infty}\\
&\qquad\times\int\bar{p}_\alpha(s-\tau_s^h,w-z)\frac{|y-y'|^{\gamma_\varepsilon}}{(t-s)^{\frac{\gamma_\varepsilon+1}{\alpha}}}(\bar{p}_\alpha(t-s,y-w)+\bar{p}_\alpha(t-s,y'-w))\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}s\\
\lesssim&\ \frac{(|y-y'|^\alpha\vee h)^{\frac{\gamma_\varepsilon}{\alpha}}}{t^{\frac{\beta_\varepsilon}{\alpha}}}(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)), \qquad (4.3.14)
\end{aligned}
$$

using the fact that the terms in $(t-s)$ can be taken out from the integral on the considered time interval.

For Δ_3, let us develop the conditional expectation as follows:

$$
\begin{aligned}
&\mathbb{E}_{\tau_s^h,z,r}\left[\nabla_y p_\alpha(t-s,y-X_s^h)-\nabla_{y'}p_\alpha(t-s,y'-X_s^h)\right]\\
&=\mathbb{E}_{\tau_s^h,z,r}\left[\nabla_y p_\alpha\left(t-s,y-(z+b\left(r,z\right)(s-\tau_s^h)+Z_s-Z_{\tau_s^h})\right)\right]\\
&\quad-\mathbb{E}_{\tau_s^h,z,r}\left[\nabla_{y'} p_\alpha\left(t-s,y'-(z+b\left(r,z\right)(s-\tau_s^h)+Z_s-Z_{\tau_s^h})\right)\right]\\
&=\nabla_y p_\alpha\left(t-\tau_s^h,y-z-b\left(r,z\right)(s-\tau_s^h)\right)-\nabla_{y'}p_\alpha\left(t-\tau_s^h,y'-z-b\left(r,z\right)(s-\tau_s^h)\right)\\
&=\nabla_{w_2}p_\alpha\left(t-\tau_s^h,w_2\right)|_{w_2=y-z-b(r,z)(s-\tau_s^h)}-\nabla_{w_1}p_\alpha\left(t-\tau_s^h,w_1\right)|_{w_1=y-z-b(r,y)(s-\tau_s^h)}\\
&\quad+\nabla_{w_1}p_\alpha\left(t-\tau_s^h,w_1\right)|_{w_1=y-z-b(r,y)(s-\tau_s^h)}-\nabla_{w_1'}p_\alpha\left(t-\tau_s^h,w_1'\right)|_{w_1'=y'-z-b(r,y')(s-\tau_s^h)}\\
&\quad+\nabla_{w_1'}p_\alpha\left(t-\tau_s^h,w_1'\right)|_{w_1'=y'-z-b(r,y')(s-\tau_s^h)}-\nabla_{w_2'}p_\alpha\left(t-\tau_s^h,w_2'\right)|_{w_2'=y'-z-b(r,z)(s-\tau_s^h)}\\
&=-\int_0^1\nabla_{w_1}^2p_\alpha\left(t-\tau_s^h,w_1-\mu[b(r,z)-b(r,y)](s-\tau_s^h)\right)|_{w_1=y-z-b(r,y)(s-\tau_s^h)}\cdot\left[b\left(r,z\right)-b\left(r,y\right)\right](s-\tau_s^h)\,\mathrm{d}\mu\\
&\quad+\nabla_{w_1}p_\alpha\left(t-\tau_s^h,w_1\right)|_{w_1=y-z-b(r,y)(s-\tau_s^h)}-\nabla_{w_1'}p_\alpha\left(t-\tau_s^h,w_1'\right)|_{w_1'=y'-z-b(r,y')(s-\tau_s^h)}\\
&\quad+\int_0^1\nabla_{w_1'}^2p_\alpha\left(t-\tau_s^h,w_1'-\mu[b(r,z)-b(r,y')](s-\tau_s^h)\right)|_{w_1'=y'-z-b(r,y')(s-\tau_s^h)}\cdot\left[b\left(r,z\right)-b\left(r,y'\right)\right](s-\tau_s^h)\,\mathrm{d}\mu,
\end{aligned}
\tag{4.3.15}
$$

yielding the corresponding terms Δ_{31},Δ_{32} and Δ_{33} once plugged into (4.3.12). For Δ_{31} and Δ_{33}, since $t-\tau_s^h\geq s-\tau_s^h$, using the boundedness and the Hölder regularity of b and then (4.2.3), for $\tilde{y}\in\{y,y'\}$,

$$
\begin{aligned}
&|\nabla_{\tilde{w}}^2p_\alpha(t-\tau_s^h,\tilde{w}-\mu\left[b(r,z)-b(r,\tilde{y})\right](s-\tau_s^h))|_{\tilde{w}=\tilde{y}-z-b(r,\tilde{y})(s-\tau_s^h)}\cdot\left[b\left(r,z\right)-b\left(r,\tilde{y}\right)\right]\\
\lesssim&(t-\tau_s^h)^{\frac{\beta-2}{\alpha}}\bar{p}_\alpha(t-\tau_s^h,\tilde{y}-z).
\end{aligned}
\tag{4.3.16}
$$

This then yields, using again $s-\tau_s^h\leq t-\tau_s^h$,

$$
\begin{aligned}
|\Delta_{31}|+|\Delta_{33}|&\lesssim\int_{t/2}^t\int\bar{p}_\alpha(\tau_s^h,z-x)(s-\tau_s^h)(t-\tau_s^h)^{\frac{\beta-2}{\alpha}}\left(\bar{p}_\alpha(t-\tau_s^h,y-z)+\bar{p}_\alpha(t-\tau_s^h,y'-z)\right)\mathrm{d}z\,\mathrm{d}s\\
&\lesssim\left(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)\right)h^{\frac{\gamma_\varepsilon}{\alpha}}\int_{t/2}^t(t-\tau_s^h)^{1+\frac{\beta-2-\gamma_\varepsilon}{\alpha}}\,\mathrm{d}s.
\end{aligned}
$$

Note that, by definition, $\gamma_\varepsilon=\big((\alpha+\beta-1)\wedge1\big)-\varepsilon\leq\alpha+\beta-1-\varepsilon$, so that

$$
1+\frac{\beta-2-\gamma_\varepsilon}{\alpha}=\frac{\alpha+\beta-2-\gamma_\varepsilon}{\alpha}=\frac{\gamma-\gamma_\varepsilon-1}{\alpha}\geq\frac{-1+\varepsilon}{\alpha}>-1,
$$

using as well that $\alpha>1$ for the last inequality. In turn, we obtain

$$
\begin{aligned}
|\Delta_{31}|+|\Delta_{33}|&\lesssim\left(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)\right)h^{\frac{\gamma_\varepsilon}{\alpha}}t^{\frac{\gamma-\gamma_\varepsilon+\alpha-1}{\alpha}}\\
&\lesssim\left(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)\right)\left(|y-y'|^\alpha\vee h\right)^{\frac{\gamma_\varepsilon}{\alpha}}T^{\frac{\gamma-\gamma_\varepsilon+\alpha-1}{\alpha}}.
\end{aligned}
\tag{4.3.17}
$$

Let us turn to Δ_{32}, which we split into two parts depending on whether the inner gradient is in diagonal (and in that case using a Taylor expansion) or off-diagonal regime:

$$\begin{aligned}\Delta_{32} =&\int_{t/2}^{t-|y-y'|^\alpha\vee h} \frac{1}{h}\int_{\tau_s^h}^{\tau_s^h+h}\int \Gamma^h(0,x,\tau_s^h,z)b(r,z)\\ &\times\int_0^1 \nabla^2_{w_1}p_\alpha\left(t-\tau_s^h, w_1+\mu\left[y'-y-(s-\tau_s^h)(b(r,y')-b(r,y))\right]\right)|_{w_1=y-z-b(r,y)(s-\tau_s^h)}\\ &\qquad\cdot\left[y'-y-(s-\tau_s^h)(b(r,y')-b(r,y))\right]\mathrm{d}\mu\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\\ &+\int_{t-|y-y'|^\alpha\vee h}^{t}\frac{1}{h}\int_{\tau_s^h}^{\tau_s^h+h}\int\Gamma^h(0,x,\tau_s^h,z)b(r,z)\\ &\times\left[\nabla_{w_1}p_\alpha\left(t-\tau_s^h,w_1\right)|_{w_1=y-z-b(r,y)(s-\tau_s^h)}-\nabla_{w_1'}p_\alpha\left(t-\tau_s^h,w_1'\right)_{w_1'=y-z-b(r,y)(s-\tau_s^h)}\right]\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\\ =:&\Delta_{321}+\Delta_{322}.\end{aligned}$$

For Δ_{321}, we use (4.3.9) with $\mathfrak{y}=y-b(r,y)(s-\tau_s^h)+\mu\left[y'-y-(b(r,y')-b(r,y))(s-\tau_s^h)\right]$. Note that the term in $h^{\beta/\alpha}\le(t-\tau_s^h)^{\beta/\alpha}$ on the associated time regime for Δ_{321}.

Using as well the fact that, in the regime $|y-y'|\le t^{\frac{1}{\alpha}}$,

$$\bar p_\alpha(t,y-x-b(r,y)(s-\tau_s^h)+\mu\left[y'-y-(b(r,y')-b(r,y))(s-\tau_s^h)\right])\lesssim\bar p_\alpha(t,y-x)$$

to obtain

$$|\Delta_{321}|\lesssim\bar p_\alpha(t,y-x)\int_{t/2}^{t-|y-y'|^\alpha\vee h}\frac{\left[|y-y'|+(s-\tau_s^h)|y-y'|^\beta\right]}{(t-\tau_s^h)^{\frac{2-\beta}{\alpha}}}\bar p_\alpha(t,y-x)\left(1+\frac{1}{(\tau_s^h)^{\frac{\beta}{\alpha}}}+\frac{g_{0,x}^{h,\varepsilon}(\tau_s^h)}{(\tau_s^h)^{\frac{\beta}{\alpha}}}\right)\mathrm{d}s.$$

Recall that on the considered time interval $t-s\ge|y'-y|^\alpha\vee h$ and $|y-y'|\le t^{\frac{1}{\alpha}}$. From the fact that $s-\tau_s^h\le t-\tau_s^h$, we then get

$$\begin{aligned}|\Delta_{321}|&\lesssim\bar p_\alpha(t,y-x)(h^{\frac{\gamma_\varepsilon}{\alpha}}t^{\frac{\beta}{\alpha}}+|y'-y|^{\gamma_\varepsilon})\int_{t/2}^{t-|y-y'|^\alpha\vee h}(t-\tau_s^h)^{\frac{\beta-1-\gamma_\varepsilon}{\alpha}}\left(1+\frac{1}{(\tau_s^h)^{\frac{\beta}{\alpha}}}+\frac{g_{0,x}^{h,\varepsilon}(\tau_s^h)}{(\tau_s^h)^{\frac{\beta}{\alpha}}}\right)\mathrm{d}s\\ &\lesssim\bar p_\alpha(t,y-x)(h^{\frac{\gamma_\varepsilon}{\alpha}}+|y'-y|^{\gamma_\varepsilon})t^{\frac{\gamma-\gamma_\varepsilon}{\alpha}}\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}g_{0,x}^{h,\varepsilon}(r)\right)\right),\end{aligned}\tag{4.3.18}$$

using as well $\frac{\beta-1-\gamma_\varepsilon}{\alpha}>-1\iff\gamma>\gamma_\varepsilon$ for the last inequality.

For Δ_{322}, denote, for $\mathfrak{y}\in\{y,y'\}$,

$$\begin{aligned}\delta_{322}(\mathfrak{y}):=&\int_{t-|y-y'|^\alpha\vee h}^{t}\frac{1}{h}\int_{\tau_s^h}^{\tau_s^h+h}\\ &\times\left|\int\Gamma^h(0,x,\tau_s^h,z)b(r,z)\cdot\mathbb{E}_{\tau_s^h,z}\left[\nabla_{\mathfrak{y}}p_\alpha\left(t-\tau_s^h,\mathfrak{y}-z-b(r,\mathfrak{y})(s-\tau_s^h)\right)\right]\mathrm{d}z\right|\mathrm{d}r\,\mathrm{d}s\end{aligned}$$

so that $|\Delta_{322}|\le\delta_{322}(y)+\delta_{322}(y')$. Similarly to Δ_{321}, we now use (4.3.9) with $\mathfrak{y}=\mathfrak{y}-b(r,\mathfrak{y})(s-\tau_s^h)$ observing that on the considered time integration interval the term in $h^{\beta/\alpha}$ of (4.3.9) remains:

$$\begin{aligned}\delta_{322}(\mathfrak{y})&\lesssim\bar p_\alpha(t,\mathfrak{y}-x)\int_{t-|y-y'|^\alpha\vee h}^{t}\left((t-\tau_s^h)^{\frac{\beta-1}{\alpha}}+h^{\frac{\beta}{\alpha}}(t-\tau_s^h)^{-\frac{1}{\alpha}}\right)\left(1+\frac{1}{(\tau_s^h)^{\frac{\beta}{\alpha}}}+\frac{g_{0,x}^{h,\varepsilon}(\tau_s^h)}{(\tau_s^h)^{\frac{\beta}{\alpha}}}\right)\mathrm{d}s\\ &\lesssim\bar p_\alpha(t,\mathfrak{y}-x)(|y-y'|^\alpha\vee h)^{\frac{\gamma}{\alpha}}\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]}g_{0,x}^{h,\varepsilon}(r)\right)\right).\end{aligned}\tag{4.3.19}$$

Gathering (4.3.17), (4.3.18) and (4.3.19), we obtain

$$\begin{aligned}|\Delta_3| &\lesssim (\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x))\,(|y-y'|^\alpha\vee h)^{\frac{\gamma_\varepsilon}{\alpha}}\\ &\quad\times\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]} g^{h,\varepsilon}_{0,x}(r)\right)\right)\left[(|y-y'|^\alpha\vee h)^{\frac{\gamma-\gamma_\varepsilon}{\alpha}}+t^{\frac{\gamma-\gamma_\varepsilon}{\alpha}}+t^{\frac{\gamma-\gamma_\varepsilon+\alpha-1}{\alpha}}\right].\end{aligned}\tag{4.3.20}$$

Along with (4.3.13) and (4.3.14), we eventually get

$$\begin{aligned}\frac{|\Gamma^h(0,x,t,y)-\Gamma^h(0,x,t,y')|t^{\frac{\gamma_\varepsilon}{\alpha}}}{(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x))\,(|y-y'|^\alpha\vee h)^{\frac{\gamma_\varepsilon}{\alpha}}} &\lesssim 1+T^{\frac{\gamma_\varepsilon-\beta_\varepsilon}{\alpha}}\\ &\quad+\left(1+t^{-\frac{\beta}{\alpha}}\left(1+\sup_{r\in(0,T]} g^{h,\varepsilon}_{0,x}(r)\right)\right)\left[(|y-y'|^\alpha\vee h)^{\frac{\gamma-\gamma_\varepsilon}{\alpha}}t^{\frac{\gamma_\varepsilon}{\alpha}}+t^{\frac{\gamma}{\alpha}}+t^{\frac{\gamma+\alpha-1}{\alpha}}\right]\\ &\lesssim 1+T^{\frac{\gamma_\varepsilon-\beta_\varepsilon}{\alpha}}+T^{\frac{\gamma-\beta}{\alpha}}+\sup_{r\in(0,T]} g^{h,\varepsilon}_{0,x}(r)\left[t^{\frac{\gamma-\beta}{\alpha}}+t^{\frac{\gamma+\alpha-1-\beta}{\alpha}}\right],\end{aligned}$$

using as well that $|y'-y|^\alpha\vee h\le t$ for the last inequality. Since $\gamma-\beta=\alpha-1>0$, Equation (4.1.19) then follows taking the supremum in time in the previous inequality provided T is small enough. □

4.3.2 Proof of (4.1.16): forward time Hölder regularity for the diffusion

Proof of (4.1.16). We proceed here with the proof of the forward time sensitivity. Importantly, the proof will use the previously proved claim (4.1.17) which actually also gives β-Hölder sensitivity since $\beta\le\gamma_\varepsilon$. W.l.o.g. we take $s=0$ for notational simplicity. Starting from the Duhamel representation (4.2.6) and using cancellation arguments, we have for $0<t<t'\le T,\ x,y\in\mathbb{R}^d$,

$$\begin{aligned}\Gamma(0,x,t,y)-\Gamma(0,x,t',y) &= p_\alpha(t,y-x)-p_\alpha(t',y-x)\\ &\quad+\int_t^{t'}\int\Gamma(0,x,s,z)b(r,z)\cdot\nabla_y p_\alpha(t'-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\\ &\quad+\int_0^{t}\int\Gamma(0,x,s,z)b(r,z)\cdot[\nabla_y p_\alpha(t'-s,y-z)-\nabla_y p_\alpha(t-s,y-z)]\,\mathrm{d}z\,\mathrm{d}s\\ &= p_\alpha(t,y-x)-p_\alpha(t',y-x)\\ &\quad+\int_t^{t'}\int[\Gamma(0,x,s,z)b(r,z)-\Gamma(0,x,s,y)b(r,y)]\cdot\nabla_y p_\alpha(t'-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\\ &\quad+\int_0^{t}\int\Gamma(0,x,s,z)b(r,z)\\ &\qquad\qquad\cdot[\nabla_y p_\alpha(t'-s,y-z)-\nabla_y p_\alpha(t-s,y-z)]\,\mathrm{d}z\,\mathrm{d}s\\ &=: H_1+H_2+H_3.\end{aligned}\tag{4.3.21}$$

For H_1, we directly use (4.3.1) to write

$$|H_1|=|p_\alpha(t,y-x)-p_\alpha(t',y-x)|\lesssim\frac{(t'-t)^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}}}p_\alpha(t,y-x).\tag{4.3.22}$$

For H_2, let us use the regularity of b and the forward spatial regularity of Γ, (4.1.17) to write:

$$|\Gamma(0,x,s,z)b(s,z)-\Gamma(0,x,s,y)b(s,y)|\lesssim\bar p_\alpha(s,z-x)|y-z|^\beta\left(1+\frac{1}{s^{\frac{\beta}{\alpha}}}\right)+\bar p_\alpha(s,y-x)\frac{|y-z|^\beta}{s^{\frac{\beta}{\alpha}}}.\tag{4.3.23}$$

Plugging this into H_2 and using the fact that for $s \geq t$, $\bar{p}_\alpha(s, y-x) \lesssim \bar{p}_\alpha(t, y-x)$ along with (4.2.2), we have

$$\begin{aligned}
|H_2| &\lesssim \int_t^{t'} \int \bar{p}_\alpha(s, z-x)|y-z|^\beta \left(1 + \frac{1}{s^{\frac{\beta}{\alpha}}}\right) |\nabla_y p_\alpha(t'-s, y-z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\quad + \int_t^{t'} \int \bar{p}_\alpha(s, y-x) \frac{|y-z|^\beta}{s^{\frac{\beta}{\alpha}}} |\nabla_y p_\alpha(t'-s, y-z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim \int_t^{t'} \int \bar{p}_\alpha(s, z-x) \left(1 + \frac{1}{s^{\frac{\beta}{\alpha}}}\right) (t'-s)^{\frac{\beta-1}{\alpha}} \bar{p}_\alpha(t'-s, y-z) \, \mathrm{d}z \, \mathrm{d}s \\
&\quad + \bar{p}_\alpha(t, y-x) \int_t^{t'} \int \frac{(t'-s)^{\frac{\beta-1}{\alpha}}}{s^{\frac{\beta}{\alpha}}} \bar{p}_\alpha(t'-s, y-z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t, y-x)(1 + t^{-\frac{\beta}{\alpha}})(t-t')^{\frac{\gamma}{\alpha}}. \qquad (4.3.24)
\end{aligned}$$

For H_3, let us split it into three parts again, using a cancellation argument on two of them:

$$\begin{aligned}
H_3 &= \int_0^{\frac{t-(t'-t)}{2}} \int \Gamma(0, x, s, z) b(r, z) \cdot [\nabla_y p_\alpha(t'-s, y-z) - \nabla_y p_\alpha(t-s, y-z)] \, \mathrm{d}z \, \mathrm{d}s \\
&\quad + \int_{\frac{t-(t'-t)}{2}}^{t-(t'-t)} \int [\Gamma(0, x, s, z) b(r, z) - \Gamma(0, x, s, y) b(r, y)] \cdot [\nabla_y p_\alpha(t'-s, y-z) - \nabla_y p_\alpha(t-s, y-z)] \, \mathrm{d}z \, \mathrm{d}s \\
&\quad + \int_{t-(t'-t)}^{t} \int [\Gamma(0, x, s, z) b(r, z) - \Gamma(0, x, s, y) b(r, y)] \cdot [\nabla_y p_\alpha(t'-s, y-z) - \nabla_y p_\alpha(t-s, y-z)] \, \mathrm{d}z \, \mathrm{d}s \\
&=: H_{31} + H_{32} + H_{33}.
\end{aligned}$$

For H_{31}, notice that for $s \leq (t-(t'-t))/2 = t - t'/2$, $t'-s \geq 3t'/2 - t$ and $t - s \geq t'/2$, there will be no time singularities in $(t-s)$ or $(t'-s)$ to integrate. There is thus no need to use a cancellation argument to derive a smoothing effect. Using simply (4.3.1), we get

$$\begin{aligned}
|H_{31}| &\lesssim \int_0^{\frac{t-(t'-t)}{2}} \int \bar{p}_\alpha(s, z-x) \|b\|_{L^\infty} \frac{(t'-t)^{\frac{\gamma}{\alpha}}}{(t-s)^{\frac{\gamma+1}{\alpha}}} [\bar{p}_\alpha(t'-s, y-z) + \bar{p}_\alpha(t-s, y-z)] \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim (t'-t)^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}} [\bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t', y-x)]. \qquad (4.3.25)
\end{aligned}$$

For H_{32}, notice that on the considered time interval, $t'-s \geq 2(t'-t)$, so we are at the right time scale to use a Taylor expansion in time:

$$\begin{aligned}
&H_{32} \\
&= \int_{\frac{t-(t'-t)}{2}}^{t-(t'-t)} \int [\Gamma(0, x, s, z) b(r, z) - \Gamma(0, x, s, y) b(r, y)] \cdot \int_0^1 \partial_t \nabla_y p_\alpha(t-s+\lambda(t'-t), y-z)(t'-t) \, \mathrm{d}\lambda \, \mathrm{d}z \, \mathrm{d}s.
\end{aligned}$$

Using (4.3.23) and then (4.2.2) along with the fact that for on the considered time interval, we have $t - s + \lambda(t'-t) \asymp t - s$ and $\bar{p}_\alpha(s, y-x) \lesssim \bar{p}_\alpha(t, y-x)$, we get

$$\begin{aligned}
|H_{32}| &\lesssim (t'-t) \int_{\frac{t-(t'-t)}{2}}^{t-(t'-t)} \int \left[\bar{p}_\alpha(s, z-x)\left(1 + s^{-\frac{\beta}{\alpha}}\right) + \bar{p}_\alpha(s, y-x) s^{-\frac{\beta}{\alpha}}\right] \\
&\qquad \times \int_0^1 |y-z|^\beta |\partial_t \nabla_y p_\alpha(t-s+\lambda(t'-t), y-z)| \, \mathrm{d}\lambda \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim (t'-t) \int_{\frac{t-(t'-t)}{2}}^{t-(t'-t)} \int \left[\bar{p}_\alpha(s, z-x)\left(1 + s^{-\frac{\beta}{\alpha}}\right) + \bar{p}_\alpha(t, y-x) s^{-\frac{\beta}{\alpha}}\right] (t-s)^{-1+\frac{\beta-1}{\alpha}} \bar{p}_\alpha(t-s, y-z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim (t'-t)^{\frac{\gamma}{\alpha}} (1 + t^{-\frac{\beta}{\alpha}}) \bar{p}_\alpha(t, y-x), \qquad (4.3.26)
\end{aligned}$$

observing that $-1+(\beta-1)/\alpha<-1$ for the last inequality.

For H_{33}, we take advantage of the fact that we integrate on a time interval whose length corresponds to the time difference. This means we can use the smoothing in time effect for each term of the difference (no need to expand in space the difference of the gradients). Namely,

$$
\begin{aligned}
|H_{33}| \le & \left|\int_{t-(t'-t)}^{t} \int [\Gamma(0,x,s,z)b(r,z)-\Gamma(0,x,s,y)b(r,y)] \cdot \nabla_y p_\alpha(t'-s,y-z)\right| \\
& + \left|\int_{t-(t'-t)}^{t} \int [\Gamma(0,x,s,z)b(r,z)-\Gamma(0,x,s,y)b(r,y)] \cdot \nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\right| \\
\le & \int_{t-(t'-t)}^{t} \int \left[\bar p_\alpha(s,z-x)\left(1+s^{-\frac{\beta}{\alpha}}\right)+\bar p_\alpha(t,y-x)s^{-\frac{\beta}{\alpha}}\right](t-s)^{\frac{\beta-1}{\alpha}} \\
& \times \left(\bar p_\alpha(t-s,y-z)+\bar p_\alpha(t'-s,y-z)\right)\mathrm{d}z\,\mathrm{d}s,
\end{aligned}
$$

using again (4.3.23) for the last inequality. This eventually yields

$$
|H_{33}| \lesssim (t'-t)^{\frac{\gamma}{\alpha}}(1+t^{-\frac{\beta}{\alpha}})(\bar p_\alpha(t,y-x)+\bar p_\alpha(t',y-x)). \tag{4.3.27}
$$

Gathering estimates (4.3.22), (4.3.24), (4.3.25), (4.3.26) and (4.3.27), we obtain

$$
|\Gamma(0,x,t,y)-\Gamma(0,x,t',y)| \lesssim \left(\frac{t'-t}{t}\right)^{\frac{\gamma}{\alpha}}\left[\bar p_\alpha(t,y-x)+\bar p_\alpha(t',y-x)\right], \tag{4.3.28}
$$

which precisely gives (4.1.16) since we have assumed $s=0$. □

4.4 About the full parabolic bootstrap in the forward variable for the diffusion

The point of this section is to provide a proof of the full parabolic bootstrap for the diffusion in its forward variable in the case $\alpha+\beta-1<1$. Indeed, when $\alpha+\beta-1\ge 1$, it cannot be expected to have an exponent greater than 1 and (4.1.17) is already sharp.

Namely, we prove the following : there exists $C:=C(d,b,\alpha,T)$ s.t. for all $0\le s<t\le T$, $(x,y,w)\in(\mathbb{R}^d)^3$ s.t. $|y-w|\le(t-s)^{\frac{1}{\alpha}}$,

$$
|\Gamma(s,x,t,y)-\Gamma(s,x,t,w)| \le C\left(\frac{|y-w|}{(t-s)^{\frac{1}{\alpha}}}\right)^{\gamma}\bar p_\alpha(t-s,w-x). \tag{4.4.1}
$$

The approach is very similar to the previous one to show (4.1.17) and we present here the result for the sake of completeness only as we do not make use of it. Equation (4.1.17) (taking ε therein s.t. $\gamma_\varepsilon=\beta$) is enough for the proof of Theorem 4.1.

Proof. Set, for η meant to be small,

$$
h^\eta_{s,x}(t) := \sup_{(z,z')\in(\mathbb{R}^d)^2}\left\{\frac{|\Gamma(s,x,t,z)-\Gamma(s,x,t,z')|(t-s)^{\frac{\gamma}{\alpha}}}{(\bar p_\alpha(t-s,z-x)+\bar p_\alpha(t-s,z'-x))\,(|z-z'|\vee\eta)^\gamma}\right\}.
$$

Since we already know from (4.1.15) that $\frac{|\Gamma(s,x,t,z)-\Gamma(s,x,t,z')|}{\bar p_\alpha(t-s,z-x)+\bar p_\alpha(t-s,z'-x)}<\infty$, we immediately have $h^\eta_{s,x}(t)\lesssim \eta^{-\gamma}<+\infty$. W.l.o.g., we take $s=0$ for simplicity and assume T is *small*, in particular $T\le 1$. Let us write for $0 < t \le T, (x,y,y') \in (\mathbb{R}^d)^3$ the following:

$$
\begin{aligned}
&\Gamma(0,x,t,y') - \Gamma(0,x,t,y) = p_\alpha(t,y'-x) - p_\alpha(t,y-x)\\
&\quad + \int_0^{t/2}\int \Gamma(0,x,s,z)b(s,z)\cdot(\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s\\
&\quad + \int_{t/2}^{t-(|y'-y|\vee\eta)^\alpha}\int \Gamma(0,x,s,z)b(s,z)\cdot(\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s\\
&\quad + \int_{t-(|y'-y|\vee\eta)^\alpha}^{t}\int \Gamma(0,x,s,z)b(s,z)\cdot(\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s\\
&=: \Delta_1+\Delta_2+\Delta_3+\Delta_4.
\end{aligned}
$$

We tacitly assume as well that $(|y'-y|\vee\eta)^\alpha \le t/2$ since otherwise, i.e. in the off-diagonal case, the expected control $[(\Gamma(0,x,t,y)-\Gamma(0,x,t,y'))t^{\frac{\gamma}{\alpha}}]/[(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x))(|y-y'|\vee\eta)^\gamma] \le C$ readily follows from the Aronson type bounds (4.1.15).

For Δ_1, we use the regularity of the stable kernel, (4.3.2) to write

$$
|\Delta_1| \lesssim \frac{|y-y'|^\gamma}{t^{\frac{\gamma}{\alpha}}}(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)). \tag{4.4.2}
$$

For Δ_2, using again (4.3.2), we write

$$
|\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z)| \lesssim \frac{|y-y'|^\gamma}{(t-s)^{\frac{\gamma+1}{\alpha}}}(\bar{p}_\alpha(t-s,y-z)+\bar{p}_\alpha(t-s,y'-z))
$$

which yields, along with (4.1.18) and the convolution property (4.1.13) of $\bar{p}_\alpha$,

$$
\begin{aligned}
|\Delta_2| &\lesssim \int_0^{t/2}\int \bar{p}_\alpha(s,z-x)\|b\|_{L^\infty}\frac{|y-y'|^\gamma}{(t-s)^{\frac{\gamma+1}{\alpha}}}(\bar{p}_\alpha(t-s,y-z)+\bar{p}_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim \frac{|y-y'|^\gamma}{t^{\frac{\beta}{\alpha}}}(\bar{p}_\alpha(t,y-x)+\bar{p}_\alpha(t,y'-x)).
\end{aligned} \tag{4.4.3}
$$

For Δ_3, using a Taylor expansion and then a cancellation argument, we have

$$
\begin{aligned}
\Delta_3 &= \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\int\int_0^1 \Gamma(0,x,s,z)b(s,z)\cdot\nabla_y^2 p_\alpha(t-s,y+\lambda(y'-y)-z)(y'-y)\,\mathrm{d}\lambda\,\mathrm{d}z\,\mathrm{d}s\\
&= \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha}\int_0^1\int [\Gamma(0,x,s,z)b(s,z) - \Gamma(0,x,s,y+\lambda(y'-y))b(s,y+\lambda(y'-y))]\\
&\qquad\qquad \cdot\nabla_y^2 p_\alpha(t-s,y+\lambda(y'-y)-z)(y'-y)\,\mathrm{d}z\,\mathrm{d}\lambda\,\mathrm{d}s.
\end{aligned} \tag{4.4.4}
$$

We then write, using (4.1.17) with $\varepsilon > 0$ s.t. $\gamma_\varepsilon = \beta$ when $|y+\lambda(y'-y)-z| \le s^{\frac{1}{\alpha}}$ and a triangular inequality when $|y+\lambda(y'-y)-z| \ge s^{\frac{1}{\alpha}}$,

$$
\begin{aligned}
&|\Gamma(0,x,s,z)b(s,z) - \Gamma(0,x,s,y+\lambda(y'-y))b(s,y+\lambda(y'-y))|\\
&\quad\lesssim |y+\lambda(y'-y)-z|^\beta \bar{p}_\alpha(s,z-x)\\
&\qquad + \|b\|_{L^\infty}\frac{|y+\lambda(y'-y)-z|^\beta}{s^{\frac{\beta}{\alpha}}}\mathbb{1}_{|y+\lambda(y'-y)-z|\le s^{\frac{1}{\alpha}}}(\bar{p}_\alpha(s,z-x)+\bar{p}_\alpha(s,y+\lambda(y'-y)-x))\\
&\quad\lesssim |y+\lambda(y'-y)-z|^\beta\left(1+s^{-\frac{\beta}{\alpha}}\right)\left(\bar{p}_\alpha(s,z-x)+\bar{p}_\alpha(s,y+\lambda(y'-y)-x)\right).
\end{aligned} \tag{4.4.5}
$$

Plugging this into (4.4.4) and using (4.2.2), recalling that on the considered time integration $(t-s) \geq |y'-y|^\alpha$ (local diagonal regime), we get for $\gamma_1 \in (\gamma, 1)$,

$$\begin{aligned}
|\Delta_3| &\lesssim \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha} \int_0^1 \int \left(\bar{p}_\alpha(s, z-x) + \bar{p}_\alpha(s, y+\lambda(y'-y)-x)\right) \left(1 + s^{-\frac{\beta}{\alpha}}\right) \\
&\quad \times \frac{|y-y'|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1-\beta}{\alpha}}} \bar{p}_\alpha(t-s, y+\lambda(y'-y)-z) \, \mathrm{d}z \, \mathrm{d}\lambda \, \mathrm{d}s \\
&\lesssim \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha} \int_0^1 \bar{p}_\alpha(t, y+\lambda(y'-y)-x) \left(1 + s^{-\frac{\beta}{\alpha}}\right) \frac{|y-y'|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1-\beta}{\alpha}}} \, \mathrm{d}\lambda \, \mathrm{d}s \\
&\lesssim \left(\bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t, y'-x)\right) \int_{t/2}^{t-(|y-y'|\vee\eta)^\alpha} \left(1 + s^{-\frac{\beta}{\alpha}}\right) \frac{|y-y'|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1-\beta}{\alpha}}} \, \mathrm{d}s,
\end{aligned}$$

where, for the two last inequalities, we use the fact that for $s \geq t/2$, up to a modification of the underlying variance in the Brownian case, $\bar{p}_\alpha(s, y+\lambda(y'-y)-x) \lesssim \bar{p}_\alpha(t, y+\lambda(y'-y)-x)$ and since $|y-y'| \leq t^{\frac{1}{\alpha}}$, $\bar{p}_\alpha(t, y+\lambda(y'-y)-x) \leq \bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t, y'-x)$ with the same previous abuse of notation if $\alpha = 2$. Finally, noting from the above γ_1, that $(1+\gamma_1-\beta)/\alpha > 1 \iff \gamma_1 > \gamma$, this yields

$$|\Delta_3| \lesssim (\bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t, y'-x)) \, |y-y'|^{\gamma_1} (|y-y'| \vee \eta)^{\gamma-\gamma_1} \lesssim (\bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t, y'-x)) \, (|y-y'| \vee \eta)^\gamma. \tag{4.4.6}$$

For Δ_4, write:

$$\begin{aligned}
|\Delta_4| &\leq \left| \int_{t-(|y'-y|\vee\eta)^\alpha}^{t} \int [\Gamma(0, x, s, z) b(s, z) - \Gamma(0, x, s, y) b(s, y)] \cdot \nabla_y p_\alpha(t-s, y-z) \, \mathrm{d}z \, \mathrm{d}s \right| \\
&\quad + \left| \int_{t-(|y'-y|\vee\eta)^\alpha}^{t} \int [\Gamma(0, x, s, z) b(s, z) - \Gamma(0, x, s, y') b(s, y')] \cdot \nabla_{y'} p_\alpha(t-s, y'-z) \, \mathrm{d}z \, \mathrm{d}s \right| \\
&\lesssim \int_{t-(|y'-y|\vee\eta)^\alpha}^{t} \int \bar{p}_\alpha(s, z-x) \left(1 + s^{-\frac{\beta}{\alpha}}\right) \frac{\bar{p}_\alpha(t-s, y-z) + \bar{p}_\alpha(t-s, y'-z)}{(t-s)^{\frac{1}{\alpha}-\frac{\beta}{\alpha}}} \, \mathrm{d}s \, \mathrm{d}z,
\end{aligned}$$

where we used (4.1.17), with $\gamma_\varepsilon = \beta$ and (4.2.2) for the second inequality. We get:

$$|\Delta_4| \lesssim (\bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t, y'-x)) (|y'-y| \vee \eta)^\gamma \left(1 + t^{-\frac{\beta}{\alpha}}\right). \tag{4.4.7}$$

Gathering estimates (4.4.2), (4.4.3), (4.4.6) and (4.4.7), we obtain

$$|\Gamma(0, x, t, y) - \Gamma(0, x, t, y')| \lesssim (\bar{p}_\alpha(t, y-x) + \bar{p}_\alpha(t, y'-x)) \frac{(|y'-y| \vee \eta)^\gamma}{t^{\frac{\gamma}{\alpha}}} \left[1 + t^{\frac{\gamma-\beta}{\alpha}}\right].$$

Noting that $\gamma - \beta = \alpha - 1 > 0$, we get, in turn

$$h^\eta_{0,x}(t) \lesssim 1.$$

Taking the limit $\eta \to 0$ concludes the proof of (4.4.1).

□

Chapter 5

Weak well-posedness and weak discretization error for stable-driven SDEs with Lebesgue drift

This chapter is based on the article [FJM24] written with Stéphane Menozzi and Benjamin Jourdain and published in the IMA Journal of Numerical Analysis. Therein, we are interested in the discretization of stable driven SDEs with additive noise for $\alpha \in (1,2)$ and $L^q - L^p$ drift under the Serrin type condition $\frac{\alpha}{q} + \frac{d}{p} < \alpha - 1$. We show weak existence and uniqueness as well as heat kernel estimates for the SDE and obtain a convergence rate of order $\frac{1}{\alpha}\big(\alpha - 1 - (\frac{\alpha}{q} + \frac{d}{p})\big)$ for the difference of the densities for the Euler scheme approximation involving suitably cutoffed and time randomized drifts.

5.1 Introduction

For a fixed time horizon $T > 0$, we are interested in the weak well-posedness and the Euler-Maruyama dicretization of the SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x, \qquad \forall t \in [0,T], \tag{5.1.1}$$

where b belongs to the Lebesgue space $L^q([0,T], L^p(\mathbb{R}^d)) := \Big\{ f : [0,T] \times \mathbb{R}^d : \|t \mapsto \|f(t,\cdot)\|_{L^p}\|_{L^q([0,T])} < \infty \Big\}$ $=: L^q - L^p$ and Z_t is a symmetric non-degenerate d-dimensional α-stable process, whose spectral measure is equivalent to the Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$ (see Subsection 5.1.4 for detailed assumptions on the noise).

We will work under the integrability condition

$$\frac{d}{p} + \frac{\alpha}{q} < \alpha - 1, \qquad \alpha \in (1,2). \tag{5.1.2}$$

This condition can be seen as the α-stable extension of the Krylov-Röckner condition for Brownian-driven SDEs (see [KR05]), although not guaranteeing *strong* well-posedness in the strictly stable setting ($\alpha < 2$). To this end, some additional smoothness conditions on the drift naturally appear, expressed in terms of Bessel potential spaces (see [XZ20]).

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In this paper, we first establish well-posedness of (5.1.1) through the study of a suitably associated Euler scheme, for which we prove heat kernel estimates. These then allow to follow the usual route to derive well-posedness: tightness, identification of a martingale problem solution and stability. As a consequence of this approach, we derive Duhamel-type expansions for the densities of the Euler scheme and the diffusion, which paves the way for an error analysis.

There has recently been a growing interest for SDEs of the type (5.1.1) which involve a singular drift, both from the theoretical and numerical points of view. Drifts of the above form indeed appear in some physics-related models, having in mind, for example, the Biot-Savart kernel or Keller-Segel-type equations.

This paper can be viewed as a stable-driven extension of [JM24b], in which the corresponding Brownian case was addressed for the weak error. Stable processes naturally appear when modelling anomalous diffusion phenomena (see [Esc06] for the fractional Keller-Segel model and [MS12] for general fractional models). It is therefore important to be able to quantify how discretization schemes can approximate (5.1.1).

5.1.1 Definition of the Euler scheme

Since we consider a potentially unbounded drift coefficient, it is natural to introduce a cutoff for the discretization scheme. For a time step size h, the two cutoffs we consider are the following:

- If $p = q = \infty$, we simply take $\forall (t, y) \in [0, T] \times \mathbb{R}^d, b_h(t, y) = \bar{b}_h(t, y) = b(t, y)$.
- Otherwise, we set

$$b_h(t,y) := \frac{\min\left\{|b(t,y)|, Bh^{-\frac{d}{\alpha p}-\frac{1}{q}}\right\}}{|b(t,y)|} b(t,y)\mathbb{1}_{|b(t,y)|>0}, \qquad (t,y) \in [0,T] \times \mathbb{R}^d, \tag{5.1.3}$$

$$\bar{b}_h(t,y) = \frac{\min\left\{|b(t,y)|, Bh^{\frac{1}{\alpha}-1}\right\}}{|b(t,y)|} b(t,y)\mathbb{1}_{t\geq h,|b(t,y)|>0}, \qquad (t,y) \in [0,T] \times \mathbb{R}^d, \tag{5.1.4}$$

  for some constant $B > 0$ which can be chosen freely as long as it does not depend on h nor T.

The first option has a cutoff level related to the integrability condition (5.1.2), while the second one is related to the auto-similarity index of the driving noise. The latter also artificially sets the drift to 0 on the first step (we will see later that this allows in particular to compute estimates on the gradient of the density of the Euler scheme). The idea behind this cutoff level is to make sure the contribution of the drift does not dominate over that of the stable noise on each time step of the scheme.

As it will be observed from Theorem 5.2, both cutoffs lead to the same convergence rate for the error associated with the densities of the corresponding schemes. Since, by (5.1.2), $\frac{1}{\alpha} - 1 < -\frac{d}{\alpha p} - \frac{1}{q}$, the first one actually cuts at a lower level, possibly yielding less singular values, and therefore more stability, from a numerical perspective. On the other hand, the auto-similarity related cutoff, which has to be reinforced on the first time-step, also is very natural.

We then define a discretization scheme with n time steps over $[0, T]$, with constant step size $h := T/n$. For the rest of this paper, we denote, $\forall k \in \{1, ..., n\}, t_k := kh$ and $\forall s > 0, \tau_s^h := h\lfloor \frac{s}{h} \rfloor \in (s - h, s]$, which is the last grid point before time s. Namely, if $s \in [t_k, t_{k+1}), \tau_s^h = t_k$.

In order to avoid assumptions on the drift b beyond integrability and measurability, we are led to randomize the evaluations of b_h (resp. $\bar{b}_h$) in the time variable. For each $k \in \{0, ..., n - 1\}$, we will draw a random variable U_k according to the uniform law on $[kh, (k+1)h]$, independently of each other and the noise $(Z_t)_{t\geq 0}$. We can then define a step of the Euler scheme as

$$X_{t_{k+1}}^h = X_{t_k}^h + (Z_{t_{k+1}} - Z_{t_k}) + hb_h(U_k, X_{t_k}^h), \tag{5.1.5}$$

and its time interpolation as the solution to

$$\mathrm{d}X_t^h = \mathrm{d}Z_t + b_h(U_{\lfloor \frac{t}{h} \rfloor}, X_{\tau_t^h}^h)\,\mathrm{d}t. \tag{5.1.6}$$

Similarly, for the alternative cutoff, we define

$$\bar{X}_{t_{k+1}}^h = \bar{X}_{t_k}^h + (Z_{t_{k+1}} - Z_{t_k}) + h\bar{b}_h(U_k, X_{t_k}^h), \tag{5.1.7}$$

and its time interpolation as the solution to

$$\mathrm{d}\bar{X}_t^h = \mathrm{d}Z_t + \bar{b}_h(U_{\lfloor \frac{t}{h} \rfloor}, \bar{X}_{\tau_t^h}^h)\,\mathrm{d}t. \tag{5.1.8}$$

As b_h and $\bar{b}_h$ are bounded, the schemes (5.1.6) and (5.1.8) are well defined and admit densities in positive times. We will denote by $\Gamma^h(0,x,t,\cdot)$ and $\bar{\Gamma}^h(0,x,t,\cdot)$ their respective densities at time $t \in (0,T]$ when starting from x at time 0.

5.1.2 Well-posedness - state of the art

Let us recall that weak well-posedness is often investigated through the parabolic PDE which is naturally associated with the SDE (5.1.1)

$$(\partial_s + b(s,x)\cdot\nabla_x + \mathcal{L}^\alpha)\,u(s,x) = f(s,x) \text{ on } [0,T)\times\mathbb{R}^d, \qquad u(t,x) = g(x) \text{ on } \mathbb{R}^d, \tag{5.1.9}$$

where $\mathcal{L}^\alpha$ is the generator of the noise and f and g are suitable functions. Bearing in mind that, in the β-Hölder setting, the associated parabolic regularity gain is $\beta+\alpha$, the regularity condition $\beta+\alpha>1$ naturally appears to define the gradient of the solution. Let us point out that this condition already appeablack in the seminal work of [TTW74]. For weak and strong well-posedness in the Hölder setting, we can e.g. refer to [MP14] and [CZZ21], which also includes the super-critical case. Since we do not have any regularity available on the drift b, we are naturally led to consider sub-critical regimes for the stability index (i.e. $\alpha > 1$).

Establishing estimates on the gradient of the solution to the PDE naturally leads to weak uniqueness in the multidimensional setting for (5.1.1) through the martingale problem. In this paper, under (5.1.2), we obtain such estimates exploiting heat kernel estimates for the density $\bar{\Gamma}^h$ of $\bar{X}^h$ and taking the limit as h goes to 0. Keep in mind that some additional smoothness is required to derive strong well-posedness in the multidimensional case.

In the strictly stable and time-homogeneous setting with mere integrability assumptions on the drift, weak existence and uniqueness of a solution to (5.1.1) was first investigated in [Por94] in $\mathbb{R}$ and extended to the multidimensional case in [PP95] under the condition $\frac{d}{p} < \alpha - 1$ by constructing the density using its parametrix expansion. When considering the embedding $L^p(\mathbb{R}^d) \hookrightarrow \mathbb{B}_{\infty,\infty}^{-\frac{d}{p}}(\mathbb{R}^d)$ (the latter being the Besov space with regularity $-\frac{d}{p}$), the previous condition is then consistent with the condition $\alpha+\beta>1$ appearing in the Hölder case.

Let us also mention the work [CdRM22a], in which weak well-posedness is proved for distributional drifts in the Besov-Lebesgue space $L^q - \mathbb{B}_{p,r}^\beta$ under the condition $\beta > \frac{1-\alpha+\frac{\alpha}{q}+\frac{d}{p}}{2}$. In view of this threshold, our well-posedness result can be seen as an extension of this work for $\beta = 0$.

Our approach to well-posedness naturally provides heat kernel estimates for both the discretization scheme and the limit SDE that quantify the behavior of their time marginal laws. Namely, as detailed in the seminal work by Kolokoltsov [Kol00], for a smooth bounded drift, the time marginals of the solution (5.1.1) and the noise behave alike. This work was then extended in various directions, mostly for Hölder continuous drifts (see [KK18], [Kul19], [CHZ20] and [MZ22]), and more recently for distributional drifts (see [PvZ22] in the Brownian setting and [Fit23] in the strictly stable case). In those works, the authors again establish that the time marginal laws of the process have a density which is "equivalent" (i.e. bounded from above and below) to the density of the noise, and that the spatial gradients exhibit the same time singularities and decay rates (see Theorem 5.1 below in the current Lebesgue setting).

5.1.3 Euler scheme - state of the art

For the discretization of singular drift diffusions, a rather vast literature exists, although it mostly focuses on the Brownian setting for an additive noise. A first approach consists in using the sewing lemma (see [Lê20]) in order to obtain results on the strong error rate, which is defined as the convergence rate of

$$\left\| \sup_{t\in(0,T)} |X_t - X_t^h| \right\|_{L^r} \tag{5.1.10}$$

for some $r > 1$. This was done in the work of Lê and Ling ([LL21]), who obtain a convergence in $h^{\frac{1}{2}}|\ln(h)|$ under the Krylov-Röckner condition $\frac{d}{p} + \frac{2}{q} < 1$ (see also [DGI22]) even with multiplicative noise (when the corresponding coefficient is Lipschitz in the spatial variable) for the semi-discrete scheme where the time-variable of the coefficients is not discretized. This is a remarkable result since, up to the logarithmic factor, this corresponds to the convergence rate for the strong error associated with a Brownian SDE with Lipschitz coefficients with non-trivial diffusion term. It remains open to understand whether the strong convergence rate improves in terms of the gap to criticity $1 - \left(\frac{d}{p} + \frac{2}{q}\right)$ in the additive noise case.

The main contribution of the sewing lemma consists in bounding L^r norms of the form

$$\mathbb{E}\left[\left|\int_0^t b(s, X_s^h) - b(s, X_{\tau_s^h}^h)\,\mathrm{d}s\right|^r\right], \tag{5.1.11}$$

that is, the strong error associated with local differences of the path along an irregular function with suitable integrability properties.

On the other hand, deriving weak error rates usually involves studying the PDE (5.1.9) or the associated Duhamel representation. Indeed, the weak error is related to the difference between the density of the SDE (5.1.1) and that of the corresponding Euler scheme (5.1.6). Using the Duhamel representations satisfied by the respective transition densities Γ and Γ^h of the diffusion and its Euler scheme, we will estimate

$$|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)|.$$

This approach allows to integrate against any type of irregular test functions enjoying suitable integrability properties.

When the coefficients of (5.1.1) are smooth, the seminal paper of Talay and Tubaro ([TT90]) gives a convergence of order 1 in h. With β-Hölder coefficients, the work of Mikulevicius and Platen ([MP91]) proves a convergence in $h^{\frac{\beta}{2}}$ in the Brownian case. In these works, for u solving (5.1.9) with smooth terminal condition g and no source term f, applying Itô's formula, the error writes

$$\begin{aligned}\mathcal{E}(g, t, x, h) &= \mathbb{E}_{0,x}[g(X_t^h) - g(X_t)] = \mathbb{E}_{0,x}[u(t, X_t^h) - u(0, x)] \\ &= \mathbb{E}_{0,x}\left[\int_0^t \left(b(r, X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h)\right)\cdot \nabla u(r, X_r^h)\,\mathrm{d}r\right],\end{aligned}$$

where the index $0, x$ of the expectation sign means that the scheme is started from $X_0^h = x$ at time 0. The authors then use classic Schauder type estimates, see e.g. [Fri64], to control $\|\nabla u\|_{L^\infty}$. From the β-Hölder continuity of the drift, the following bound is then derived

$$\mathcal{E}(g, t, x, h) \le C\|\nabla u\|_{L^\infty} \int_0^t \mathbb{E}_{0,x}\left[|X_r^h - X_{\tau_r^h}^h|^\beta\right] \mathrm{d}r \le C\|\nabla u\|_{L^\infty} h^{\frac{\beta}{2}}. \tag{5.1.12}$$

The above final rate then comes from the magnitude of the increment of the Euler scheme on one time step in the $L^\beta(\mathbb{P})$ norm. However, one can see that this essentially consists in using strong error analysis techniques to derive a weak error rate, which does not seem adequate.

In the current work, we want to investigate errors of the form $\mathcal{E}(\delta_y, t, x, h)$ (where δ_y is the dirac mass at point y). This formally writes

$$\mathcal{E}(\delta_y, t, x, h) = \mathbb{E}_{0,x}\left[\int_0^t \left(b(r, X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h))\right) \cdot \nabla_z \Gamma(r, z, t, y)|_{z=X_r^h}\, \mathrm{d}r\right], \tag{5.1.13}$$

where Γ is the density of (5.1.1). When comparing this equation to (5.1.11), we see that, in the weak setting, an additional gradient term appears. Whenever this term is not regular enough, which is the case in the current Lebesgue setting, it lowers the time integrability properties of the irregular function that we want to investigate. However, in the specific case of a Hölder continuous drift and terminal condition g, this additional term can be handled using sewing techniques. Doing so in [Hol24], the author improves the convergence rate in (5.1.12) to $h^{\frac{\beta+1}{2}}$. The study of the weak error for Hölder coefficients and a final Dirac mass will be the topic of an upcoming work.

In the irregular setting, for the weak error associated with the densities, two additional tricks turn out to be useful for the error analysis. The first one is the randomization of the time variable which permits to replace $b(\tau_r^h, X_{\tau_r^h}^h)$ by $b(r, X_{\tau_r^h}^h)$ in (5.1.13) (up to some error term on $[\tau_t^h, t]$). The second one, introduced in [BJ22] in order to tackle mere bounded drifts, consists in writing

$$\begin{aligned}
&\mathbb{E}_{0,x}[b(r, X_r^h) \cdot \nabla\Gamma(r, X_r^h, t, y) - b(r, X_{\tau_r^h}^h) \cdot \nabla\Gamma(r, X_{\tau_r^h}^h, t, y)] \\
&\qquad = \int [\Gamma^h(0, x, r, z) - \Gamma^h(0, x, \tau_r^h, z)] b(r, z) \cdot \nabla\Gamma(r, z, t, y)\, \mathrm{d}z
\end{aligned} \tag{5.1.14}$$

and exploiting the regularity in the forward time variable of Γ^h instead of that of $b \cdot \nabla\Gamma$. In [JM24b], authors use this technique with a drift in $L^q - L^p$ to derive a rate of order $\frac{\gamma}{2}$, where $\gamma := 1 - \frac{d}{p} - \frac{2}{q}$ is the so-called "gap to singularity" or "gap to criticity". Note that, with respect to the rate obtained in [LL21], due to the additional *gradient* term in $\nabla\Gamma$ in (5.1.14) (as opposed to (5.1.11)), an order $\frac{1}{2}$ is lost on the convergence rate. However, the techniques developed therein allow to take advantage of the gap to singularity.

As mentioned, the rate for the strong error under the Krylov-Röckner condition is (at least) $\frac{1}{2}$, up to a logarithmic factor. Since we expect the weak error rate to be at least as good, it remains to understand how to improve it beyond $\frac{1}{2}$.

In Theorem 5.2, we obtain a weak error rate in $h^{\frac{\gamma}{\alpha}}$, where our "gap to singularity" is now defined as $\gamma := \alpha - 1 - \left(\frac{d}{p} + \frac{\alpha}{q}\right) > 0$. Importantly, there is continuity w.r.t. the stability index for the associated error rates.

5.1.4 Driving noise and related density properties

Let us denote by $\mathcal{L}^\alpha$ the generator of the driving noise Z. In the case $\alpha = 2$, $\mathcal{L}^\alpha$ is the usual Laplacian $\frac{1}{2}\Delta$. When $\alpha \in (1, 2)$, in whole generality, the generator of a symmetric stable process writes, $\forall \phi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ (smooth compactly supported functions),

$$\begin{aligned}
\mathcal{L}^\alpha \phi(x) &= \text{p.v.} \int_{\mathbb{R}^d} [\phi(x+z) - \phi(x)]\, \nu(\mathrm{d}z) := \lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} [\phi(x+z) - \phi(x)]\, \nu(\mathrm{d}z) \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} [\phi(x+\rho\xi) - \phi(x)]\, \mu(\mathrm{d}\xi) \frac{\mathrm{d}\rho}{\rho^{1+\alpha}},
\end{aligned}$$

i.e. p.v. stands for the *principal value* and μ is a finite measure on the unit sphere $\mathbb{S}^{d-1}$ such that $\mu(A) = \mu(\{\xi \in \mathbb{S}^{d-1} : -\xi \in A\})$ for each Borel subset A of $\mathbb{S}^{d-1}$. We refer to [Sat99] for the polar decomposition of the spectral measure.

This general setting will not allow us to derive heat kernel estimates, because it does not lead to global estimates of the noise density. In [Wat07], Watanabe investigates the behavior of the density of an α-stable process in terms of properties fulfilled by the support of its spectral measure. From this work, we know that whenever the measure μ is not equivalent to the Lebesgue measure m on the unit sphere, accurate estimates on the density of the stable process are delicate to obtain. However, Watanabe (see [Wat07], Theorem 1.5) and Kolokoltsov ([Kol00], Propositions 2.1–2.5) showed that if

$$c^{-1}m(\mathrm{d}\xi) \le \mu(\mathrm{d}\xi) \le cm(\mathrm{d}\xi) \text{ for some } c \in [1,+\infty), \tag{5.1.15}$$

the following estimates hold for the density $z \mapsto p_\alpha(v,z)$ of Z_v with respect to the Lebesgue measure on $\mathbb{R}^d$ when $v > 0$: there exists a constant C depending only on α, d, s.t. $\forall v \in \mathbb{R}_+^*, z \in \mathbb{R}^d$,

$$C^{-1}v^{-\frac{d}{\alpha}}\left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \le p_\alpha(v,z) \le Cv^{-\frac{d}{\alpha}}\left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}. \tag{5.1.16}$$

As our approach heavily relies on these global bounds, we assume that μ satisfies (5.1.15).

Note that in Section 5.2.1 and Appendix 5.5 which are dedicated to technical lemmas, we will be using the proxy notation

$$\bar{p}_\alpha(v,z) := \frac{C_\alpha}{v^{\frac{d}{\alpha}}}\left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}, \qquad v>0, z \in \mathbb{R}^d, \tag{5.1.17}$$

where C_α is chosen so that $\forall v > 0, \int \bar{p}_\alpha(v,y)\,\mathrm{d}y = 1$, because we therein explicitly rely on the global bounds provided by $\bar{p}_\alpha$. In the rest of the paper, we will prefer the notation p_α, directly referring to the density of the noise. Note as well that, with these notations at hand, equation (5.1.16) then also yields:

$$\bar{C}^{-1}\bar{p}_\alpha(v,z) \le \frac{C^{-1}}{C_\alpha}\bar{p}_\alpha(v,z) \le p_\alpha(v,z) \le \frac{C}{C_\alpha}\bar{p}_\alpha(v,z) \le \bar{C}\bar{p}_\alpha(v,z),$$

for some constant $\bar{C} \ge 1$.

Further properties related to the density of the driving noise are stated in Lemmas 5.1 and 5.2 below.

5.1.5 Main results

We are now in position to state the main results of the current work. The first result concerns the well-posedness of (5.1.1).

Theorem 5.1 (Weak existence and density estimates for the diffusion)**.** *Assume* (5.1.2)*. The stochastic differential equation* (5.1.1) *admits a weak solution such that for each* $t \in (0,T]$*,* X_t *admits a density* $y \mapsto \Gamma(0,t,x,y)$ *w.r.t. the Lebesgue measure such that* $\exists C := C(b,T) < \infty : \forall t \in (0,T], \forall (x,y) \in (\mathbb{R}^d)^2$,

$$\Gamma(0,x,t,y) \le Cp_\alpha(t,y-x), \tag{5.1.18}$$

and this density is the unique solution to the following Duhamel representation among functions of $(t,y) \in [0,T]\times\mathbb{R}^d$ *satisfying* (5.1.18)*:*

$$\forall t \in (0,T],\ \forall y \in \mathbb{R}^d,\ \Gamma(0,x,t,y) = p_\alpha(t,y-x) - \int_0^t \int_{\mathbb{R}^d} \Gamma(0,x,r,z)b(r,z).\nabla_y p_\alpha(t-r,y-z)\,\mathrm{d}z\,\mathrm{d}r. \tag{5.1.19}$$

Furthermore, there exists a unique solution to the martingale problem related to $b\cdot\nabla + \mathcal{L}^\alpha$ *starting from* x *at time* 0 *in the sense of Definition 5.1 (see page 102 below).*

Finally, let us define the "gap to singularity" as

$$\gamma := \alpha - 1 - \left(\frac{d}{p}+\frac{\alpha}{q}\right) > 0. \tag{5.1.20}$$

Then, Γ *has the following regularity in the forward spatial variable:* $\forall t \in (0,T], \forall (x,y,y') \in (\mathbb{R}^d)^3$,

$$|\Gamma(0,x,t,y) - \Gamma(0,x,t,y')| \le C\frac{|y-y'|^\gamma \wedge t^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}}}\left(p_\alpha(t,y-x)+p_\alpha(t,y'-x)\right). \tag{5.1.21}$$

The proof of the heat kernel estimates for the diffusion heavily relies on the following heat kernel estimates for the density the Euler schemes (5.1.6) and (5.1.8).

Proposition 5.1 (Density estimates for the Euler scheme)**.** *Assume* (5.1.2)*. Set* $h = \frac{T}{n},\ n \in \mathbb{N}^*$*. Let* X^h *be the scheme defined in* (5.1.6) *(resp.* $\bar{X}^h$ *the scheme defined in* (5.1.8)*) starting from* $x_0 \in \mathbb{R}^d$ *at time* 0*. Then, for all* $0 \le t_k := kh < t \le T,\ k \in \{0, \cdots, n-1\}, (x, y) \in (\mathbb{R}^d)^2$*, the random variable* X_t^h *admits, conditionally to* $X_{t_k}^h = x$*, a density* $\Gamma^h(t_k, x, t, \cdot)$*, which enjoys the following Duhamel representation: for all* $y \in \mathbb{R}^d$*,*

$$\Gamma^h(t_k, x, t, y) = p_\alpha(t - t_k, y - x) - \int_{t_k}^{t} \mathbb{E}_{t_k,x}\left[b_h(U_{\lfloor \frac{r}{h} \rfloor}, X_{\tau_r^h}^h) \cdot \nabla_y p_\alpha(t - r, y - X_r^h) \right] \mathrm{d}r, \tag{5.1.22}$$

where the index t_k, x *of the expectation sign means that the scheme* $(X_r^h)_{r \in [t_k, T]}$ *is started from* $X_{t_k}^h = x$ *at time* t_k*. Similarly, the random variable* $\bar{X}_t^h$ *admits, conditionally to* $\bar{X}_{t_k}^h = x$*, a transition density* $\bar{\Gamma}^h(t_k, x, t, \cdot)$*, which enjoys the following Duhamel representation: for all* $y \in \mathbb{R}^d$*,*

$$\bar{\Gamma}^h(t_k, x, t, y) = p_\alpha(t - t_k, y - x) - \int_{t_k}^{t} \mathbb{E}_{t_k,x}\left[\bar{b}_h(U_{\lfloor \frac{r}{h} \rfloor}, \bar{X}_{\tau_r^h}^h) \cdot \nabla_y p_\alpha(t - r, y - \bar{X}_r^h) \right] \mathrm{d}r. \tag{5.1.23}$$

Furthermore, there exists a finite constant C *not depending on* $h = \frac{T}{n}$ *such that for all* $k \in [\![0, n-1]\!], t \in (t_k, T], x, y, y' \in \mathbb{R}^d$*,*

$$\Gamma^h(t_k, x, t, y) \le C p_\alpha(t - t_k, y - x) \tag{5.1.24}$$

and

$$\begin{aligned} &|\Gamma^h(t_k, x, t, y') - \Gamma^h(t_k, x, t, y)| \\ &\quad \le C \frac{|y - y'|^\gamma \wedge (t - t_k)^{\frac{\gamma}{\alpha}}}{(t - t_k)^{\frac{\gamma}{\alpha}}} \big(p_\alpha(t - t_k, y - x) + p_\alpha(t - t_k, y' - x) \big), \end{aligned} \tag{5.1.25}$$

for γ *defined in* (5.1.20)*. Also, for all* $0 \le k < \ell < n,\ t \in [t_\ell, t_{\ell+1}], x, y \in \mathbb{R}^d$*,*

$$|\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t_\ell, y)| \le C \frac{(t - t_\ell)^{\frac{\gamma}{\alpha}}}{(t_\ell - t_k)^{\frac{\gamma}{\alpha}}} p_\alpha(t - t_k, y - x), \tag{5.1.26}$$

and the same estimates hold with $\bar{\Gamma}^h$ *replacing* Γ^h*.*

Our second main result, requiring the results of Theorem 5.1, states a weak convergence rate bound for the Euler schemes (5.1.6) and (5.1.8) :

Theorem 5.2 (Convergence Rate for the stable-driven Euler-Maruyama scheme with $L^q - L^p$ drift)**.** *Assume that* (5.1.2) *holds. There exists a constant* $C < \infty$ *s.t. for all* $h = T/n$ *with* $n \in \mathbb{N}^*$*, and all* $t \in (0, T]$*,* $x, y \in \mathbb{R}^d$

$$\begin{aligned} &|\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)| \le C h^{\frac{\gamma}{\alpha}} p_\alpha(t, y - x), \\ \text{resp.} \quad &|\bar{\Gamma}^h(0, x, t, y) - \Gamma(0, x, t, y)| \le C h^{\frac{\gamma}{\alpha}} p_\alpha(t, y - x). \end{aligned}$$

5.1.6 Notations and conventions

We will use the following notations :

- $A \lesssim B$ if there exists a constant $C \ge 1$, which depends only on α, d, p, q, b, T, such that $A \le CB$.
- $A \asymp B$ if there exists a constant $C \ge 1$, which depends only on α, d, p, q, b, T, such that $C^{-1}B \le A \le CB$.
- For $\ell \in [1, +\infty]$, we always denote by $\ell' \in [1, +\infty]$ its conjugate exponent, i.e. $\frac{1}{\ell} + \frac{1}{\ell'} = 1$.

Also,

- The symbol $\star$ stands for the convolution operator in space.
- The integrals appearing in the computations for which we omit the integration domain must be understood as integral over the whole spatial domain $\mathbb{R}^d$. It will be clear from context and shortens the writing.
- For $a, b \in \mathbb{N}$ with $a \leq b$, the notation $[\![a, b]\!] := [a, b] \cap \mathbb{N}$ is the corresponding integer valued interval $\{a, a+1, \cdots, b\}$.

The article is organized as follows. The proof of Theorem 5.2 is developed in Section 5.2 (assuming that the controls of Theorem 5.1 hold). Section 5.3 is dedicated to the proof of the estimates for the schemes. The proof of Theorem 5.1 is presented in Section 5.4.1. The proof of some technical results are gathered in Appendix 5.5.

5.2 Proof of the convergence rate for the error (Theorem 5.2)

5.2.1 Technical tools

We will profusely use the following technical lemmas which hold for any stability index $\alpha \in (1, 2)$ and are proved in Appendix 5.5:

Lemma 5.1 (Stable sensitivities - Estimates on the α-stable kernel)**.** *For each multi-index ζ with length $|\zeta| \leq 2$, and for all $0 < u \leq u' \leq T$, $(x, x') \in (\mathbb{R}^d)^2$,*

- *Bounds for space and time derivatives: for all $\beta \in \{0, 1\}$,*

$$|\partial_u^\beta \nabla_x^\zeta p_\alpha(u, x)| \lesssim \frac{p_\alpha(u, x)}{u^{\beta + \frac{|\zeta|}{\alpha}}}. \tag{5.2.1}$$

- *Spatial Hölder regularity: for all $\theta \in (0, 1]$,*

$$\left|\nabla_x^\zeta p_\alpha(u, x) - \nabla_x^\zeta p_\alpha(u, x')\right| \lesssim \left(\frac{|x - x'|^\theta}{u^{\frac{\theta}{\alpha}}} \wedge 1\right) \frac{1}{u^{\frac{|\zeta|}{\alpha}}} \left(p_\alpha(u, x) + p_\alpha(u, x')\right). \tag{5.2.2}$$

- *Time Hölder regularity: for all $\theta \in (0, 1]$,*

$$\left|\nabla_x^\zeta p_\alpha(u, x) - \nabla_x^\zeta p_\alpha(u', x)\right| \lesssim \frac{|u - u'|^\theta}{u^{\theta + \frac{|\zeta|}{\alpha}}} \left(p_\alpha(u, x) + p_\alpha(u', x)\right). \tag{5.2.3}$$

- *Time scales for spatial moments: for all $\ell \in [1, +\infty]$ and $\delta \in [0, \frac{d}{\ell} + \alpha)$,*

$$\|p_\alpha(u, \cdot)| \cdot |^\delta\|_{L^{\ell'}} \leq C u^{-\frac{d}{\alpha \ell} + \frac{\delta}{\alpha}}. \tag{5.2.4}$$

- *Convolution: for all $(x, y) \in (\mathbb{R}^d)^2$, $0 < s < u < t \leq T$, $\ell \geq 1$,*

$$\|p_\alpha(t - u, \cdot - y) p_\alpha(u - s, x - \cdot)\|_{L^{\ell'}} \lesssim \left[\frac{1}{(t - u)^{\frac{d}{\alpha \ell}}} + \frac{1}{(u - s)^{\frac{d}{\alpha \ell}}}\right] p_\alpha(t - s, x - y). \tag{5.2.5}$$

- *Integration of an L^ℓ function in a spatial stable convolution: for all $(x, y) \in (\mathbb{R}^d)^2$, $0 \leq s < u < t \leq T$, $\ell \geq 1, \phi \in L^\ell(\mathbb{R}^d, \mathbb{R})$,*

$$\int p_\alpha(t - u, z - x) |\phi(z)| p_\alpha(u - s, y - z) \, \mathrm{d}z \lesssim \left[\frac{1}{(t - u)^{\frac{d}{\alpha \ell}}} + \frac{1}{(u - s)^{\frac{d}{\alpha \ell}}}\right] p_\alpha(t - s, y - x) \|\phi\|_{L^\ell}. \tag{5.2.6}$$

Lemma 5.2 (Feynman-Kac partial differential equation)**.** *Let $t > 0$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function with bounded derivatives. Then the function $v(s, y) = \mathbb{1}_{s<t} p_\alpha(t - s, \cdot) \star \phi(y) + \mathbb{1}_{s=t}\phi(y)$ is $\mathcal{C}^{1,2}$ on $[0, t] \times \mathbb{R}^d$ and satisfies the Feynman-Kac partial differential equation*

$$\forall (s, y) \in [0, t) \times \mathbb{R}^d,\ \partial_s v(s, y) + \mathcal{L}^\alpha v(s, y) = 0,\ v(t, y) = \phi(y).$$

Lemma 5.3 (Integration of the drift in a spatial stable time-space convolution)**.** *Let $0 \le u < v \le t \le T$ and $\beta_1, \beta_2 \in \mathbb{R}_+$. Let $b \in L^q([0, T], L^p(\mathbb{R}^d))$ with p, q such that (5.1.2) holds.*

- *Singular case. If $v < t$ and*

$$q'\left(\frac{d}{\alpha p} + \beta_1\right) > 1 \qquad \text{and} \qquad q'\left(\frac{d}{\alpha p} + \beta_2\right) < 1,$$

then,

$$\int_u^v \int p_\alpha(r, z - x)|b(r, z)|p_\alpha(t - r, y - z)\frac{1}{(t - r)^{\beta_1}}\frac{1}{r^{\beta_2}}\,\mathrm{d}r \lesssim p_\alpha(t, y - x)\Big((v - u)^{\frac{\gamma+1}{\alpha} - (\beta_1+\beta_2)} + (v - u)^{-\beta_2}(t - v)^{\frac{\gamma+1}{\alpha} - \beta_1}\Big). \tag{5.2.7}$$

- *Integrable case. If*

$$q'\left(\frac{d}{\alpha p} + \beta_1\right) < 1 \qquad \text{and} \qquad q'\left(\frac{d}{\alpha p} + \beta_2\right) < 1,$$

then,

$$\int_u^v \int p_\alpha(r, z - x)|b(r, z)|p_\alpha(t - r, y - z)\frac{1}{(t - r)^{\beta_1}}\frac{1}{r^{\beta_2}}\,\mathrm{d}r \lesssim p_\alpha(t, y - x)[(v - u)^{\frac{\gamma+1}{\alpha} - (\beta_1+\beta_2)}]. \tag{5.2.8}$$

The previous lemma will be used to treat the main error terms in the analysis of the error. The most common use case is when $\beta_2 = 0$ and $\beta_1 = \frac{1}{\alpha}$ (we are thus in case (5.2.8)) and $u = h, v = \tau_t^h - h$. This configuration appears when we previously used (5.2.1) to bound the gradient of $p_\alpha(t - r, y - z)$ and that no other singularities come into play. The case $\beta_2 > 0$ with an additional singular in r factor is needed for the proof of Theorem 5.2 (which will require setting $\beta_2 = \frac{\gamma}{\alpha}$).
We will also use (5.2.7) whenever there is an additional singularity in $(t - r)$ which makes the previous integral non-convergent (see e.g. (5.2.22)). This will actually happen in order to obtain exactly the gap γ defined (5.1.20) in the convergence rate or in the Hölder exponents for the density, see e.g. Section 5.3.3 for the proof of the Hölder regularity of the density of the scheme stated in Proposition 5.1.

Remark 5.1. *From the definition of $\bar{p}_\alpha(u, x) = \frac{C_\alpha}{u^{\frac{d}{\alpha}}}\left(1 + \frac{|x|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}$ introduced in* (5.1.17)*, one can gather the following:*
Let $x \in \mathbb{R}^d$ and $u > 0$.

- *If $|x| \ge u^{\frac{1}{\alpha}}$ (off-diagonal regime),*

$$\bar{p}_\alpha(u, x) \asymp \frac{u}{|x|^{d+\alpha}}. \tag{5.2.9}$$

- *If $|x| \le u^{\frac{1}{\alpha}}$ (diagonal regime),*

$$\bar{p}_\alpha(u, x) \asymp \frac{1}{u^{\frac{d}{\alpha}}}. \tag{5.2.10}$$

Those two regimes will be central in our proofs. The scales which we consider for these regimes derive from the self-similarity of the noise. Let us as well point out that the diagonal bound in (5.2.10) is also a global upper bound for both $\bar{p}_\alpha$ and p_α.

The next lemma is very important since it precisely emphasizes that the drift b_h (resp. $\bar{b}_h$) is actually a *negligible* term w.r.t. the scale of the underlying noise for a one-step transition of the corresponding scheme.

Lemma 5.4 (About the cutoff on a one-step transition). *Here, $\mathfrak{b}_h \in \{b_h, \bar{b}_h\}$ stands for one of the two drifts considered for the schemes.*

- *For all $(u,r) \in (0,T]^2, s \le \min(u,h), (x,y) \in (\mathbb{R}^d)^2$, and each multi-index $\zeta \in \mathbb{N}^d$ with length $|\zeta| \le 1$,*

$$|\nabla^\zeta p_\alpha(u, y - s\mathfrak{b}_h(r,x))| \lesssim \frac{p_\alpha(u,y)}{u^{\frac{|\zeta|}{\alpha}}}. \tag{5.2.11}$$

- *For all $(u,r) \in (0,T]^2, s \le \min(u,h), (x,y,y') \in (\mathbb{R}^d)^3$, for each multi-index $\zeta \in \mathbb{N}^d$ with length $|\zeta| \le 1$, and for all $\delta \in (0,1]$,*

$$\left|\nabla^\zeta p_\alpha(u, y - s\mathfrak{b}_h(r,x)) - \nabla^\zeta p_\alpha(u, y' - s\mathfrak{b}_h(r,x))\right| \lesssim \left(\frac{|y-y'|^\delta}{u^{\frac{\delta}{\alpha}}} \wedge 1\right) \frac{1}{u^{\frac{|\zeta|}{\alpha}}} \left(p_\alpha(u,y) + p_\alpha(u,y')\right). \tag{5.2.12}$$

5.2.2 Proof of the error bounds of Theorem 5.2

Comparing the Duhamel formula of the scheme, (5.1.22), to that of the diffusion, (5.1.19), we get

$$\begin{aligned}
&\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y) \\
&= \int_0^t \int_{\mathbb{R}^d} \Gamma(0,x,s,z) b(s,z).\nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}s - \mathbb{E}_{0,x}\left[\int_0^t b_h(U_{\lfloor \frac{s}{h} \rfloor}, X^h_{\tau^h_s}) \cdot \nabla_y p_\alpha(t-s, y - X^h_s)\,\mathrm{d}s\right].
\end{aligned}$$

Respectively, for the alternative scheme involving $\bar{b}_h$,

$$\begin{aligned}
&\bar{\Gamma}^h(0,x,t,y) - \Gamma(0,x,t,y) \\
&= \int_0^t \int_{\mathbb{R}^d} \Gamma(0,x,s,z) b(s,z).\nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}s - \mathbb{E}_{0,x}\left[\int_0^t \bar{b}_h(U_{\lfloor \frac{s}{h} \rfloor}, X^h_{\tau^h_s}) \cdot \nabla_y p_\alpha(t-s, y - X^h_s)\,\mathrm{d}s\right].
\end{aligned}$$

The error admits the following decomposition:

$$\begin{aligned}
\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y) &= \int_0^t \int [\Gamma(0,x,s,z) - \Gamma^h(0,x,s,z)] b(s,z) \cdot \nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}s \\
&+ \mathbb{1}_{\{t \ge 3h\}} \int_{t_1}^{\tau^h_t - h} \int \Gamma^h(0,x,s,z)(b(s,z) - b_h(s,z)) \cdot \nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}s \\
&+ \mathbb{1}_{\{t \ge 3h\}} \int_{t_1}^{\tau^h_t - h} \int [\Gamma^h(0,x,s,z) - \Gamma^h(0,x,\tau^h_s,z)] b_h(s,z) \cdot \nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}s \\
&+ \mathbb{1}_{\{t \ge 3h\}} \int_{t_1}^{\tau^h_t - h} \mathbb{E}_{0,x}\left[b_h(U_{\lfloor s/h \rfloor}, X^h_{\tau^h_s}) \cdot (\nabla_y p_\alpha(t - U_{\lfloor s/h \rfloor}, y - X^h_{\tau^h_s}) - \nabla_y p_\alpha(t-s, y - X^h_s))\right] \mathrm{d}s \\
&+ \frac{1}{h} \int_0^{t_1 \wedge t} \int_0^h \int p_\alpha(s, z - x - b_h(r,x)s)\,(b(s,z) - b_h(r,x)) \cdot \nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s \\
&+ \mathbb{1}_{\{t \ge h\}} \frac{1}{h} \int_{(\tau^h_t - h) \vee t_1}^t \int_{\tau^h_s}^{\tau^h_s + h} \int \int \Gamma^h(0,x,\tau^h_s,w) p_\alpha\left(s - \tau^h_s, z - w - b_h(r,w)(s - \tau^h_s)\right) \\
&\qquad\qquad \times (b(s,z) - b_h(r,w)) \cdot \nabla_y p_\alpha(t-s, y-z)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
&=: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6,
\end{aligned} \tag{5.2.13}$$

where, for the last term, we use that for $s \in (t_1, T]$, not belonging to the discretization grid and $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ measurable and bounded, since $X^h_s = X^h_{\tau^h_s} + Z_s - Z_{\tau^h_s} + b_h(U_{\lfloor \frac{s}{h} \rfloor}, X^h_{\tau^h_s})(s - \tau^h_s)$ with $X^h_{\tau^h_s}$, $Z_s - Z_{\tau^h_s}$ and $U_{\lfloor \frac{s}{h} \rfloor}$ independent, we can write

$$\begin{aligned}
&\mathbb{E}_{0,x}\left[\phi(X^h_{\tau^h_s}, X^h_s, U_{\lfloor \frac{s}{h} \rfloor})\right] \\
&\qquad = \frac{1}{h} \int_{\tau^h_s}^{\tau^h_s + h} \int \int \phi(w,z,r) \Gamma^h(0,x,\tau^h_s,w) p_\alpha(s - \tau^h_s, z - w - b_h(r,w)(s - \tau^h_s))\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r.
\end{aligned} \tag{5.2.14}$$

Similarly, we define

$$\bar{\Gamma}^h(0,x,t,y) - \Gamma(0,x,t,y) =: \bar{\Delta}_1 + \bar{\Delta}_2 + \bar{\Delta}_3 + \bar{\Delta}_4 + \bar{\Delta}_5 + \bar{\Delta}_6,$$

where b_h is replaced by $\bar{b}_h$.

For $\Delta_2, \Delta_3, \Delta_4$, we suppose that $t \geq 3h$ (otherwise these contributions vanish) and rely on the fact that the current integration time is distinct from 0 and from t, meaning that we can rely on the smoothness properties of the integrands on the considered time intervals. For Δ_5, Δ_6, on the opposite, we rely on the smallness of the considered time intervals. The figure below shows the nature of the various terms in the error expansion.

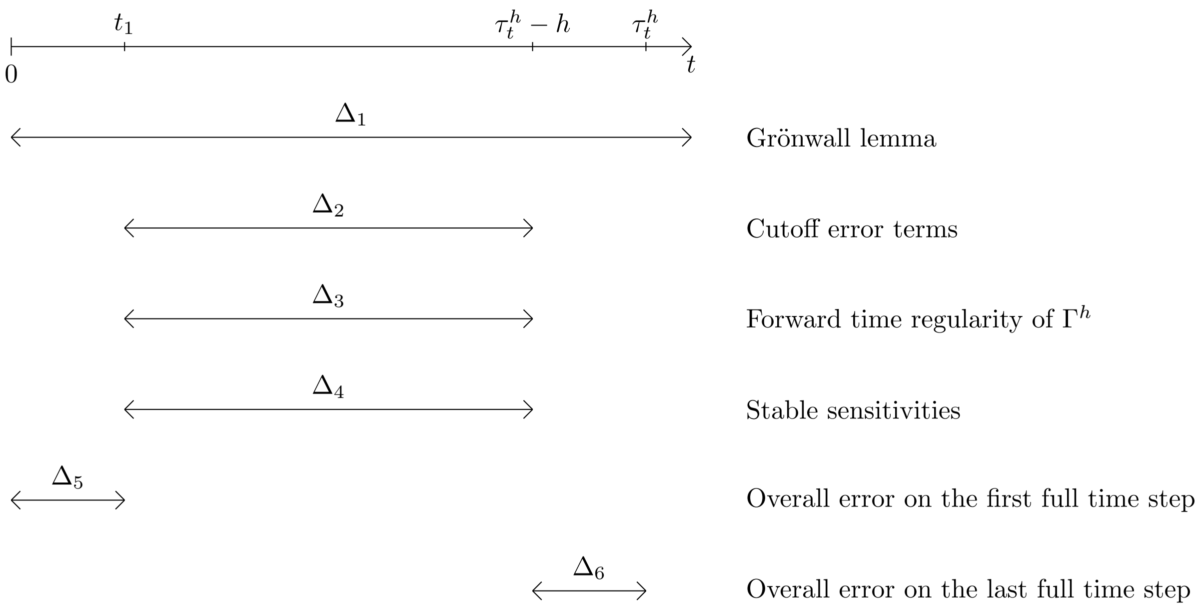


Figure 5.1: Splitting of the error (holds for both Δ_i and $\bar{\Delta}_i$, $i \in [\![1, 6]\!]$).

Let us first deal with Δ_2. Since this term vanishes when $p = q = \infty$ (the same is true for $\bar{\Delta}_2$ when $h \le (\|b\|_{L^\infty - L^\infty}/B)^{\frac{\alpha}{1-\alpha}}$), we assume that either $p < \infty$ or $q < \infty$. Let $\lambda \ge 1$. Using the fact that $\forall y \in \mathbb{R}_+, \mathbb{1}_{\{y \ge 1\}} \le y^{\lambda - 1}$, we obtain that $\forall f : \mathbb{R} \to \mathbb{R}_+$, $\forall C > 0$,

$$f \mathbb{1}_{\{f \ge C\}} \le f^\lambda C^{1-\lambda}.$$

This allows us to control the cutoff error in the following way:

$$|b - b^h| = \left(|b| - B h^{-\frac{d}{\alpha p} - \frac{1}{q}}\right)_+ \le |b| \mathbb{1}_{|b| \ge B h^{-\frac{d}{\alpha p} - \frac{1}{q}}} \le |b|^\lambda B^{1-\lambda} h^{(\frac{d}{\alpha p} + \frac{1}{q})(\lambda - 1)}. \tag{5.2.15}$$

Respectively

$$|b - \bar{b}^h| = \left(|b| - B h^{\frac{1}{\alpha} - 1}\right)_+ \le |b| \mathbb{1}_{|b| \ge B h^{\frac{1}{\alpha} - 1}} \le |b|^\lambda B^{1-\lambda} h^{(1 - \frac{1}{\alpha})(\lambda - 1)}.$$

Along with the use of (5.1.24) and (5.2.1), we obtain

$$\begin{aligned} |\Delta_2| &= \left| \mathbb{1}_{\{t \ge 3h\}} \int_{t_1}^{\tau_t^h - h} \int \Gamma^h(0, x, s, z)(b(s, z) - b_h(s, z)) \cdot \nabla_y p_\alpha(t - s, y - z) \, \mathrm{d}z \, \mathrm{d}s \right| \\ &\lesssim h^{-(\frac{d}{\alpha p} + \frac{1}{q})(1 - \lambda)} \int_{t_1}^{\tau_t^h - h} \int p_\alpha(s, z - x) |b(s, z)|^\lambda \frac{p_\alpha(t - s, y - z)}{(t - s)^{\frac{1}{\alpha}}} \, \mathrm{d}z \, \mathrm{d}s. \end{aligned}$$

Let us check that we can choose

$$\lambda = 1 + \frac{\frac{\gamma_1}{\alpha}}{\frac{d}{\alpha p} + \frac{1}{q}} = 1 + \frac{\gamma_1}{\frac{d}{p} + \frac{\alpha}{q}} \text{ with } \gamma_1 \in (\gamma, 1], \tag{5.2.16}$$

small enough so that $\tilde{p} = \frac{p}{\lambda}$ and $\tilde{q} = \frac{q}{\lambda}$ satisfy $\tilde{p} \ge 1$ and $\tilde{q} \ge 1$. This is indeed possible since, by the definition (5.1.20) of γ and (5.1.2),

$$\frac{p}{1 + \frac{\gamma}{\frac{d}{p} + \frac{\alpha}{q}}} = \frac{d + \frac{\alpha p}{q}}{\alpha - 1} > 1 \text{ and } \frac{q}{1 + \frac{1}{\frac{d}{p} + \frac{\alpha}{q}}} > \frac{\frac{dq}{p} + \alpha}{\alpha} \ge 1.$$

Morever, in order to estimate the time integrals that will appear below after the application of Hölder's inequality, let us observe that $\lambda > 1$ and since

$$\frac{q}{q-1-\frac{\gamma}{\frac{d}{p}+\frac{\alpha}{q}}}\left[\frac{1}{\alpha}+\frac{d}{\alpha p}\left(1+\frac{\gamma}{\frac{d}{p}+\frac{\alpha}{q}}\right)\right]=1,$$

$$\text{we have } -\left(\frac{q}{\lambda}\right)'\left[\frac{1}{\alpha}+\frac{d\lambda}{\alpha p}\right]=-\left(1-\frac{\lambda}{q}\right)^{-1}\left[\frac{1}{\alpha}+\frac{d\lambda}{\alpha p}\right]<-1.$$

Using the identity $\forall f:\mathbb{R}\to\mathbb{R}, \forall \mu\in\mathbb{R}_+, \forall \mathfrak{p}\geq\mu, \|f^\mu\|_{L^{\frac{\mathfrak{p}}{\mu}}}=\|f\|_{L^{\mathfrak{p}}}^{\mu}$ and (5.2.6) with $\ell=\tilde{p}$ then Young's inequality, Hölder's inequality in time and the last inequality combined with $(t-\tau_t^h+h)\geq h$, we get

$$\begin{aligned}
|\Delta_2| &\lesssim h^{(\frac{d}{\alpha p}+\frac{1}{q})(\lambda-1)}\int_{t_1}^{\tau_t^h-h}\int p_\alpha(s,z-x)|b(s,z)|^\lambda\frac{p_\alpha(t-s,y-z)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim h^{(\frac{d}{\alpha p}+\frac{1}{q})(\lambda-1)}p_\alpha(t,y-x)\int_{t_1}^{\tau_t^h-h}\frac{1}{(t-s)^{\frac{1}{\alpha}}}\|b(s,\cdot)\|_{L^p}^{\lambda}\left[\frac{1}{s^{\frac{d\lambda}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d\lambda}{\alpha p}}}\right]\mathrm{d}s\\
&\lesssim h^{(\frac{d}{\alpha p}+\frac{1}{q})(\lambda-1)}p_\alpha(t,y-x)\int_{t_1}^{\tau_t^h-h}\|b(s,\cdot)\|_{L^p}^{\lambda}\left[\frac{1}{s^{\frac{1}{\alpha}+\frac{d\lambda}{\alpha p}}}+\frac{1}{(t-s)^{\frac{1}{\alpha}+\frac{d\lambda}{\alpha p}}}\right]\mathrm{d}s\\
&\lesssim h^{(\frac{d}{\alpha p}+\frac{1}{q})(\lambda-1)}p_\alpha(t,y-x)\|s\mapsto\|b(s,\cdot)\|_{L^p}^{\lambda}\|_{L^{\frac{q}{\lambda}}}\\
&\qquad\times\left(\int_{t_1}^{\tau_t^h-h}\left[\frac{1}{s^{(\frac{q}{\lambda})'(\frac{1}{\alpha}+\frac{d\lambda}{\alpha p})}}+\frac{1}{(t-s)^{(\frac{q}{\lambda})'(\frac{1}{\alpha}+\frac{d\lambda}{\alpha p})}}\right]\mathrm{d}s\right)^{\frac{1}{(\frac{q}{\lambda})'}}\\
&\lesssim h^{(\frac{d}{\alpha p}+\frac{1}{q})(\lambda-1)}p_\alpha(t,y-x)\left(h^{\frac{1}{(\frac{q}{\lambda})'}-\frac{1}{\alpha}-\frac{d\lambda}{\alpha p}}+(t-\tau_t^h+h)^{\frac{1}{(\frac{q}{\lambda})'}-\frac{1}{\alpha}-\frac{d\lambda}{\alpha p}}\right)\\
&\lesssim h^{(\frac{d}{\alpha p}+\frac{1}{q})(\lambda-1)+1-\frac{1}{\alpha}-\lambda(\frac{d}{\alpha p}+\frac{1}{q})}p_\alpha(t,y-x)\\
&\lesssim h^{\frac{\gamma}{\alpha}}p_\alpha(t,y-x). \qquad (5.2.17)
\end{aligned}$$

The same computations with the same choice of λ yield $|\bar{\Delta}_2|\lesssim h^{(1-\frac{1}{\alpha})(\lambda-1)}h^{1-\frac{1}{\alpha}-\lambda(\frac{d}{\alpha p}+\frac{1}{q})}p_\alpha(t,y-x)\lesssim h^{\frac{\lambda\gamma}{\alpha}}p_\alpha(t,y-x)\lesssim h^{\frac{\gamma}{\alpha}}p_\alpha(t,y-x)$.

We now turn our attention to Δ_3, for which we mainly rely on the Hölder regularity of Γ^h in time (equation (5.1.26) of Proposition 5.1). We assume $t\geq 3h$, otherwise this contribution vanishes. Using (5.1.26), we can write

$$\forall s\geq t_1,\ |\Gamma^h(0,x,s,z)-\Gamma^h(0,x,\tau_s^h,z)|\lesssim\frac{(s-\tau_s^h)^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}}p_\alpha(s,z-x).$$

We plug this into the definition of Δ_3, using as well $|b_h|\leq|b|$ (resp. $|\bar{b}_h|\leq|b|$):

$$\begin{aligned}
|\Delta_3| &\lesssim\int_{t_1}^{\tau_t^h-h}\int\frac{(s-\tau_s^h)^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}}p_\alpha(s,z-x)|b(s,z)||\nabla_y p_\alpha(t-s,y-z)|\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim h^{\frac{\gamma}{\alpha}}\int_{t_1}^{\tau_t^h-h}\int\frac{1}{(\tau_s^h)^{\frac{\gamma}{\alpha}}}p_\alpha(s,z-x)|b(s,z)||\nabla_y p_\alpha(t-s,y-z)|\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

We now deal with the gradient using (5.2.1) and notice that since $s\geq t_1\geq h$ then $(\tau_s^h)^{-1}\leq 2s^{-1}$ to write

$$\begin{aligned}
|\Delta_3| &\lesssim h^{\frac{\gamma}{\alpha}}\int_{t_1}^{\tau_t^h-h}\int\frac{1}{(t-s)^{\frac{1}{\alpha}}s^{\frac{\gamma}{\alpha}}}p_\alpha(s,z-x)|b(s,z)|p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim h^{\frac{\gamma}{\alpha}}\int_0^t\int\frac{1}{(t-s)^{\frac{1}{\alpha}}s^{\frac{\gamma}{\alpha}}}p_\alpha(s,z-x)|b(s,z)|p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

Using (5.2.8) (with $u = 0, v = t, \beta_1 = 1/\alpha, \beta_2 = \gamma/\alpha$ so that $\frac{\gamma+1}{\alpha} - (\beta_1 + \beta_2) = 0$), we obtain

$$|\Delta_3| \lesssim h^{\frac{\gamma}{\alpha}} p_\alpha(t, y - x). \tag{5.2.18}$$

For Δ_4, we first expand the expectation with the known densities using (5.2.14):

$$\begin{aligned}
\Delta_4 &= \int_{t_1}^{\tau_t^h - h} \mathbb{E}_{0,x}\left[b_h(U_{\lfloor s/h \rfloor}, X^h_{\tau^h_s}) \cdot (\nabla p_\alpha(t - U_{\lfloor s/h \rfloor}, y - X^h_{\tau^h_s}) - \nabla p_\alpha(t - s, y - X^h_s)) \right] \mathrm{d}s \\
&= \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} \Gamma^h(0, x, t_j, z) \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(s - t_j, w - z - b_h(r, z)(s - t_j)) \\
&\qquad \times b_h(r, z) \cdot (\nabla p_\alpha(t - r, y - z) - \nabla p_\alpha(t - s, y - w)) \, \mathrm{d}z \, \mathrm{d}w \, \mathrm{d}r \, \mathrm{d}s.
\end{aligned}$$

We then derive, using (5.1.24) and (5.2.11),

$$\begin{aligned}
|\Delta_4| &\lesssim \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int \Gamma^h(0, x, t_j, z) p_\alpha(s - t_j, w - z - b_h(r, z)(s - t_j)) |b(r, z)| \\
&\quad \times (|\nabla p_\alpha(t - r, y - z) - \nabla p_\alpha(t - s, y - z)| + |\nabla p_\alpha(t - s, y - z) - \nabla p_\alpha(t - s, y - w)|) \, \mathrm{d}z \, \mathrm{d}w \, \mathrm{d}r \, \mathrm{d}s \\
&\lesssim \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j, z - x) p_\alpha(s - t_j, w - z) |b(r, z)| \\
&\quad \times (|\nabla p_\alpha(t - r, y - z) - \nabla p_\alpha(t - s, y - z)| + |\nabla p_\alpha(t - s, y - z) - \nabla p_\alpha(t - s, y - w)|) \, \mathrm{d}z \, \mathrm{d}w \, \mathrm{d}r \, \mathrm{d}s.
\end{aligned} \tag{5.2.19}$$

Next, we use (5.2.3) to write

$$|\nabla p_\alpha(t - r, y - z) - \nabla p_\alpha(t - s, y - z)| \lesssim \frac{|r - s|}{(t - r \vee s)^{1 + \frac{1}{\alpha}}} \left(p_\alpha(t - r, y - z) + p_\alpha(t - s, y - z) \right).$$

Since $r - s < h$, $t \geq 3h$ and $t - r \vee s \geq h$, we can use (5.1.16) to deduce that

$$|\nabla p_\alpha(t - r, y - z) - \nabla p_\alpha(t - s, y - z)| \lesssim \frac{|r - s|}{(t - r)^{1 + \frac{1}{\alpha}}} p_\alpha(t - r, y - z),$$

which also yields, for any $\gamma_1 \in (\gamma, \alpha]$, recalling that $r - s < t - r$,

$$|\nabla p_\alpha(t - r, y - z) - \nabla p_\alpha(t - s, y - z)| \lesssim \frac{|r - s|^{\frac{\gamma_1}{\alpha}}}{(t - r)^{\frac{\gamma_1}{\alpha} + \frac{1}{\alpha}}} p_\alpha(t - r, y - z). \tag{5.2.20}$$

For the second term in (5.2.19), assuming that $\gamma_1 \in (\gamma, 1]$, we deduce from (5.2.2) that

$$|\nabla p_\alpha(t - s, y - z) - \nabla p_\alpha(t - s, y - w)| \lesssim \left(\frac{|z - w|^{\gamma_1}}{(t - s)^{\frac{\gamma_1}{\alpha}}} \wedge 1 \right) \frac{1}{(t - s)^{\frac{1}{\alpha}}} \left(p_\alpha(t - s, y - z) + p_\alpha(t - s, y - w) \right). \tag{5.2.21}$$

Plugging (5.2.21) and (5.2.20) into (5.2.19), we can write

$$\begin{aligned}
\Delta_4 \lesssim & \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j, z-x) p_\alpha(s-t_j, w-z) |b(r,z)| p_\alpha(t-r, y-z) \frac{|r-s|^{\frac{\gamma_1}{\alpha}}}{(t-r)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
& + \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j, z-x) p_\alpha(s-t_j, w-z) |b(r,z)| p_\alpha(t-s, y-z) \frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
& + \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j, z-x) p_\alpha(s-t_j, w-z) |b(r,z)| \\
& \qquad \times p_\alpha(t-s, y-w) \left(\frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{\gamma_1}{\alpha}}} \wedge 1 \right) \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
=: & \ \Delta_4^1 + \Delta_4^2 + \Delta_4^3.
\end{aligned} \tag{5.2.22}$$

Let us treat Δ_4^1. From the Fubini theorem, we integrate first in w using the fact that p_α is a probability density:

$$\Delta_4^1 = \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j, z-x) |b(r,z)| p_\alpha(t-r, y-z) \frac{|r-s|^{\frac{\gamma_1}{\alpha}}}{(t-r)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s. \tag{5.2.23}$$

Then, using $t_j^{-1} \le 2r^{-1}$ and (5.1.16) along with the fact that $|r-s| < h$, we get

$$\begin{aligned}
\Delta_4^1 &\lesssim \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} h^{\frac{\gamma_1}{\alpha}-1} \int_{t_j}^{t_{j+1}} \int p_\alpha(r, z-x) |b(r,z)| p_\alpha(t-r, y-z) \frac{1}{(t-r)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim \int_{t_1}^{t_{\lfloor \frac{t}{h} \rfloor -1}} h^{\frac{\gamma_1}{\alpha}} \int p_\alpha(r, z-x) |b(r,z)| p_\alpha(t-r, y-z) \frac{1}{(t-r)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r.
\end{aligned}$$

Using (5.2.7) (singular case with $u = t_1, v = \tau_t^h - h, \beta_1 = (1+\gamma_1)/\alpha, \beta_2 = 0$ and noting that since $t \ge 3h$, $v - u \ge h$), we get

$$\begin{aligned}
\Delta_4^1 &\lesssim p_\alpha(t, y-x) h^{\frac{\gamma_1}{\alpha}} \left(h^{\frac{\gamma-\gamma_1}{\alpha}} + (t - t_{\lfloor \frac{t}{h} \rfloor -1})^{\frac{\gamma-\gamma_1}{\alpha}} \right) \\
&\lesssim p_\alpha(t, y-x) h^{\frac{\gamma}{\alpha}}.
\end{aligned} \tag{5.2.24}$$

Let us treat Δ_4^2 defined in (5.2.22). We integrate in w using (5.2.4) and use the fact that $s - t_j \le h$:

$$\begin{aligned}
\Delta_4^2 &= \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j, z-x) p_\alpha(s-t_j, w-z) |b(r,z)| p_\alpha(t-s, y-z) \frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j, z-x) |b(r,z)| p_\alpha(t-s, y-z) \frac{(s-t_j)^{\frac{\gamma_1}{\alpha}}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim h^{\frac{\gamma_1}{\alpha}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j, z-x) |b(r,z)| p_\alpha(t-s, y-z) \frac{1}{(t-s)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}$$

Notice that in the previous integral, as above $p_\alpha(t_j, z-x) \lesssim p_\alpha(r, z-x)$ and $p_\alpha(t-s, y-z) \lesssim p_\alpha(t-r, y-z)$, $\frac{1}{(t-s)^{\frac{1+\gamma_1}{\alpha}}} \lesssim \frac{1}{(t-r)^{\frac{1+\gamma_1}{\alpha}}}$. This yields

$$\Delta_4^2 \lesssim h^{\frac{\gamma_1}{\alpha}} \int_{t_1}^{t_{\lfloor \frac{t}{h} \rfloor -1}} \int p_\alpha(r, z-x) |b(r,z)| p_\alpha(t-r, y-z) \frac{1}{(t-r)^{\frac{1+\gamma_1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r,$$

which is the right form to use (5.2.7) with the same parameters as for Δ_4^1. Doing so, we obtain similarly

$$\begin{aligned}\Delta_4^2 &\lesssim p_\alpha(t,y-x)h^{\frac{\gamma_1}{\alpha}}(t-t_{\lfloor \frac{t}{h}\rfloor-1})^{\frac{\gamma-\gamma_1}{\alpha}}\\ &\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}. \end{aligned}\tag{5.2.25}$$

Let us now turn to the term Δ_4^3 in (5.2.22):

- **Global off-diagonal case:** $|x-y|>t^{\frac{1}{\alpha}}$, then, since $|y-x|\le|y-w|+|z-w|+|z-x|$, at least one of the stable transitions in Δ_4^3 will be off-diagonal as well. In this case, we will actually manage to retrieve the global final regime for $p_\alpha(t,y-x)$ from the inner densities in Δ_4^3.
  - If $|z-x|>\frac{1}{3}|x-y|>\frac{1}{3}t^{\frac{1}{\alpha}}\gtrsim t_j^{\frac{1}{\alpha}}$, we can write

    $$p_\alpha(t_j,z-x)\lesssim\frac{t_j}{|z-x|^{d+\alpha}}\lesssim\frac{t}{|x-y|^{d+\alpha}}\lesssim p_\alpha(t,x-y).$$

    We can then compute

    $$\begin{aligned}\Delta_4^{3,1}:=&\mathbb{1}_{|x-y|>t^{\frac{1}{\alpha}}}\sum_{j=1}^{\lfloor\frac{t}{h}\rfloor-2}\int_{t_j}^{t_{j+1}}\frac{1}{h}\int_{t_j}^{t_{j+1}}\int\int\mathbb{1}_{|z-x|>\frac{1}{3}|x-y|}p_\alpha(t_j,z-x)\\ &\times p_\alpha(s-t_j,w-z)|b(r,z)|p_\alpha(t-s,y-w)\left(\frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{\gamma_1}{\alpha}}}\wedge 1\right)\frac{1}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s\\ \lesssim&\, p_\alpha(t,y-x)\sum_{j=1}^{\lfloor\frac{t}{h}\rfloor-2}\int_{t_j}^{t_{j+1}}\frac{1}{h}\int_{t_j}^{t_{j+1}}\int\int p_\alpha(s-t_j,w-z)\\ &\times|b(r,z)|p_\alpha(t-s,y-w)\frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s\\ \lesssim&\, p_\alpha(t,y-x)\sum_{j=1}^{\lfloor\frac{t}{h}\rfloor-2}\int_{t_j}^{t_{j+1}}\frac{1}{h}\int_{t_j}^{t_{j+1}}\|b(r,\cdot)\|_{L^p}\,\mathrm{d}r\\ &\times\int\|p_\alpha(s-t_j,w-\cdot)|\cdot-w|^{\gamma_1}\|_{L^{p'}}\frac{p_\alpha(t-s,y-w)}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}w\,\mathrm{d}s.\end{aligned}\tag{5.2.26}$$

    Note that $\gamma_1\le d+\alpha-\frac{d}{p'}$, allowing us to use (5.2.4):

    $$\|p_\alpha(s-t_j,w-\cdot)|\cdot-w|^{\gamma_1}\|_{L^{p'}}\lesssim(s-t_j)^{\frac{\gamma_1}{\alpha}-\frac{d}{\alpha p}},$$

    yielding, once integrating in w,

    $$\begin{aligned}\Delta_4^{3,1}&\lesssim p_\alpha(t,y-x)\sum_{j=1}^{\lfloor\frac{t}{h}\rfloor-2}\int_{t_j}^{t_{j+1}}\frac{1}{h}\int_{t_j}^{t_{j+1}}\|b(r,\cdot)\|_{L^p}\,\mathrm{d}r\frac{(s-t_j)^{\frac{\gamma_1}{\alpha}-\frac{d}{\alpha p}}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}s\\ &\lesssim p_\alpha(t,y-x)\int_{t_1}^{t_{\lfloor\frac{t}{h}\rfloor-1}}h^{\frac{\gamma_1}{\alpha}-\frac{d}{\alpha p}-\frac{1}{q}}\frac{1}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}s\\ &\lesssim p_\alpha(t,y-x)h^{\frac{\gamma_1}{\alpha}-\frac{d}{\alpha p}-\frac{1}{q}}h^{1-\frac{1}{\alpha}-\frac{\gamma_1}{\alpha}}\\ &\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}},\end{aligned}\tag{5.2.27}$$

    the last inequality being true only if $1-\frac{1}{\alpha}-\frac{\gamma_1}{\alpha}<0$, which is always possible to satisfy since the choice of $\gamma_1\in(\gamma,1]$ is free.

– If $|z-w| > \frac{1}{3}|x-y| > \frac{1}{3}t^{\frac{1}{\alpha}}$, remarking that $s - t_j \le h$ and $0 < \frac{\gamma}{\alpha} + \frac{1}{q} < 1$, we can write

$$p_\alpha(s-t_j, w-z) \lesssim \frac{s-t_j}{|w-z|^{d+\alpha}} \lesssim \frac{s-t_j}{t} \times \frac{t}{|x-y|^{d+\alpha}} \lesssim \frac{s-t_j}{t} p_\alpha(t, x-y) \lesssim \frac{h^{\frac{\gamma}{\alpha}+\frac{1}{q}}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} p_\alpha(t, x-y),$$

and then compute

$$\begin{aligned}
\Delta_4^{3,2} &:= \mathbb{1}_{|x-y|>t^{\frac{1}{\alpha}}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t_j}^{t_{j+1}} \int\int \mathbb{1}_{|z-w|>\frac{1}{3}|x-y|} p_\alpha(t_j, z-x) \\
&\quad \times p_\alpha(s-t_j, w-z)|b(r,z)| p_\alpha(t-s, y-w) \left(\frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{\gamma_1}{\alpha}}} \wedge 1 \right) \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim p_\alpha(t, y-x) \frac{h^{\frac{\gamma}{\alpha}+\frac{1}{q}-1}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j, z-x) \\
&\quad \times |b(r,z)| p_\alpha(t-s, y-w) \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim p_\alpha(t, y-x) \frac{h^{\frac{\gamma}{\alpha}+\frac{1}{q}-1}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j, z-x)|b(r,z)| \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim p_\alpha(t, y-x) \frac{h^{\frac{\gamma}{\alpha}+\frac{1}{q}-1}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \|b(r,\cdot)\|_{L^p} \,\mathrm{d}r \int_{t_j}^{t_{j+1}} \|p_\alpha(t_j, \cdot - x)\|_{L^{p'}} \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}s \\
&\lesssim p_\alpha(t, y-x) \frac{h^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{t_j^{\frac{d}{\alpha p}}} \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}s,
\end{aligned}$$

using (5.2.4) (with $\delta = 0$) and the Hölder inequality in space for the antepenultimate inequality and the Hölder inequality in time for the last one.
Next, remarking that $h \le t_j$ and therefore $t_j^{-1} \le 2s^{-1}$, we can write

$$\Delta_4^{3,2} \lesssim p_\alpha(t, y-x) \frac{h^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} \int_0^t \frac{1}{s^{\frac{d}{\alpha p}}} \frac{1}{(t-s)^{\frac{1}{\alpha}}} \,\mathrm{d}s.$$

Hence,

$$\begin{aligned}
\Delta_4^{3,2} &\lesssim p_\alpha(t, y-x) \frac{h^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}+\frac{1}{q}}} t^{1-\frac{d}{\alpha p}-\frac{1}{\alpha}} \int_0^1 \frac{1}{\lambda^{\frac{d}{\alpha p}}} \frac{1}{(1-\lambda)^{\frac{1}{\alpha}}} \,\mathrm{d}\lambda \\
&\lesssim p_\alpha(t, y-x) h^{\frac{\gamma}{\alpha}}, \qquad (5.2.28)
\end{aligned}$$

recalling the definition of γ in (5.1.20) for the last inequality.

– If $|y-w| > \frac{1}{3}|x-y| > \frac{1}{3}t^{\frac{1}{\alpha}} \gtrsim (t-s)^{\frac{1}{\alpha}}$, we can write

$$p_\alpha(t-s, y-w) \lesssim \frac{t-s}{|y-w|^{d+\alpha}} \lesssim \frac{t}{|x-y|^{d+\alpha}} \lesssim p_\alpha(t, x-y).$$

This yields, using (5.2.4) to bound $\|p_\alpha(s-t_j,z-\cdot)|z-\cdot|^{\gamma_1}\|_{L^1}$,

$$\begin{aligned}
\Delta_4^{3,3} &:= \mathbb{1}_{|x-y|>t^{\frac{1}{\alpha}}} \sum_{j=1}^{\lfloor \frac{t}{h}\rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h}\int_{t_j}^{t_{j+1}} \int\int \mathbb{1}_{|y-w|>\frac{1}{3}|x-y|} p_\alpha(t_j,z-x) \\
&\qquad \times p_\alpha(s-t_j,w-z)|b(r,z)|p_\alpha(t-s,y-w)\left(\frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{\gamma_1}{\alpha}}}\wedge 1\right)\frac{1}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim p_\alpha(t,y-x)\sum_{j=1}^{\lfloor \frac{t}{h}\rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h}\int_{t_j}^{t_{j+1}} \int\int p_\alpha(t_j,z-x)p_\alpha(s-t_j,w-z) \\
&\qquad\qquad \times |b(r,z)|\frac{|z-w|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s \qquad (5.2.29)\\
&\lesssim p_\alpha(t,y-x)\sum_{j=1}^{\lfloor \frac{t}{h}\rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{h}\int_{t_j}^{t_{j+1}} \frac{(s-t_j)^{\frac{\gamma_1}{\alpha}}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\int p_\alpha(t_j,z-x)|b(r,z)|\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}$$

Next, we use $s-t_j\le h$, a Hölder inequality in z and (5.2.4):

$$\begin{aligned}
\Delta_4^{3,3} &\lesssim p_\alpha(t,y-x)h^{\frac{\gamma_1}{\alpha}-1}\sum_{j=1}^{\lfloor \frac{t}{h}\rfloor -2} \int_{t_j}^{t_{j+1}} \|b(r,\cdot)\|_{L^p}\,\mathrm{d}r \int_{t_j}^{t_{j+1}} \|p_\alpha(t_j,\cdot-x)\|_{L^{p'}}\frac{1}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}s \\
&\lesssim p_\alpha(t,y-x)h^{\frac{\gamma_1}{\alpha}-\frac{1}{q}}\sum_{j=1}^{\lfloor \frac{t}{h}\rfloor -2} \int_{t_j}^{t_{j+1}} \frac{1}{t_j^{\frac{d}{\alpha p}}}\frac{1}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}s \\
&\lesssim p_\alpha(t,y-x)h^{\frac{\gamma_1}{\alpha}-\frac{1}{q}-\frac{d}{\alpha p}}\int_h^{\tau_t^h-h}\frac{1}{(t-s)^{\frac{1+\gamma_1}{\alpha}}}\,\mathrm{d}s.
\end{aligned}$$

Choosing $\gamma_1\in(\gamma,1]$ such that $\frac{1+\gamma_1}{\alpha}>1$ we conclude that

$$\Delta_4^{3,3}\lesssim p_\alpha(t,y-x)h^{\frac{\gamma_1}{\alpha}-\frac{1}{q}-\frac{d}{\alpha p}}(t-\tau_t^h+h)^{1-\frac{1+\gamma_1}{\alpha}}\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}. \qquad (5.2.30)$$

- **Global diagonal case:** $|x-y|<t^{\frac{1}{\alpha}}$. We will use the fact that $p_\alpha(t,y-x)\asymp t^{-\frac{d}{\alpha}}$ to replace one of the local transitions with $p_\alpha(t,y-x)$, and then the computations will be the same as in the global off-diagonal case:
  - if $t_j<t/2$, $p_\alpha(t-s,y-w)\lesssim(t-s)^{-\frac{d}{\alpha}}\lesssim t^{-\frac{d}{\alpha}}\asymp p_\alpha(t,y-x)$, and the computations are the same as from (5.2.29),
  - if $t_j\ge t/2$, $p_\alpha(t_j,z-x)\lesssim t_j^{-\frac{d}{\alpha}}\lesssim t^{-\frac{d}{\alpha}}\asymp p_\alpha(t,y-x)$, and the computations are the same as from (5.2.26).

Overall, gathering the estimates (5.2.27), (5.2.28) and (5.2.30) as well as the estimates from the global diagonal case, we obtain $\Delta_4^3\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}$, which together with (5.2.25), (5.2.24) and (5.2.22) eventually yields

$$\Delta_4\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}, \qquad (5.2.31)$$

as intended. As we only used $|b_h|\lesssim b$ for Δ_3 and Δ_4, the estimations remain valid for $\bar\Delta_3$ and $\bar\Delta_4$.

Let us turn our attention to Δ_5 in (5.2.13) (first time step). Note that, even though a term $b(s,z)-b_h(r,x)$ appears in Δ_5, its smallness actually follows from the fact that it only covers the first time step (over $(0,t_1\wedge t)$). Thus, we will bound Δ_5 using the triangular inequality $|b(s,z)-b_h(r,x)|\lesssim|b(s,z)|+h^{-\frac{d}{\alpha p}-\frac{1}{q}}$ (resp. using

$|\bar b_h(r,x)| = 0$ for $r \le h$), and then compute a bound for each term. Namely,

$$\begin{aligned}|\Delta_5| &= \left|\frac{1}{h}\int_0^{t_1\wedge t}\int_0^h\int p_\alpha(s, z-x-b_h(r,x)s)(b(s,z)-b_h(r,x))\cdot\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\right|\\ &\lesssim \frac{1}{h}\int_0^{t_1\wedge t}\int_0^h\int p_\alpha(s,z-x-b_h(r,x)s)\left(|b(s,z)|+h^{-\frac{d}{\alpha p}-\frac{1}{q}}\right)\frac{1}{(t-s)^{\frac{1}{\alpha}}}p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

Since in our current integrals, using (5.2.11), $p_\alpha(s,z-x-b_h(r,x)s) \lesssim p_\alpha(s,z-x)$, we can write

$$\begin{aligned}|\Delta_5| \lesssim& \int_0^{t_1\wedge t}\int p_\alpha(s,z-x)|b(s,z)|\frac{1}{(t-s)^{\frac{1}{\alpha}}}p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\\ &+\int_0^{t_1\wedge t}\int p_\alpha(s,z-x)h^{-\frac{d}{\alpha p}-\frac{1}{q}}\frac{1}{(t-s)^{\frac{1}{\alpha}}}p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s.\end{aligned}$$

We then use (5.2.8) with $u=0$, $v=t_1\wedge t$, $\beta_1=\frac{1}{\alpha}$ and $\beta_2=0$ for the first term in the right-hand side and the convolution properties of the stable kernel for the second one to conclude that :

$$|\Delta_5| \lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}. \tag{5.2.32}$$

$$\text{resp.}\qquad |\bar\Delta_5| \lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}. \tag{5.2.33}$$

Let us now turn to Δ_6 in (5.2.13), for which the same reasoning as for Δ_5 applies, although this time we are working on the last time step, over $((\tau_t^h-h)\vee t_1, t)$. Let $t\ge h$ (otherwise, Δ_6 vanishes). Using (5.1.24), (5.2.11) and (5.2.1), we can write

$$\begin{aligned}\Delta_6 =& \frac{1}{h}\bigg|\int_{(\tau_t^h-h)\vee t_1}^t\int_{\tau_s^h}^{\tau_s^h+h}\int\int\Gamma^h(0,x,\tau_s^h,w)p_\alpha\left(s-\tau_s^h,z-w-b_h(r,w)(s-\tau_s^h)\right)\\ &\qquad\times(b(s,z)-b_h(r,w))\cdot\nabla p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s\bigg|\\ \lesssim& \int_{(\tau_t^h-h)\vee t_1}^t\int\int p_\alpha(\tau_s^h,w-x)p_\alpha(s-\tau_s^h,z-w)|b(s,z)|\frac{p_\alpha(t-s,y-z)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\\ &+\frac{1}{h}\int_{(\tau_t^h-h)\vee t_1}^t\int_{\tau_s^h}^{\tau_s^h+h}\int\int p_\alpha(\tau_s^h,w-x)p_\alpha(s-\tau_s^h,z-w)|b(r,w)|\frac{p_\alpha(t-s,y-z)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s\\ =:& \Delta_6^1+\Delta_6^2.\end{aligned}$$

For Δ_6^1, we first use the convolution properties of the stable kernel in w and then apply (5.2.8) with $u=(\tau_t^h-h)\vee t_1$, $v=t$, $\beta_1=\frac{1}{\alpha}$ and $\beta_2=0$ to obtain

$$\Delta_6^1 \lesssim \int_{(\tau_t^h-h)\vee t_1}^t\int p_\alpha(s,z-x)|b(s,z)|\frac{p_\alpha(t-s,y-z)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}s \lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}.$$

For Δ_6^2, we use the convolution properties of the stable kernel in z and (5.2.6)

$$\begin{aligned}\Delta_6^2 &\lesssim \frac{1}{h}\int_{(\tau_t^h-h)\vee t_1}^t\int_{\tau_s^h}^{\tau_s^h+h}\int p_\alpha(\tau_s^h,w-x)|b(r,w)|\frac{p_\alpha(t-\tau_s^h,y-w)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s\\ &\lesssim p_\alpha(t,y-x)\frac{1}{h}\int_{(\tau_t^h-h)\vee t_1}^t\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\frac{1}{(t-s)^{\frac{1}{\alpha}}}\int_{\tau_s^h}^{\tau_s^h+h}\|b(r,\cdot)\|_{L^p}\,\mathrm{d}r\,\mathrm{d}s\\ &\lesssim p_\alpha(t,y-x)h^{-\frac{1}{q}}\int_{(\tau_t^h-h)\vee t_1}^t\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\frac{1}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}s.\end{aligned}$$

Remarking that $t-\tau_s^h \geq t-s$, that $\tau_s^h \geq h$ and $(t-(\tau_t^h-h)\vee t_1) \leq 2h$ we get

$$\begin{aligned}
\Delta_6^2 &\lesssim p_\alpha(t,y-x)h^{-\frac{1}{q}}\int_{(\tau_t^h-h)\vee t_1}^t \left[h^{-\frac{d}{\alpha p}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\frac{1}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}s\\
&\lesssim p_\alpha(t,y-x)h^{-\frac{1}{q}}\left(h^{-\frac{d}{\alpha p}}(t-(\tau_t^h-h)\vee t_1)^{1-\frac{1}{\alpha}}+(t-(\tau_t^h-h)\vee t_1)^{1-\frac{1}{\alpha}-\frac{d}{\alpha p}}\right)\\
&\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}.
\end{aligned}$$

This is also a valid bound for $|\bar{\Delta}_6|$ as we only used $|b_h|\lesssim|b|$.

Now that, plugging the above computations for Δ_6 and (5.2.17), (5.2.18), (5.2.31), (5.2.32) in (5.2.13) and using (5.2.1) for Δ_1, we obtain

$$\begin{aligned}
|\Gamma^h(0,x,t,y)-\Gamma(0,x,t,y)| &\lesssim p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}\\
&\quad+\int_0^t\int|\Gamma^h(0,x,s,z)-\Gamma(0,x,s,z)||b(s,z)|\frac{p_\alpha(t-s,y-z)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}s. \qquad (5.2.34)
\end{aligned}$$

Setting for all $u\in(0,T]$,

$$f(u):=\sup_{x,z\in\mathbb{R}^d}\frac{|\Gamma^h(0,x,u,z)-\Gamma(0,x,u,z)|}{p_\alpha(u,z-x)}, \qquad (5.2.35)$$

we use (5.2.34) then (5.2.5) and Hölder's inequality in time to obtain :

$$\begin{aligned}
f(t) &\lesssim h^{\frac{\gamma}{\alpha}}+\sup_{x,y\in\mathbb{R}^d}\frac{1}{p_\alpha(t,y-x)}\int_0^t\int f(s)p_\alpha(s,z-x)|b(s,z)|\frac{p_\alpha(t-s,y-z)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim h^{\frac{\gamma}{\alpha}}+\sup_{x,y\in\mathbb{R}^d}\frac{1}{p_\alpha(t,y-x)}\int_0^t\|p_\alpha(s,\cdot-x)p_\alpha(t-s,y-\cdot)\|_{L^{p'}}\|b(s,\cdot)\|_{L^p}\frac{f(s)}{(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}s\\
&\lesssim h^{\frac{\gamma}{\alpha}}+\left(\int_0^t\left(\frac{f(s)}{(t-s)^{\frac{1}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\right)^{q'}\mathrm{d}s\right)^{\frac{1}{q'}}.
\end{aligned}$$

Up to a convexity inequality, we thus obtain the estimate

$$f(t)^{q'}\lesssim h^{\frac{\gamma q'}{\alpha}}+\int_0^t f(s)^{q'}\frac{1}{(t-s)^{\frac{q'}{\alpha}}}\left[\frac{1}{s^{\frac{dq'}{\alpha p}}}+\frac{1}{(t-s)^{\frac{dq'}{\alpha p}}}\right]\mathrm{d}s.$$

Since $\frac{q'}{\alpha}+\frac{dq'}{\alpha p}<1$, this permits to conclude by a suitable Grönwall-Volterra lemma (see e.g. Lemma 2.2 and Example 2.4 [Zha10]) which ensures that

$$f(t)\lesssim h^{\frac{\gamma}{\alpha}}.$$

The same reasoning applies for scheme involving $\bar{b}_h$, which concludes the proof of Theorem 5.2.

5.3 Proof of Proposition 5.1: Duhamel representation for the density of the schemes and associated controls

5.3.1 Duhamel representation for the density of the scheme

Let us first prove (5.1.22). Let $t\in(t_k,T]$, ϕ be a $\mathcal{C}^2$ function with compact support and $v(s,y)=\mathbb{1}_{s<t}p_\alpha(t-s,\cdot)\star\phi(y)+\mathbb{1}_{s=t}\phi(y)$. According to Lemma 5.2, v is $\mathcal{C}^{1,2}$ on $[0,t]\times\mathbb{R}^d$ and satisfies the Feynman-Kac partial differential equation

$$\forall(s,y)\in[0,t)\times\mathbb{R}^d,\ \partial_s v(s,y)+\mathcal{L}^\alpha v(s,y)=0.$$

Applying Itô's formula between t_k and t to $v(s, X_s^h)$ where $(X_s^h)_{s\in[t_k,T]}$ denotes the Euler scheme started from $X_{t_k}^h = x$ and evolving according to (5.1.6), we obtain :

$$\phi(X_t^h) = v(t_k, x) + M_{t_k,t}^h + \int_{t_k}^t \nabla v(s, X_s^h) \cdot b_h\left(U_{\lfloor \frac{s}{h}\rfloor}, X_{\tau_s^h}^h\right) \mathrm{d}s,$$

where $M_{t_k,t}^h = \int_{t_k}^t \int_{\mathbb{R}^d\setminus\{0\}} \left(v(s,X_{s^-}^h + x) - v(s,X_{s^-}^h)\right)\tilde{N}(\mathrm{d}s, \mathrm{d}x)$, in which $\tilde{N}$ is the compensated Poisson measure associated with Z. Taking now the expectation (recalling that $(M_{t_k,s}^h)_{s\in[t_k,t]}$ is a martingale) and using Fubini's theorem, we derive

$$\int \phi(y)\Gamma^h(t_k, x, t, y)\,\mathrm{d}y = v(t_k, x) + \int_{t_k}^t \mathbb{E}_{t_k,x}\left[\nabla v(s, X_s^h) \cdot b_h\left(U_{\lfloor \frac{s}{h}\rfloor}, X_{\tau_s^h}^h\right)\right] \mathrm{d}s.$$

Using the definition of v, we get

$$\begin{aligned}&\int \phi(y)\Gamma^h(t_k, x, t, y)\,\mathrm{d}y \\ &\quad= \int p_\alpha(t-t_k, x-y)\phi(y)\,\mathrm{d}y + \int \phi(y) \int_{t_k}^t \mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha(t-s, X_s^h - y) \cdot b_h\left(U_{\lfloor \frac{s}{h}\rfloor}, X_{\tau_s^h}^h\right)\right] \mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Since ϕ is arbitrary and $p_\alpha(t-s, \cdot)$ is even, we deduce that $\mathrm{d}y$ a.e.,

$$\Gamma^h(t_k, x, t, y) = p_\alpha(t-t_k, x-y) - \int_{t_k}^t \mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha(t-s, y - X_s^h) \cdot b_h\left(U_{\lfloor \frac{s}{h}\rfloor}, X_{\tau_s^h}^h\right)\right] \mathrm{d}s. \tag{5.3.1}$$

We will see later that (5.3.1) actually holds for all $y \in \mathbb{R}^d$ as a consequence of the Hölder regularity of Γ^h in the forward space variable. This concludes the proof of (5.1.22).

5.3.2 Heat kernel bounds for the scheme

We will now prove inequality (5.1.24), upper stable bound for the density of the scheme, in 3 steps. First, we will prove it for $t \in (t_k, t_{k+1}]$, using only the definition of the cutoffed drift and assuming $h < 1$. Then, we will prove it between t_k and t_ℓ, when $t_k - t_\ell$ is small enough at a *macro* scale. We will finally chain the previous estimates to obtain (5.1.24) for any time interval $(t_k, t] \subset [0, T]$.

Step 1 : $t \in (t_k, t_{k+1}]$

Remarking that when $t \in (t_k, t_{k+1}]$, $\forall z \in \mathbb{R}^d$, $\Gamma^h(t_k, x, t, z) = \frac{1}{h}\int_{t_k}^{t_{k+1}} p_\alpha(t-t_k, z-x-(t-t_k)b_h(r, x))\,\mathrm{d}r$ (resp. $\bar{\Gamma}^h(t_k, x, t, z) = \frac{1}{h}\int_{t_k}^{t_{k+1}} p_\alpha(t-t_k, z-x-(t-t_k)\bar{b}_h(r, x))\,\mathrm{d}r$), we obtain (5.1.24) in the case $t \in (t_k, t_{k+1}]$ using (5.2.11) from Lemma 5.4 to get rid of the drift.

Step 2 : $t - t_k$ ***small enough***

Recall that for $j \in \{k, \cdots, \lceil t/h\rceil - 1\}$ and $r \in [t_j, t_{j+1}]$, $X_r^h = X_{t_j}^h + (Z_r - Z_{t_j}) + b_h(U_j, X_{t_j}^h)(r - t_j)$. Using this and the independence between $X_{t_j}^h$, $(Z_r - Z_{t_j})$ and U_j, we have, starting from the Duhamel representation (5.1.22),

$$\begin{aligned}\Gamma^h(t_k, x, t, y) &= p_\alpha(t-t_k, y-x) - \sum_{j=k}^{\lceil \frac{t}{h}\rceil - 1} \int_{t_j}^{t_{j+1}\wedge t} \mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha(t-r, y - X_r^h) \cdot b_h\left(U_j, X_{t_j}^h\right)\right] \mathrm{d}r \\ &= p_\alpha(t-t_k, y-x) \\ &\quad - \sum_{j=k}^{\lceil \frac{t}{h}\rceil - 1} \int_{t_j}^{t_{j+1}\wedge t} \frac{1}{h}\int_{t_j}^{t_{j+1}} \mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha\left(t-r, y - X_{t_j}^h - (Z_r - Z_{t_j}) - b_h(s, X_{t_j}^h)(r-t_j)\right) \cdot b_h\left(s, X_{t_j}^h\right)\right] \mathrm{d}s\,\mathrm{d}r.\end{aligned}$$

Using Fubini's and Lebesgue's theorems and the convolution property of the stable density,

$$
\begin{aligned}
&\mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha\left(t-r, y-X^h_{t_j}-(Z_r-Z_{t_j})-b_h(s,X^h_{t_j})(r-t_j)\right)\cdot b_h\left(s,X^h_{t_j}\right)\right]\\
&\qquad= \int \mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha\left(t-r, y-X^h_{t_j}-z-b_h(s,X^h_{t_j})(r-t_j)\right)\cdot b_h\left(s,X^h_{t_j}\right)\right] p_\alpha(r-t_j,z)\,\mathrm{d}z\\
&\qquad= \mathbb{E}_{t_k,x}\left[\nabla_y\left(\int p_\alpha\left(t-r, y-X^h_{t_j}-z-b_h(s,X^h_{t_j})(r-t_j)\right)p_\alpha(r-t_j,z)\,\mathrm{d}z\right)\cdot b_h\left(s,X^h_{t_j}\right)\right]\\
&\qquad= \mathbb{E}_{t_k,x}\left[\nabla_y p_\alpha\left(t-t_j, y-X^h_{t_j}-b_h(s,X^h_{t_j})(r-t_j)\right)\cdot b_h\left(s,X^h_{t_j}\right)\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Gamma^h(t_k,x,t,y) &= p_\alpha(t-t_k,y-x) - \int_{t_k}^{t_{k+1}\wedge t}\frac{1}{h}\int_{t_k}^{t_{k+1}} \nabla_y p_\alpha\left(t-t_k, y-x-b_h(s,x)(r-t_k)\right)\cdot b_h(s,x)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r\\
&- \sum_{j=k+1}^{\lceil \frac{t}{h}\rceil-1}\int_{t_j}^{t_{j+1}\wedge t}\frac{1}{h}\int_{t_j}^{t_{j+1}}\int \Gamma^h(t_k,x,t_j,z)\nabla_y p_\alpha\left(t-t_j,y-z-b_h(s,z)(r-t_j)\right)\cdot b_h(s,z)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r.
\end{aligned}
\tag{5.3.2}
$$

Note that we have not used any property related to b_h here, so the same holds with $(\bar{\Gamma}^h, \bar{b}_h)$ in place of (Γ^h, b_h).

Set for $j \in \{k+1,\cdots,n\}$, $m_{k,j} := \sup_{x,y\in\mathbb{R}^d} \frac{\Gamma^h(t_k,x,t_j,y)}{p_\alpha(t_j-t_k,y-x)}$. Observe from the previous one-step part that there exists $C \geq 1$ s.t. $m_{k,j} \leq C^{n-k} < +\infty$. The point of step 2 is to make this bound uniform in n. Using (5.2.11) to get rid of the *negligible* cutoffed drift, we get, for $n \geq \ell \geq k+1 \geq 1$:

$$
\begin{aligned}
\frac{\Gamma^h(t_k,x,t_\ell,y)}{p_\alpha(t_\ell-t_k,y-x)} &\lesssim 1 + \int_{t_k}^{t_{k+1}}\frac{1}{h(t_\ell-t_k)^{\frac{1}{\alpha}}}\int_{t_k}^{t_{k+1}}|b_h(s,x)|\,\mathrm{d}s\,\mathrm{d}r\\
&+ \sum_{j=k+1}^{\ell-1}\int_{t_j}^{t_{j+1}}\frac{1}{h}\int_{t_j}^{t_{j+1}}\int m_{k,j}\frac{p_\alpha(t_j-t_k,z-x)}{p_\alpha(t_\ell-t_k,y-x)}\times\frac{1}{(t_\ell-t_j)^{\frac{1}{\alpha}}}p_\alpha(t_\ell-t_j,y-z)\,|b_h(s,z)|\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r.
\end{aligned}
$$

In the first integral, we use the bound $|b_h| \leq h^{-\frac{d}{\alpha p}-\frac{1}{q}}$ (the bound remains valid for $\bar{b}_h$ since the latter vanishes on the first time step) and in the second we use $t_\ell - t_j \geq t_\ell - s$ for $s \in [t_j, t_{j+1}]$ and then bound $m_{k,j}$ from above:

$$
\begin{aligned}
&\frac{\Gamma^h(t_k,x,t_\ell,y)}{p_\alpha(t_\ell-t_k,y-x)}\\
&\lesssim 1 + \frac{h^{1-\frac{d}{\alpha p}-\frac{1}{q}}}{(t_\ell-t_k)^{\frac{1}{\alpha}}} + \max_{j\in[\![k+1,\ell-1]\!]} m_{k,j}\int_{t_{k+1}}^{t_\ell}\int\frac{1}{(t_\ell-s)^{\frac{1}{\alpha}}}\frac{p_\alpha(s-t_k,z-x)}{p_\alpha(t_\ell-t_k,y-x)}p_\alpha(t_\ell-s,y-z)\,|b(s,z)|\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}
$$

We are now in the right setting to apply (5.2.8) (with $u = t_{k+1}, v = t_\ell, \beta_1 = 1/\alpha, \beta_2 = 0$), which readily gives

$$
\begin{aligned}
\frac{\Gamma^h(t_k,x,t_\ell,y)}{p_\alpha(t_\ell-t_k,y-x)} &\lesssim 1 + \frac{h^{1-\frac{d}{\alpha p}-\frac{1}{q}}}{(t_\ell-t_k)^{\frac{1}{\alpha}}} + \max_{j\in[\![k+1,\ell-1]\!]} m_{k,j}(t_\ell-t_{k+1})^{\frac{\gamma}{\alpha}}\\
&\lesssim 1 + (t_\ell-t_k)^{\frac{\gamma}{\alpha}} + \max_{j\in[\![k+1,\ell-1]\!]} m_{k,j}(t_\ell-t_{k+1})^{\frac{\gamma}{\alpha}}\\
&\lesssim 1 + \max_{j\in[\![k+1,\ell-1]\!]} m_{k,j}(t_\ell-t_{k+1})^{\frac{\gamma}{\alpha}}.
\end{aligned}
$$

Taking the supremum over $(x,y) \in \mathbb{R}^d$ in the l.h.s., and remarking that the r.h.s. is non-decreasing with ℓ, along with the definition of $\lesssim$, we get

$$
\max_{j\in[\![k+1,\ell]\!]} m_{k,j} \leq C + C(t_\ell-t_k)^{\frac{\gamma}{\alpha}}\max_{j\in[\![k+1,\ell]\!]} m_{k,j}
$$

for some constant C not depending on h. Thus, if $C(t_\ell - t_k)^{\frac{\gamma}{\alpha}} < 1$, then

$$\max_{j\in[\![k+1,l]\!]} m_{k,j} \leq \frac{C}{1-C(t_\ell - t_k)^{\frac{\gamma}{\alpha}}}. \tag{5.3.3}$$

In particular, it is bounded uniformly in h for k,ℓ s.t. $(t_\ell - t_k) < C^{-\frac{\alpha}{\gamma}}$.
As we only used the fact that $|b_h| \lesssim b$ for the main term, which remains true with $\bar{b}_h$ instead of b_h, the same estimates hold for $\bar{m}_{k,j} := \sup_{x,y\in\mathbb{R}^d} \frac{\bar{\Gamma}^h(t_k,x,t_j,y)}{p_\alpha(t_j-t_k,y-x)}$.

Remark 5.2. *Note that in the Gaussian setting, a precise control of the variance was required because of the exponential structure of the Gaussian tails (see [JM24b]). In the stable setting, as the tails of the stable kernel are polynomial, these controls are not required.*

Step 3 : chaining the previous estimates

In order to obtain the result for any arbitrary time interval, we will now chain the previous estimates. This will be done in the following way: denote $\theta = C^{-\frac{\alpha}{\gamma}}$ and let us first suppose that $h \leq \min\{\theta, 1\}$, which implies that $\tau_\theta^h \geq \frac{\theta}{2}$. Let $t > 0$ s.t. $t - t_k > \theta$ and let $J = \lceil \frac{t-t_k}{\tau_\theta^h} \rceil - 1 \leq \frac{2T}{\theta}$. We will first divide (t_k, t) into a main term (over $(t_k, t_k + J\tau_\theta^h)$) composed of J slices of size τ_θ^h (and thus on which we can use (5.3.3)) and a remainder term (over $(t_k + J\tau_\theta^h, t)$). This remainder term will then be split into two terms again ($(t_k + J\tau_\theta^h, \tau_t^h)$ and (τ_t^h, t)), in order to account for the fact that t does not necessarily belong to the discretization grid. Over $(t_k + J\tau_\theta^h, \tau_t^h)$, as we work on the grid, we will use (5.3.3) again, and over (τ_t^h, t), we will use the cutoff and (5.5.3).

With the convention $y_0 = x$,

$$\begin{aligned}
\Gamma^h(t_k,x,t,y) &= \int_{(\mathbb{R}^d)^J} \prod_{j=1}^J \Gamma^h(t_k + (j-1)\tau_\theta^h, y_{j-1}, t_k + j\tau_\theta^h, y_j)\Gamma^h(t_k + J\tau_\theta^h, y_J, t, y)\,\mathrm{d}y_1 ... \mathrm{d}y_J \\
&\leq \int_{(\mathbb{R}^d)^J} \prod_{j=1}^J m_{k+(j-1)\lfloor\frac{\theta}{h}\rfloor, k+j\lfloor\frac{\theta}{h}\rfloor} p_\alpha(\tau_\theta^h, y_j - y_{j-1})\Gamma^h(t_k + J\tau_\theta^h, y_J, t, y)\,\mathrm{d}y_1 ... \mathrm{d}y_J.
\end{aligned}$$

Using the boundedness of $m_{k+(j-1)\lfloor\frac{\theta}{h}\rfloor, k+j\lfloor\frac{\theta}{h}\rfloor}$, we get

$$\begin{aligned}
\Gamma^h(t_k,x,t,y) &\lesssim \int_{(\mathbb{R}^d)^J} \prod_{j=1}^J p_\alpha(\tau_\theta^h, y_j - y_{j-1})\Gamma^h(t_k + J\tau_\theta^h, y_J, t, y)\,\mathrm{d}y_1 ... \mathrm{d}y_J \\
&\lesssim \int p_\alpha(J\tau_\theta^h, y_J - y_0)\Gamma^h(t_k + J\tau_\theta^h, y_J, t, y)\,\mathrm{d}y_J.
\end{aligned}$$

Pay attention that the constants grow exponentially fast with J, but $J \leq \frac{2T}{\theta}$. Remarking that

$$\begin{aligned}
\Gamma^h(t_k + J\tau_\theta^h, y_J, t, y) &= \int \Gamma^h(t_k + J\tau_\theta^h, y_J, \tau_t^h, z)\Gamma^h(\tau_t^h, z, t, y)\,\mathrm{d}z \\
&\lesssim m_{k+J\lfloor\frac{\theta}{h}\rfloor, \tau_t^h} \int p_\alpha(t_k + J\tau_\theta^h, y_J, \tau_t^h, z) p_\alpha(\tau_t^h, z, t, y)\,\mathrm{d}z \\
&\lesssim p_\alpha(t_k + J\tau_\theta^h, y_J, t, y),
\end{aligned}$$

we obtain, by convolution,

$$\Gamma^h(t_k,x,t,y) \lesssim p_\alpha(t_k,x,t,y).$$

When $h > \theta$, then $\frac{T}{h} < \frac{T}{\theta}$ and the conclusion remains valid by chaining in a similar way with τ_θ^h replaced by h the estimate derived in Step 1. The same reasoning applied to $\bar{\Gamma}^h$ gives

$$\bar{\Gamma}^h(t_k,x,t,y) \lesssim p_\alpha(t_k,x,t,y),$$

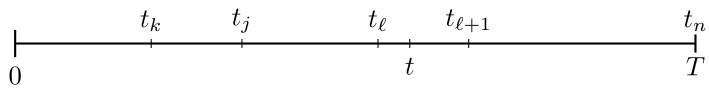
which concludes the proof of (5.1.24).

5.3.3 Hölder regularity of Γ^h in the forward variables

We will establish here the Hölder properties for the density of the scheme stated in Proposition 5.1. We begin with the forward time variable and discuss the forward space variable later on.

Hölder regularity of Γ^h in the forward time variable

Let us now prove (5.1.26). Let $0 \le k < \ell < n$, $x, y \in \mathbb{R}^d$ and $t \in [t_\ell, t_{\ell+1}]$.



Going back to (5.3.2), we can write:

$$
\begin{aligned}
&\Gamma^h(t_k, x, t_\ell, y) - \Gamma^h(t_k, x, t, y) = p_\alpha(t_\ell - t_k, y - x) - p_\alpha(t - t_k, y - x)\\
&\quad - \frac{1}{h}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \left[\nabla p_\alpha\left(t_\ell - t_k, w\right) - \nabla p_\alpha\left(t - t_k, w\right)\right]_{w=y-x-b_h(s,x)(r-t_k)} \cdot b_h\left(s, x\right)\, \mathrm{d}s\, \mathrm{d}r\\
&\quad - \frac{\mathbb{1}_{\{\ell \ge k+2\}}}{h}\sum_{j=k+1}^{\ell-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}}\int \Gamma^h(t_k, x, t_j, z)[\nabla p_\alpha\left(t_\ell - t_j, w\right)\\
&\qquad\qquad\qquad\qquad - \nabla p_\alpha\left(t - t_j, w\right)]_{w=y-z-b_h(s,z)(r-t_j)} \cdot b_h\left(s, z\right)\, \mathrm{d}z\, \mathrm{d}s\, \mathrm{d}r\\
&\quad + \frac{1}{h}\int_{t_\ell}^{t}\int_{t_\ell}^{t_{\ell+1}}\int \Gamma^h(t_k, x, t_\ell, z) b_h(s, z)\cdot \nabla p_\alpha(t - t_\ell, y - z - b_h(s, z)(r - t_\ell))\, \mathrm{d}z\, \mathrm{d}s\, \mathrm{d}r\\
&\quad =: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4.
\end{aligned}
\tag{5.3.4}
$$

Resp. $\bar{\Gamma}^h(t_k, x, t_\ell, y) - \bar{\Gamma}^h(t_k, x, t, y) =: \bar{\Delta}_1 + \bar{\Delta}_2 + \bar{\Delta}_3 + \bar{\Delta}_4$ for the scheme involving $\bar{b}_h$.

For Δ_1 (which is actually the same as $\bar{\Delta}_1$), we use (5.2.3) and $t - t_k \asymp t_\ell - t_k$ then $t - t_\ell < t - t_k$:

$$
|\Delta_1| \lesssim \frac{t - t_\ell}{t - t_k} p_\alpha(t - t_k, y - x) \lesssim \left(\frac{t - t_\ell}{t - t_k}\right)^{\frac{\gamma}{\alpha}} p_\alpha(t - t_k, y - x). \tag{5.3.5}
$$

For Δ_2, let us first bound $\left[\nabla p_\alpha\left(t_\ell - t_k, w\right) - \nabla p_\alpha\left(t - t_k, w\right)\right]_{w=y-x-b_h(s,x)(r-t_k)}$, using again (5.2.3) along with $t - t_k \asymp t_\ell - t_k$:

$$
\left|\nabla p_\alpha\left(t_\ell - t_k, w\right) - \nabla p_\alpha\left(t - t_k, w\right)\right|_{w=y-x-b_h(s,x)(r-t_k)} \lesssim \frac{|t - t_\ell|}{(t - t_k)^{1+\frac{1}{\alpha}}} p_\alpha(t - t_k, y - x - b_h(s, x)(r - t_k))
$$

In our current integral, $r - t_k \le t - t_k$, which means that, using (5.2.11), we get $p_\alpha(t - t_k, y - x - b_h(s, x)(r - t_k)) \lesssim p_\alpha(t - t_k, x - y)$. We can thus compute the following bound for Δ_2 (recalling that $|b_h| \lesssim h^{-\frac{d}{\alpha p} - \frac{1}{q}}$):

$$
\begin{aligned}
|\Delta_2| &= \frac{1}{h}\left|\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \left[\nabla p_\alpha\left(t_\ell - t_k, w\right) - \nabla p_\alpha\left(t - t_k, w\right)\right]_{w=y-x-b_h(s,x)(r-t_k)} \cdot b_h\left(s, x\right)\, \mathrm{d}s\, \mathrm{d}r\right|\\
&\lesssim \frac{1}{h}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \frac{t - t_\ell}{(t - t_k)^{1+\frac{1}{\alpha}}} p_\alpha(t - t_k, y - x)|b_h(s, x)|\, \mathrm{d}s\, \mathrm{d}r\\
&\lesssim \frac{t - t_\ell}{(t - t_k)^{1+\frac{1}{\alpha}}} p_\alpha(t - t_k, y - x)\int_{t_k}^{t_{k+1}} |b_h(s, x)|\, \mathrm{d}s\\
&\lesssim \frac{t - t_\ell}{(t - t_k)^{1+\frac{1}{\alpha}}} p_\alpha(t - t_k, y - x) h^{1-\frac{d}{\alpha p}-\frac{1}{q}}.
\end{aligned}
$$

Using the fact that $t - t_k \ge t_\ell - t_k \ge h$, we get

$$
|\Delta_2| \lesssim \frac{t - t_\ell}{t - t_k} p_\alpha(t - t_k, y - x) h^{1-\frac{1}{\alpha}-\frac{d}{\alpha p}-\frac{1}{q}} \lesssim \left(\frac{t - t_\ell}{t - t_k}\right)^{\frac{\gamma}{\alpha}} p_\alpha(t - t_k, y - x) h^{\frac{\gamma}{\alpha}}. \tag{5.3.6}
$$

For the alternative scheme, we would have used the inequality $|\bar{b}_h| \lesssim h^{\frac{1}{\alpha}-1}$, which yields

$$|\bar{\Delta}_2| \lesssim \frac{t-t_\ell}{(t-t_k)^{1+\frac{1}{\alpha}}} p_\alpha(t-t_k, y-x) \lesssim \left(\frac{t-t_\ell}{t-t_k}\right)^{\frac{\gamma}{\alpha}} p_\alpha(t-t_k, y-x).$$

For Δ_3, note that for all $j \in [\![k+1, \ell-1]\!]$, denoting $u = t_\ell - t_j$ and $u' = t - t_j$, we can see $u' - u = t - t_\ell \in [0, h]$ as a small perturbation at the scale of u or u'. This allows us to use (5.2.3) along with $t - t_j \asymp t_\ell - t_j$ and then (5.2.11):

$$\begin{aligned}
|\nabla p_\alpha(t_\ell - t_j, w) - \nabla p_\alpha(t - t_j, w)|_{w=y-z-b_h(s,z)(r-t_j)} &\lesssim \frac{t-t_\ell}{(t-t_j)^{1+\frac{1}{\alpha}}} p_\alpha(t-t_j, y-z-b_h(s,z)(r-t_j)) \\
&\lesssim \frac{t-t_\ell}{(t-t_j)^{1+\frac{1}{\alpha}}} p_\alpha(t-t_j, y-z).
\end{aligned}$$

For the computations on Δ_3, we assume that $\ell \geq k+2$ and introduce an exponent $\gamma_1 \in (\gamma, \alpha]$. Here, we singularize some of the estimates in order to obtain the expected Hölder rate involving γ. This is somehow a flexibility of the scheme: since we stay away from the final time t for this contribution, we can afford to make non-integrable exponents appear. Those terms will be handled with Lemma 5.3 (eq. (5.2.7)). Namely, using $|b_h| \leq b$ and the stable upper-bound (5.1.24), we get

$$\begin{aligned}
|\Delta_3| &\lesssim \frac{1}{h} \sum_{j=k+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} \int \Gamma^h(t_k, x, t_j, z) \frac{t-t_\ell}{(t-t_j)^{1+\frac{1}{\alpha}}} p_\alpha(t-t_j, y-z)|b(s,z)| \, \mathrm{d}z \, \mathrm{d}s \, \mathrm{d}r \\
&\lesssim \sum_{j=k+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \frac{(t-t_\ell)^{\frac{\gamma_1}{\alpha}}}{(t-t_j)^{\frac{\gamma_1+1}{\alpha}}} \int p_\alpha(t_j - t_k, z-x) p_\alpha(t-t_j, y-z)|b(s,z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim \sum_{j=k+1}^{\ell-2} \int_{t_j}^{t_{j+1}} \frac{(t-t_\ell)^{\frac{\gamma_1}{\alpha}}}{(t-t_j)^{\frac{\gamma_1+1}{\alpha}}} \int p_\alpha(t_j - t_k, z-x) p_\alpha(t-t_j, y-z)|b(s,z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\quad + \int_{t_{\ell-1}}^{t_\ell} \frac{(t-t_\ell)^{\frac{\gamma_1}{\alpha}}}{(t-t_{\ell-1})^{\frac{\gamma_1+1}{\alpha}}} \int p_\alpha(t_{\ell-1} - t_k, z-x) p_\alpha(t-t_{\ell-1}, y-z)|b(s,z)| \, \mathrm{d}z \, \mathrm{d}s \\
&=: \Delta_3^1 + \Delta_3^2.
\end{aligned}$$

Assume $\ell \geq k+3$ (otherwise Δ_3^1 vanishes). In Δ_3^1, which only contains non-singular integrals, we now approximate the discrete $(t_j)_{j \in [\![k+1, l-2]\!]}$ with s in the corresponding time integrals to apply (5.2.7) with $u = t_{k+1}, v = t_{\ell-1}, \beta_1 = (\gamma_1+1)/\alpha, \beta_2 = 0$:

$$\begin{aligned}
\Delta_3^1 &\lesssim \mathbb{1}_{\{\ell \geq k+3\}} \int_{t_{k+1}}^{t_{\ell-1}} \frac{(t-t_\ell)^{\frac{\gamma_1}{\alpha}}}{(t-s)^{\frac{\gamma_1+1}{\alpha}}} \int p_\alpha(s-t_k, z-x) p_\alpha(t-s, z-y)|b(s,z)| \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim p_\alpha(t-t_k, y-x)(t-t_\ell)^{\frac{\gamma_1}{\alpha}} \left[(t_{\ell-1} - t_{k+1})^{\frac{\gamma-\gamma_1}{\alpha}} + (t-t_{\ell-1})^{\frac{\gamma-\gamma_1}{\alpha}}\right] \\
&\lesssim p_\alpha(t-t_k, y-x)(t-t_\ell)^{\frac{\gamma_1}{\alpha}} h^{\frac{\gamma-\gamma_1}{\alpha}},
\end{aligned}$$

where, for the last inequality, we used $t_{\ell-1} - t_{k+1} \geq h$ since $\ell - 1 \geq (k+1)+1$, $t - t_{\ell-1} = t - t_\ell + h \geq h$ and $\gamma - \gamma_1 < 0$.
For Δ_3^2 (last time step), let us first use the convolution estimate (5.2.6):

$$\Delta_3^2 \lesssim p_\alpha(t-t_k, y-x) \frac{(t-t_\ell)^{\frac{\gamma_1}{\alpha}}}{(t-t_{\ell-1})^{\frac{\gamma_1+1}{\alpha}}} \left[\frac{1}{(t_{\ell-1}-t_k)^{\frac{d}{\alpha p}}} + \frac{1}{(t-t_{\ell-1})^{\frac{d}{\alpha p}}}\right] \int_{t_{\ell-1}}^{t_\ell} \|b(s,\cdot)\|_{L^p} \, \mathrm{d}s.$$

Using $\left[\frac{1}{(t_{\ell-1}-t_k)^{\frac{d}{\alpha p}}} + \frac{1}{(t-t_{\ell-1})^{\frac{d}{\alpha p}}}\right] \frac{1}{(t-t_{\ell-1})^{\frac{\gamma_1+1}{\alpha}}} \leq h^{-\frac{d}{\alpha p} - \frac{\gamma_1+1}{\alpha}}$ and applying Hölder's inequality to the integral, we obtain:

$$\begin{aligned}
\Delta_3^2 &\lesssim p_\alpha(t-t_k, y-x)(t-t_\ell)^{\frac{\gamma_1}{\alpha}} h^{-\frac{d}{\alpha p} - \frac{\gamma_1+1}{\alpha}} \|\mathbb{1}_{(t_{\ell-1}, t_\ell)}\|_{L^{q'}} \\
&\lesssim p_\alpha(t-t_k, y-x)(t-t_\ell)^{\frac{\gamma_1}{\alpha}} h^{\frac{\gamma-\gamma_1}{\alpha}}.
\end{aligned}$$

Gathering both estimates and recalling that $t-t_\ell \le h$ and $\gamma-\gamma_1<0$, we obtain

$$|\Delta_3| \lesssim p_\alpha(t-t_k,y-x)(t-t_\ell)^{\frac{\gamma}{\alpha}}. \tag{5.3.7}$$

Let us now bound Δ_4 in (5.3.4). Recalling that from its definition, $|b_h|\le |b|$ (resp. $|\bar b_h|\le |b|$), we can write, using also (5.1.24) and (5.2.11):

$$|\Delta_4| \lesssim \frac{(t-t_\ell)^{1-\frac{1}{\alpha}}}{h}\int_{t_\ell}^{t_{\ell+1}}\int p_\alpha(t_\ell-t_k,z-x)|b(s,z)|p_\alpha(t-t_\ell,z-y)\,\mathrm{d}z\,\mathrm{d}s.$$

We can now bound $|\Delta_4|$ using (5.2.6), then $(t_\ell-t_k)^{-1}\le (t-t_\ell)^{-1}$ and finally $t-t_\ell\le h$:

$$\begin{aligned}
|\Delta_4| &\lesssim \frac{(t-t_\ell)^{1-\frac{1}{\alpha}}}{h}\left[\frac{1}{(t_\ell-t_k)^{\frac{d}{\alpha p}}}+\frac{1}{(t-t_\ell)^{\frac{d}{\alpha p}}}\right]p_\alpha(t-t_k,y-x)\int_{t_\ell}^{t_{\ell+1}}\|b(s,\cdot)\|_{L^p}\,\mathrm{d}s\\
&\lesssim \frac{(t-t_\ell)^{1-\frac{1}{\alpha}-\frac{d}{\alpha p}}}{h}p_\alpha(t-t_k,y-x)\|b\|_{L^q-L^p}\|\mathbb{1}_{(t_\ell,t_{\ell+1})}\|_{L^{q'}}\\
&\lesssim (t-t_\ell)^{1-\frac{1}{\alpha}-\frac{d}{\alpha p}}h^{1-\frac{1}{q}-1}p_\alpha(t-t_k,y-x)\\
&\lesssim (t-t_\ell)^{1-\frac{1}{\alpha}-\frac{d}{\alpha p}-\frac{1}{q}}p_\alpha(t-t_k,y-x)\\
&\lesssim (t-t_\ell)^{\frac{\gamma}{\alpha}}p_\alpha(t-t_k,y-x).
\end{aligned} \tag{5.3.8}$$

As, for Δ_3 and Δ_4, we only used the fact that $|b_h|\le|b|$, the same estimations still hold for the alternative scheme involving $\bar b_h$. Plugging the estimates (5.3.5)-(5.3.8) into (5.3.4) concludes the proof of (5.1.26).

Hölder regularity of Γ^h in the forward space variable

Let us now prove (5.1.25). This property is important to prove that any limit point of the law induced by the Euler scheme solves the martingale problem and that its marginals will satisfy heat kernel estimates through a compactness type argument (see Section 5.4.1 for details).

- **Off-diagonal regime:** $|y-y'|\gtrsim (t-t_k)^{1/\alpha}$.
  In this case, using the stable upper bound (5.1.24), we only need to write

$$\begin{aligned}
|\Gamma^h(t_k,x,t,y')-\Gamma^h(t_k,x,t,y)| &\lesssim \Gamma^h(t_k,x,t,y')+\Gamma^h(t_k,x,t,y)\\
&\lesssim p_\alpha(t-t_k,y'-x)+p_\alpha(t-t_k,y-x)\\
&\lesssim \frac{|y-y'|^\gamma\wedge(t-t_k)^{\frac{\gamma}{\alpha}}}{(t-t_k)^{\frac{\gamma}{\alpha}}}\left(p_\alpha(t-t_k,y'-x)+p_\alpha(t-t_k,y-x)\right).
\end{aligned}$$

- **Diagonal regime:** $|y-y'|\lesssim (t-t_k)^{1/\alpha}$. Note that in this setting, $p_\alpha(t-t_k,y-x)\asymp p_\alpha(t-t_k,y'-x)$. In this case, we go back to (5.3.2), denoting $\ell=\lceil t/h\rceil-1$ (so that $t\in(t_\ell,t_{\ell+1}]$) and we write, similarly to (5.3.4):

$$\begin{aligned}
&\Gamma^h(t_k,x,t,y')-\Gamma^h(t_k,x,t,y) = p_\alpha(t-t_k,y'-x)-p_\alpha(t-t_k,y-x)\\
&-\frac{1}{h}\int_{t_k}^{t_{k+1}\wedge t}\int_{t_k}^{t_{k+1}}[\nabla p_\alpha(t-t_k,y'-w)-\nabla p_\alpha(t-t_k,y-w)]_{w=x+b_h(s,x)(r-t_k)}\cdot b_h(s,x)\,\mathrm{d}s\,\mathrm{d}r\\
&-\frac{1}{h}\sum_{j=k+1}^{\ell-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}}\int\Gamma^h(t_k,x,t_j,z)\\
&\qquad\qquad[\nabla p_\alpha(t-t_j,y'-w)-\nabla p_\alpha(t-t_j,y-w)]_{w=z+b_h(s,z)(r-t_j)}\cdot b_h(s,z)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r\\
&-\frac{\mathbb{1}_{\ell\ge k+1}}{h}\int_{t_\ell}^{t}\int_{t_\ell}^{t_{\ell+1}}\int\Gamma^h(t_k,x,t_\ell,z)\\
&\qquad\qquad[\nabla p_\alpha(t-t_\ell,y'-w)-\nabla p_\alpha(t-t_\ell,y-w)]_{w=z+b_h(s,z)(r-t_\ell)}\cdot b_h(s,z)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r\\
&=:\Delta_1+\Delta_2+\Delta_3+\Delta_4.
\end{aligned} \tag{5.3.9}$$

Resp. $\bar{\Gamma}^h(t_k,x,t,y') - \bar{\Gamma}^h(t_k,x,t,y) =: \bar{\Delta}_1 + \bar{\Delta}_2 + \bar{\Delta}_3 + \bar{\Delta}_4$ for the scheme with $\bar{b}_h$.
Those terms will be treated in similar way than for the time sensitivities, up to the fact that we will use the Hölder regularity in space of p_α (5.2.2) instead of its Hölder regularity in time (5.2.3).

For Δ_1 (which is the same as $\bar{\Delta}_1$), we directly use (5.2.2) with $\theta = 1$ and the diagonal regime:

$$\begin{aligned}\Delta_1 &\lesssim \frac{|y-y'|}{(t-t_k)^{\frac{1}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x)) \\ &\lesssim \frac{|y-y'|^\gamma}{(t-t_k)^{\frac{\gamma}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x)). \end{aligned} \tag{5.3.10}$$

For Δ_2, we first use (5.2.12) to bound $[\nabla p_\alpha(t-t_k, x+b_h(s,x)(r-t_k)-y') - \nabla p_\alpha(t-t_k, x+b_h(s,x)(r-t_k)-y)]$ and get rid of the drift:

$$\begin{aligned}&|\nabla p_\alpha(t-t_k, y'-x-b_h(s,x)(r-t_k)) - \nabla p_\alpha(t-t_k, y-x-b_h(s,x)(r-t_k))| \\ &\quad\lesssim \frac{|y-y'|}{(t-t_k)^{\frac{2}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x)).\end{aligned}$$

We then compute, recalling from the definition (5.1.3) that $|b_h| \lesssim h^{-\frac{d}{\alpha p}-\frac{1}{q}}$ (resp. $|\bar{b}_h| \lesssim h^{\frac{1}{\alpha}-1}$),

$$\begin{aligned}\Delta_2 &\lesssim \frac{|y-y'|}{(t-t_k)^{\frac{2}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x)) \int_{t_k}^{t_{k+1}\wedge t} |b_h(s,x)|\,\mathrm{d}s \\ &\lesssim \frac{|y-y'|}{(t-t_k)^{\frac{1}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x)) \frac{(t_{k+1}\wedge t - t_k)h^{-\frac{d}{\alpha p}-\frac{1}{q}}}{(t-t_k)^{\frac{1}{\alpha}}} \\ &\lesssim \frac{|y-y'|}{(t-t_k)^{\frac{1}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x))(t_{k+1}\wedge t - t_k)^{1-\frac{1}{\alpha}} h^{-\frac{d}{\alpha p}-\frac{1}{q}} \\ \text{resp.}\qquad \bar{\Delta}_2 &\lesssim \frac{|y-y'|}{(t-t_k)^{\frac{1}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x))(t_{k+1}\wedge t - t_k)^{1-\frac{1}{\alpha}} h^{\frac{1}{\alpha}-1}.\end{aligned}$$

In our current diagonal regime, we can write $\frac{|y-y'|}{(t-t_k)^{\frac{1}{\alpha}}} \lesssim \frac{|y-y'|^\gamma}{(t-t_k)^{\frac{\gamma}{\alpha}}}$, which, along with $t_{k+1}\wedge t - t_k \le h$, yields

$$\Delta_2 \lesssim \frac{|y-y'|^\gamma}{(t-t_k)^{\frac{\gamma}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x))h^{\frac{\gamma}{\alpha}}. \tag{5.3.11}$$

$$\text{resp.}\qquad \bar{\Delta}_2 \lesssim \frac{|y-y'|^\gamma}{(t-t_k)^{\frac{\gamma}{\alpha}}}(p_\alpha(t-t_k,y'-x) + p_\alpha(t-t_k,y-x)).$$

For Δ_3, we first use (5.2.12) and (5.1.24) to write, for $\gamma_1 \in (\gamma, 1]$

$$\begin{aligned}\Delta_3 \lesssim \sum_{j=k+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j-t_k, z-x) \left(\frac{|y-y'|^{\gamma_1}}{(t-t_j)^{\frac{\gamma_1}{\alpha}}} \wedge 1\right) \frac{1}{(t-t_j)^{\frac{1}{\alpha}}} \\ \times (p_\alpha(t-t_j, y-z) + p_\alpha(t-t_j, y'-z))\,|b_h(s,z)|\,\mathrm{d}z\,\mathrm{d}s.\end{aligned}$$

Then, we will proceed differently depending on whether, at the current time t_j, the spatial difference $|y-y'|$ we are interested in is in the diagonal regime w.r.t. the corresponding time scale $t-t_j$. To this end, let us split between what we call *meso-scale* diagonal and off-diagonal regimes (respectively $|y-y'| < (t-t_j)^{\frac{1}{\alpha}}$ and $|y-y'| \ge (t-t_j)^{\frac{1}{\alpha}}$). This meso-scale dichotomy did not appear in the proof of the Hölder time-regularity of Γ^h. It does now because of technical reasons: we need to retrieve the loss induced by the introduction of $\gamma_1 \in (\gamma, \alpha]$, which can only be done in the mesoscopic diagonal case.

Let us point out that $|y-y'| \le (t-t_j)^{\frac{1}{\alpha}} \iff j \le \frac{t-|y-y'|^\alpha}{h}$. Set $j_{\max} := \left\lfloor \frac{t-|y-y'|^\alpha}{h} \right\rfloor \wedge (\ell-2)$. We recall that, when t_j is close to t, a *local* off-diagonal regime might appear. With the previous notations it will precisely be the case from $t_{j_{\max}+1}$ to t_ℓ whenever $j_{\max} < \ell - 2$. We can thus write

$$
\begin{aligned}
\Delta_3 \lesssim& \mathbb{1}_{j_{\max}=\ell-2} \frac{|y-y'|^\gamma}{(t-t_{\ell-1})^{\frac{1+\gamma}{\alpha}}} \int_{t_{\ell-1}}^{t_\ell} \int p_\alpha(t_{\ell-1}-t_k, z-x) \\
&\qquad\qquad (p_\alpha(t-t_{\ell-1}, y-z) + p_\alpha(t-t_{\ell-1}, y'-z)) |b_h(s,z)| \,\mathrm{d}z\,\mathrm{d}s \\
&+ \mathbb{1}_{j_{\max}<\ell-2} \sum_{j=j_{\max}+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j-t_k, z-x) \frac{1}{(t-t_j)^{\frac{1}{\alpha}}} \\
&\qquad \times (p_\alpha(t-t_j, y-z) + p_\alpha(t-t_j, y'-z)) |b_h(s,z)| \,\mathrm{d}z\,\mathrm{d}s \\
&+ \sum_{j=k+1}^{j_{\max}} \int_{t_j}^{t_{j+1}} \int p_\alpha(t_j-t_k, z-x) \frac{|y-y'|^{\gamma_1}}{(t-t_j)^{\frac{1+\gamma_1}{\alpha}}} \\
&\qquad \times (p_\alpha(t-t_j, y-z) + p_\alpha(t-t_j, y'-z)) |b_h(s,z)| \,\mathrm{d}z\,\mathrm{d}s \\
=:& \Delta_3^{EDGE} + \Delta_3^{OD} + \Delta_3^{D}.
\end{aligned}
$$

For the first term, we first use $|b_h| \lesssim |b|$ and (5.2.6), then Hölder's inequality for the time integral and last $t - t_{\ell-1} = t - t_\ell + h \ge h$ to obtain:

$$
\begin{aligned}
\Delta_3^{EDGE} \lesssim& \frac{p_\alpha(t-t_k, y-x) + p_\alpha(t-t_k, y'-x)}{(t-t_{\ell-1})^{\frac{1}{\alpha}}} \left[\frac{1}{(t_{\ell-1}-t_k)^{\frac{d}{\alpha p}}} + \frac{1}{(t-t_{\ell-1})^{\frac{d}{\alpha p}}} \right] \int_{t_{\ell-1}}^{t_\ell} \|b(s,\cdot)\|_{L^p} \,\mathrm{d}s \\
&\times \frac{|y-y'|^\gamma}{(t-t_{\ell-1})^{\frac{\gamma}{\alpha}}} \\
&\lesssim (p_\alpha(t-t_k, y-x) + p_\alpha(t-t_k, y'-x)) \frac{1}{(t-t_{\ell-1})^{\frac{1}{\alpha}}} \left[\frac{1}{(t_{\ell-1}-t_k)^{\frac{d}{\alpha p}}} + \frac{1}{(t-t_{\ell-1})^{\frac{d}{\alpha p}}} \right] h^{1-\frac{1}{q}} \\
&\times \frac{|y-y'|^\gamma}{(t-t_{\ell-1})^{\frac{\gamma}{\alpha}}} \\
&\lesssim (p_\alpha(t-t_k, y-x) + p_\alpha(t-t_k, y'-x)) |y-y'|^\gamma.
\end{aligned}
$$

Next, note that in the integrals appearing in Δ_3^{OD} and Δ_3^D, we can use $t - s \le t - t_j$ for $s \in [t_j, t_{j+1}]$. Together with $|\bar{b}_h| \le |b|$, this yields

$$
\begin{aligned}
\Delta_3^{OD} + \Delta_3^D \lesssim& \mathbb{1}_{j_{\max}<\ell-2} \int_{t_{j_{\max}+1}}^{t_\ell} \int p_\alpha(s-t_k, z-x) \frac{1}{(t-s)^{\frac{1}{\alpha}}} \\
&\qquad \times (p_\alpha(t-s, y-z) + p_\alpha(t-s, y'-z)) |b(s,z)| \,\mathrm{d}z\,\mathrm{d}s \\
&+ \int_{t_{k+1}}^{t_{j_{\max}}} \int p_\alpha(s-t_k, z-x) \frac{|y-y'|^{\gamma_1}}{(t-s)^{\frac{1+\gamma_1}{\alpha}}} \\
&\qquad \times (p_\alpha(t-s, y-z) + p_\alpha(t-s, y'-z)) |b(s,z)| \,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}
$$

For Δ_3^{OD}, we simply use (5.2.8), with $u = t_{j_{\max}+1}, v = t_\ell, \beta_1 = 1/\alpha, \beta_2 = 0$.
For Δ_3^D,

- if $\frac{1}{2}(t_{j_{\max}} - t_{k+1}) > t - t_{j_{\max}}$, we use (5.2.7) with $u = t_{k+1}, v = t_{j_{\max}}, \beta_1 = (\gamma_1+1)/\alpha, \beta_2 = 0$.
- if $\frac{1}{2}(t_{j_{\max}} - t_{k+1}) \le t - t_{j_{\max}}$, we use the bound $(t-s)^{-\frac{\gamma_1}{\alpha}} \le (t-t_{j_{\max}})^{-\frac{\gamma_1}{\alpha}}$ for $s \le t_{j_{\max}}$ then apply (5.2.8) with $u = t_{k+1}, v = t_{j_{\max}}, \beta_1 = 1/\alpha, \beta_2 = 0$.

This yields

$$\begin{aligned}\Delta_3^{OD}+\Delta_3^{D} \lesssim & (p_\alpha(t-t_k,y-x)+p_\alpha(t-t_k,y'-x))\Big[(t-t_{j_{\max}+1})^{\frac{\gamma}{\alpha}}\mathbb{1}_{j_{\max}<\ell-2}\\ &+|y-y'|^{\gamma_1}\big(\{(t_{j_{\max}}-t_{k+1})^{\frac{\gamma-\gamma_1}{\alpha}}+(t-t_{j_{\max}})^{\frac{\gamma-\gamma_1}{\alpha}}\}\mathbb{1}_{\frac{1}{2}(t_{j_{\max}}-t_{k+1})>t-t_{j_{\max}}}\\ &+(t-t_{j_{\max}})^{-\frac{\gamma_1}{\alpha}}(t_{j_{\max}}-t_{k+1})^{\frac{\gamma}{\alpha}}\mathbb{1}_{\frac{1}{2}(t_{j_{\max}}-t_{k+1})\le t-t_{j_{\max}}}\big)\Big]\\ \lesssim & (p_\alpha(t-t_k,y-x)+p_\alpha(t-t_k,y'-x))\Big[(t-t_{j_{\max}+1})^{\frac{\gamma}{\alpha}}\mathbb{1}_{j_{\max}<\ell-2}\\ &+|y-y'|^{\gamma_1}(t-t_{j_{\max}})^{\frac{\gamma-\gamma_1}{\alpha}}\Big].\end{aligned}$$

Since $(t-t_{j_{\max}+1})\le|y-y'|^\alpha$ if $j_{\max}<\ell-2$ and $(t-t_{j_{\max}})\ge|y-y'|^\alpha$, we obtain

$$\Delta_3^{OD}+\Delta_3^{D}\lesssim(p_\alpha(t-t_k,y-x)+p_\alpha(t-t_k,y'-x))\,|y-y'|^\gamma. \tag{5.3.12}$$

Finally, for Δ_4, we suppose that $\ell\ge k+1$ since otherwise this term vanishes. Using again (5.2.12), we get

$$\begin{aligned}|\nabla_{y'}p_\alpha(t-t_\ell,y'-w)-\nabla_y p_\alpha(t-t_\ell,y-w)|_{w=z+b_h(s,z)(r-t_\ell)}&\\ \lesssim\frac{|y-y'|^\gamma}{(t-t_\ell)^{\frac{1+\gamma}{\alpha}}}(p_\alpha(t-t_\ell,y'-z)+p_\alpha(t-t_\ell,y-z)),&\end{aligned}$$

yielding, along with (5.1.24) and $|\bar b_h|\le|b|$,

$$\Delta_4\lesssim\frac{|y-y'|^\gamma}{(t-t_\ell)^{\frac{1+\gamma}{\alpha}}}\int_{t_\ell}^t\int p_\alpha(t_\ell-t_k,z-x)(p_\alpha(t-t_\ell,y'-z)+p_\alpha(t-t_\ell,y-z))|b(s,z)|\,\mathrm{d}z\,\mathrm{d}s.$$

Let

$$d_4:\{y,y'\}\ni\mathfrak{y}\mapsto\int_{t_\ell}^t\int p_\alpha(t_\ell-t_k,z-x)p_\alpha(t-t_\ell,\mathfrak{y}-z)|b(s,z)|\,\mathrm{d}z\,\mathrm{d}s,$$

so that $|\Delta_4|\lesssim\frac{|y-y'|^\gamma}{(t-t_\ell)^{\frac{1+\gamma}{\alpha}}}(d_4(y)+d_4(y'))$. Let us then bound d_4 using the convolution inequality (5.2.6). For $\mathfrak{y}\in\{y,y'\}$,

$$\begin{aligned}d_4(\mathfrak{y})&\lesssim\int_{t_\ell}^t\int p_\alpha(t_\ell-t_k,z-x)p_\alpha(t-t_\ell,\mathfrak{y}-z)|b(s,z)|\,\mathrm{d}z\,\mathrm{d}s\\ &\lesssim\left[\frac{1}{(t_\ell-t_k)^{\frac{d}{\alpha p}}}+\frac{1}{(t-t_\ell)^{\frac{d}{\alpha p}}}\right]p_\alpha(t-t_k,\mathfrak{y}-x)\int_{t_\ell}^t\|b(s,\cdot)\|_{L^p}\,\mathrm{d}s\\ &\lesssim p_\alpha(t-t_k,\mathfrak{y}-x)\|b\|_{L^q-L^p}\|\mathbb{1}_{(t_\ell,t)}\|_{L^{q'}}\left[\frac{1}{(t_\ell-t_k)^{\frac{d}{\alpha p}}}+\frac{1}{(t-t_\ell)^{\frac{d}{\alpha p}}}\right]\\ &\lesssim p_\alpha(t-t_k,\mathfrak{y}-x)(t-t_\ell)^{1-\frac{1}{q}-\frac{d}{\alpha p}},\end{aligned}$$

where, for the last inequality, we used the fact that $t_\ell-t_k\ge t-t_\ell$. Plugging this into $|\Delta_4|$ yields

$$\begin{aligned}|\Delta_4|&\lesssim(p_\alpha(t-t_k,y-x)+p_\alpha(t-t_k,y'-x))|y-y'|^\gamma(t-t_\ell)^{1-\frac{1}{q}-\frac{d}{\alpha p}-\frac{1}{\alpha}-\frac{\gamma}{\alpha}}\\ &\lesssim(p_\alpha(t-t_k,y-x)+p_\alpha(t-t_k,y'-x))|y-y'|^\gamma.\end{aligned}\tag{5.3.13}$$

The estimates for Δ_3 and Δ_4 remain valid for $\bar\Delta_3$ and $\bar\Delta_4$ since we only used $|b_h|\lesssim|b|$. Plugging the estimations (5.3.10)-(5.3.13) into (5.3.9) concludes the proof of (5.1.25) and of Proposition 5.1.

5.4 Proof of existence of a unique weak solution and heat kernel estimates for the SDE (5.1.1) (Theorem 5.1)

5.4.1 Uniqueness of solutions to the Duhamel formulation (5.1.19) satisfying the estimation (5.1.18)

Assume that Γ_1 and Γ_2 both satisfy the estimation (5.1.18) and the Duhamel formula. Then

$$\mu_x(t) := \sup_{y\in\mathbb{R}^d} \frac{|\Gamma_1(0,x,t,y)-\Gamma_2(0,x,t,y)|}{p_\alpha(t,y-x)}.$$

is bounded on $(0,T]$ and we can write for all $(t,y)\in(0,T]\times\mathbb{R}^d$:

$$\Gamma_1(0,x,t,y)-\Gamma_2(0,x,t,y) = \int_0^t \int b(r,z)\cdot\nabla_y p_\alpha(t-r,y-z)\,[\Gamma_2(0,x,r,z)-\Gamma_1(0,x,r,z)]\,\mathrm{d}z\,\mathrm{d}r.$$

We deduce that for $(t,y)\in(0,T]\times\mathbb{R}^d$,

$$\left|\frac{\Gamma_1(0,x,t,y)-\Gamma_2(0,x,t,y)}{p_\alpha(t,y-x)}\right| \le \frac{1}{p_\alpha(t,y-x)}\int_0^t\int |b(r,z)||\nabla_y p_\alpha(t-r,y-z)|p_\alpha(r,z-x)\mu_x(r)\,\mathrm{d}z\,\mathrm{d}r.$$

Using (5.2.1) and (5.2.6), we get:

$$\left|\frac{\Gamma_1(0,x,t,y)-\Gamma_2(0,x,t,y)}{p_\alpha(t,y-x)}\right| \le \int_0^t \frac{\|b(r,\cdot)\|_{L^p}}{(t-r)^{\frac{1}{\alpha}}}\left[\frac{1}{r^{\frac{d}{\alpha p}}}+\frac{1}{(t-r)^{\frac{d}{\alpha p}}}\right]\mu_x(r)\,\mathrm{d}r.$$

Taking the supremum over $y\in\mathbb{R}^d$ on the l.h.s. and applying Hölder's inequality in time, we get like in the last step of the proof of Theorem 5.2

$$\forall t\in(0,T],\ \mu_x(t)^{q'} \lesssim \int_0^t \frac{\mu_x(r)^{q'}}{(t-r)^{\frac{q'}{\alpha}}}\left[\frac{1}{r^{\frac{dq'}{\alpha p}}}+\frac{1}{(t-r)^{\frac{dq'}{\alpha p}}}\right]\mathrm{d}r.$$

Since $\frac{q'}{\alpha}+\frac{dq'}{\alpha p}<1$, Lemma 2.2 and Example 2.4 [Zha10] ensure that $\forall t\in(0,T]$, $\mu_x(t)=0$, from which we immediately deduce $\Gamma_1=\Gamma_2$.

5.4.2 Tightness of the laws P^h of $((X_s^h)_{s\in[0,T]})_h$ and $\bar{P}^h$ of $((\bar{X}_s^h)_{s\in[0,T]})_h$

Let $\mathfrak{B}^\eta := \mathbb{E}_{0,x}\left[\int_0^T \left|b_h(U_{\lfloor\frac{s}{h}\rfloor}, X^h_{\tau_s^h})\right|^\eta \mathrm{d}s\right]$, where $\eta>1$ is chosen sufficiently close to 1 in order that, under (5.1.2), $p/\eta>1$, $q/\eta>1$ and $\eta(d/p+\alpha/q)<\alpha$. Using $|b_h|\le Bh^{-\frac{d}{\alpha p}-\frac{1}{q}}$ on the first time step then $|b_h|\le|b|$, (5.1.24), Hölder's inequality and (5.2.4) with $\delta=0$, we obtain

$$\begin{aligned}
\mathfrak{B}^\eta &\lesssim h^{1-\frac{d}{\alpha p}-\frac{1}{q}} + \sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}}\frac{1}{h}\int_{t_k}^{t_{k+1}}\int |b(r,y)|^\eta\,\Gamma^h(0,x,t_k,y)\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\\
&\lesssim h^{1-\frac{d}{\alpha p}-\frac{1}{q}} + \sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}}\int |b(r,y)|^\eta\,p_\alpha(t_k,y-x)\,\mathrm{d}y\,\mathrm{d}r\\
&\lesssim h^{1-\frac{d}{\alpha p}-\frac{1}{q}} + \sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}}\|b(r,\cdot)^\eta\|_{L^{\frac{p}{\eta}}}\|p_\alpha(t_k,\cdot-x)\|_{L^{(\frac{p}{\eta})'}}\,\mathrm{d}r\\
&\lesssim h^{1-\frac{d}{\alpha p}-\frac{1}{q}} + \sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}} t_k^{-\frac{d\eta}{\alpha p}}\|b(r,\cdot)\|^\eta_{L^p}\,\mathrm{d}r.
\end{aligned}$$

We then write

$$\int_{t_k}^{t_{k+1}} t_k^{-\frac{d\eta}{\alpha p}} \|b(r,\cdot)\|_{L^p}^\eta \,\mathrm{d}r \le \int_{t_k}^{t_{k+1}} r^{-\frac{d\eta}{\alpha p}} \|b(r,\cdot)\|_{L^p}^\eta \,\mathrm{d}r,$$

and use a Hölder inequality to obtain from the condition $\eta(d/p+\alpha/q) < \alpha$ which ensures that $-(\frac{q}{\eta})' \frac{d\eta}{\alpha p} > -1$,

$$\begin{aligned}\mathfrak{B}^\eta &\lesssim \left(\int_{t_1}^{t_n} r^{-(\frac{q}{\eta})'\frac{d\eta}{\alpha p}} \,\mathrm{d}r\right)^{\frac{1}{(\frac{q}{\eta})'}} \times \|r \mapsto \|b(r,\cdot)\|_{L^p}^\eta\|_{L^{\frac{q}{\eta}}} \\ &\lesssim h^{1-\frac{d}{\alpha p}-\frac{1}{q}} + T^{1-\eta(\frac{d}{\alpha p}+\frac{1}{q})}\|b\|_{L^q-L^p}^\eta. \end{aligned} \tag{5.4.1}$$

In the same way,

$$\bar{\mathfrak{B}}^\eta := \mathbb{E}_{0,x}\left[\int_0^T \left|\bar{b}_h(U_{\lfloor \frac{s}{h}\rfloor}, \bar{X}^h_{\tau^h_s})\right|^\eta \mathrm{d}s\right] = \mathbb{E}_{0,x}\left[\int_h^T \left|\bar{b}_h(U_{\lfloor \frac{s}{h}\rfloor}, \bar{X}^h_{\tau^h_s})\right|^\eta \mathrm{d}s\right] \lesssim T^{1-\eta(\frac{d}{\alpha p}+\frac{1}{q})}\|b\|_{L^q-L^p}^\eta.$$

By Hölder's inequality, we have

$$\forall 0 \le u \le t \le T, \left|\int_u^t b_h(U_{\lfloor \frac{s}{h}\rfloor}, X^h_{\tau^h_s})\,\mathrm{d}s\right| \lesssim (t-u)^{\frac{\eta-1}{\eta}} \left(\int_0^T \left|b_h(U_{\lfloor \frac{s}{h}\rfloor}, X^h_{\tau^h_s})\right|^\eta \mathrm{d}s\right)^{\frac{1}{\eta}}. \tag{5.4.2}$$

and the same estimation holds with (b_h, X^h) replaced by $(\bar{b}_h, \bar{X}^h)$. Since by (5.2.4) applied with $\delta = 1$, $\mathbb{E}[|Z_t - Z_u|] \lesssim (t-u)^{\frac{1}{\alpha}}$, setting $\zeta = \left(1 - \frac{1}{\eta}\right) \wedge \frac{1}{\alpha} > 0$, we deduce that

$$\forall 0 \le u \le t \le T,\ \mathbb{E}[|X^h_t - X^h_u|] + \mathbb{E}[|\bar{X}^h_t - \bar{X}^h_u|] \lesssim (t-u)^\zeta, \tag{5.4.3}$$

where the constant associated with the $\lesssim$ symbol is independent of h. This ensures the tightness of the laws P^h of X^h and $\bar{P}^h$ of $(\bar{X}^h)$ on the space $\mathcal{D}([0,T],\mathbb{R}^d)$ of càdlàg functions endowed with the Skorokhod topology (see Proposition 34.9 from [Bas11] for example). Let $\left(\xi_s\right)_{s\in[0,T]}$ denote the canonical process on this space.
We may then extract a subsequence, still denoted by (P^h) (resp. $(\bar{P}^h)$), such that P^h (resp. $(\bar{P}^h)$) weakly converges to some limit probability P on $\mathcal{D}([0,T],\mathbb{R}^d)$ as $h \to 0$. For $u,t \in [0,T]$ outside the at most countable set $\{s \in (0,T] : P(|\xi_s - \xi_{s-}| > 0) > 0\}$, the law of (X^h_u, X^h_t) (resp. $(\bar{X}^h_u, \bar{X}^h_t)$) converges to $P \circ (\xi_u, \xi_t)^{-1}$ so that (5.4.3) combined with the right-continuity of sample-paths ensures that $\sup_{0\le u<t\le T}(t-u)^{-\zeta}\int_{\mathcal{D}([0,T],\mathbb{R}^d)} |\xi_t - \xi_u| P(\mathrm{d}\xi) < \infty$. As a consequence

$$\{s \in (0,T] : P(|\xi_s - \xi_{s-}| > 0) > 0\} = \emptyset \tag{5.4.4}$$

and for each $t \in (0,T]$, the distribution $\Gamma^h(0,x,t,y)\,\mathrm{d}y$ of X^h_t (resp. $\bar{\Gamma}^h(0,x,t,y)\,\mathrm{d}y$ of $\bar{X}^h_t$) converges weakly to $P_t = P \circ \xi_t^{-1}$. By (5.1.24) and (5.1.25), the Ascoli-Arzelà theorem ensures that we can extract a further subsequence such that $y \mapsto \Gamma^h(0,x,t,y)$ (resp. $y \mapsto \bar{\Gamma}^h(0,x,t,y)$) converges uniformly on the compact subsets of $\mathbb{R}^d$ to some limit $y \mapsto \Gamma(0,x,t,y)$ so that $P_t(\mathrm{d}y) = \Gamma(0,x,t,y)\,\mathrm{d}y$. Taking the limit $h \to 0$ into (5.1.24) and (5.1.25) ensures that Γ satisfies (5.1.18) and (5.1.21).

We are next going to prove that the limit probability measure P solves the following martingale problem.

Definition 5.1 (Martingale Problem). *A probability measure P on the space $\mathcal{D}([0,T],\mathbb{R}^d)$ of càdlàg functions with time-marginals $(P_t)_{t\in[0,T]}$, solves the martingale problem related to $b\cdot\nabla + \mathcal{L}^\alpha$ and $x \in \mathbb{R}^d$ if :*

(i) $P_0 = \delta_x$,

(ii) for a.a. $t \in (0,T]$, $P_t(\mathrm{d}y) = \rho(t,y)\,\mathrm{d}y$ *for some* $\rho \in L^{q'}((0,T], L^{p'}(\mathbb{R}^d))$,

(iii) for all $\mathcal{C}^{1,2}$ *function* f *on* $[0,T]\times\mathbb{R}^d$ *bounded together with its derivatives, the process*

$$\left\{M^f_t = f(t,\xi_t) - f(0,\xi_0) - \int_0^t \Big((\partial_s + \mathcal{L}^\alpha)f(s,\xi_s) + b(s,\xi_s)\cdot\nabla f(s,\xi_s)\Big)\,\mathrm{d}s\right\}_{0\le t\le T}, \tag{M}$$

is a P *martingale.*

Let us point out that, in the current singular drift setting, condition (ii) which guarantees that

$$\int_{\mathcal{D}([0,T],\mathbb{R}^d)} \int_0^T |b(s,\xi_s)| \,\mathrm{d}s P(\mathrm{d}\xi) < \infty$$

is somehow the minimal one required for all the terms in (M) to be well defined.

Before checking that the limit probability measure P solves the martingale problem, let us prove that this implies that Γ solves (5.1.19), which concludes the proof of Theorem 5.1 (in fact, for this purpose, it would be enough to check that the limit probability measure associated with either the schemes X^h or the schemes $\bar{X}^h$ solves the martingale problem). Let $t \in (0,T]$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^4$ function with compact support. Choosing $f(s,z) = \mathbb{1}_{[0,t)}(s) p_\alpha(t-s,\cdot) \star \phi(z) + \mathbb{1}_{[t,T]}(s)(\phi(z) - (s-t)\mathcal{L}^\alpha \phi(z))$ which, according to Lemma 5.2, satisfies $(\partial_s + \mathcal{L}^\alpha) f(s,z) = 0$ for $(s,z) \in [0,t] \times \mathbb{R}^d$ and writing the centering of M_t^f (introduced in Definition 5.1) under P, we obtain that

$$\int_{\mathbb{R}^d} \phi(y)\Gamma(0,x,t,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \phi(y) p_\alpha(t, x-y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} \Gamma(0,x,s,z) b(s,z)\cdot \nabla_z f(s,z)\,\mathrm{d}z\,\mathrm{d}s.$$

Using (5.1.18) and (5.2.1) to justify the use of Fubini's theorem and the fact that for $s \in (0,T]$, $p_\alpha(s,\cdot)$ is an even function, we deduce that

$$\int_{\mathbb{R}^d} \phi(y)\Gamma(0,x,t,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \phi(y) \left(p_\alpha(t,y-x) - \int_0^t \Gamma(0,x,s,z) b(s,z) \cdot \nabla_y p_\alpha(s, y-z)\,\mathrm{d}s \right) \mathrm{d}y$$

Since ϕ is arbitrary, we conclude that $(0,T] \times \mathbb{R}^d \ni (t,y) \mapsto \Gamma(0,x,t,y)$ satisfies (5.1.19).

5.4.3 Any limit point solves the martingale problem

Let us now prove that the limit point P solves the martingale problem associated with (5.1.1) and introduced in Definition 5.1. Since for each h, $X_0^h = x = \bar{X}_0^h$, one has $P_0 = \delta_x$. Moreover, for $t \in (0,T]$, $P_t(dy) = \Gamma(0,x,t,y)dy$ with Γ satisfying (5.1.18). By (5.2.4) applied with $\delta = 0$, $\|\Gamma(0,x,t,\cdot)\|_{L^{p'}} \le C t^{-\frac{d}{\alpha p}}$ where the right-hand side belongs to $L^{q'}([0,T])$ since $q' \frac{d}{\alpha p} < 1$ by (5.1.2). As a consequence, $\Gamma(0,x,\cdot,\cdot) \in L^{q'}((0,T], L^{p'}(\mathbb{R}^d))$. Therefore properties (i) and (ii) in Definition 5.1 hold.

Let $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{C}^{1,2}$ and bounded together with its derivatives, $\psi : (\mathbb{R}^d)^p \to \mathbb{R}$ be continuous and bounded, $0 \le s_1 \le ... \le s_p < u \le t \le T$ with $u > 0$ and $F : \mathcal{D}([0,T],\mathbb{R}^d) \to \mathbb{R}$ be defined by

$$F(\xi) := \left(f(t,\xi_t) - f(u,\xi_u) - \int_u^t \left[(\partial_s + \mathcal{L}^\alpha) f(s,\xi_s) + b(s,\xi_s)\cdot \nabla f(s,\xi_s) \right] \mathrm{d}s \right) \psi(\xi_{s_1}, ..., \xi_{s_p}). \tag{5.4.5}$$

In order to prove that P satisfies (iii) in Definition 5.1, we will show that $\int_{\mathcal{D}([0,T],\mathbb{R}^d)} F(\xi) P(\mathrm{d}\xi) = 0$.

Proof of $\lim_{h\to 0} \mathbb{E}[F(X^h)] = 0 = \lim_{h \to 0} \mathbb{E}[F(\bar{X}^h)]$**.**

Using Itô's formula, we can write

$$f(t,X_t^h) - f(u,X_u^h) = M_t^h - M_u^h + \int_u^t \nabla f(s,X_s^h)\cdot b_h(U_{\lfloor s/h \rfloor}, X^h_{\tau_s^h})\,\mathrm{d}s + \int_u^t (\partial_s + \mathcal{L}^\alpha) f(s,X_s^h)\,\mathrm{d}s,$$

with $M^h_s = \int_0^s \int_{\mathbb{R}^d\setminus\{0\}} \Big(f(r, X^h_{r^-} + x) - f(r, X^h_{r^-})\Big)\tilde{N}(\,\mathrm{d}r, \,\mathrm{d}x)$ where $\tilde{N}$ is the compensated Poisson measure associated with Z. Since M^h is a martingale, taking expectations, we get:

$$\begin{aligned}
\mathbb{E}[F(X^h)] &= \mathbb{E}\left[\left(\int_u^t \Big(b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s}) - b(s, X^h_s)\Big)\cdot\nabla f(s, X^h_s)\,\mathrm{d}s\right)\psi(X^h_{s_1}, ..., X^h_{s_p})\right] \\
&= \mathbb{E}\left[\left(\int_u^t b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s})\cdot\Big(\nabla f(s, X^h_s) - \nabla f(s, X^h_{\tau^h_s})\Big)\,\mathrm{d}s\right)\psi(X^h_{s_1}, ..., X^h_{s_p})\right] \\
&\quad + \mathbb{E}\left[\left(\int_u^t \Big(b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s}) - b_h(s, X^h_{\tau^h_s})\Big)\cdot\nabla f(s, X^h_{\tau^h_s})\,\mathrm{d}s\right)\psi(X^h_{s_1}, ..., X^h_{s_p})\right] \\
&\quad + \mathbb{E}\left[\left(\int_u^t \Big(b_h(s, X^h_{\tau^h_s}) - b(s, X^h_{\tau^h_s})\Big)\cdot\nabla f(s, X^h_{\tau^h_s})\,\mathrm{d}s\right)\psi(X^h_{s_1}, ..., X^h_{s_p})\right] \\
&\quad + \mathbb{E}\left[\left(\int_u^t \Big(b(s, X^h_{\tau^h_s})\cdot\nabla f(s, X^h_{\tau^h_s}) - b(s, X^h_s)\nabla f(s, X^h_s)\Big)\,\mathrm{d}s\right)\psi(X^h_{s_1}, ..., X^h_{s_p})\right] \\
&=: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4.
\end{aligned} \tag{5.4.6}$$

In the same way, $\mathbb{E}[F(\bar{X}^h)] = \bar{\Delta}_1 + \bar{\Delta}_2 + \bar{\Delta}_3 + \bar{\Delta}_4$ where $\bar{\Delta}_i$ is defined like Δ_i with (X^h, b_h) replaced by $(\bar{X}^h, \bar{b}_h)$ for $i \in \{1, \cdots, 4\}$.

For Δ_1, we first write, using $|b_h| \lesssim h^{-\frac{d}{\alpha p}-\frac{1}{q}}$ and conditioning w.r.t. $\mathcal{F}_{\tau^h_s} = \sigma\big(X^h_u,\ 0 \le u \le \tau^h_s\big)$,

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_{\tau^h_s}}|\nabla f(s, X^h_s) - \nabla f(s, X^h_{\tau^h_s})| &\le \|\nabla^2 f\|_{L^\infty}\mathbb{E}_{\mathcal{F}_{\tau^h_s}}|X^h_s - X^h_{\tau_s}| \\
&\le \|\nabla^2 f\|_{L^\infty}\mathbb{E}_{\mathcal{F}_{\tau^h_s}}\left[\int_{\tau^h_s}^s |b_h(U_{\lfloor r/h\rfloor}, X^h_{\tau^h_r})|\,\mathrm{d}r + |Z_s - Z_{\tau^h_s}|\right] \\
&\lesssim \|\nabla^2 f\|_{L^\infty}\left[\int_{\tau^h_s}^s h^{-\frac{d}{\alpha p}-\frac{1}{q}}dr + (s-\tau^h_s)^{\frac{1}{\alpha}}\right] \\
&\lesssim \|\nabla^2 f\|_{L^\infty} h^{\frac{1}{\alpha}}.
\end{aligned}$$

Using this bound along with $|b_h| \le |b|$ and (5.4.1), we can compute

$$|\Delta_1| \lesssim \|\psi\|_{L^\infty}\|\nabla^2 f\|_{L^\infty}h^{\frac{1}{\alpha}}\,\mathbb{E}\left[\int_0^T |b(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s})|\,\mathrm{d}s\right] \lesssim \|\psi\|_{L^\infty}\|\nabla^2 f\|_{L^\infty}h^{\frac{1}{\alpha}}.$$

The same bound holds for $|\bar{\Delta}_1|$ since the larger cutoff $|\bar{b}_h| \lesssim h^{\frac{1}{\alpha}-1}$ does not deteriorate the estimation of $\mathbb{E}_{\bar{\mathcal{F}}_{\tau^h_s}}|\nabla f(s, \bar{X}^h_s) - \nabla f(s, \bar{X}^h_{\tau^h_s})|$ where $\bar{\mathcal{F}}_{\tau^h_s} = \sigma\big(\bar{X}^h_u,\ 0 \le u \le \tau^h_s\big)$.

For Δ_2, supposing that h is small enough to ensure that $\tau^h_u < \tau^h_t$, we split the time integral into three terms: a main term over $(\tau^h_u + h, \tau^h_t)$ which matches the time grid, and two terms around the edges, over $(u, \tau^h_u + h)$ and (τ^h_t, t) respectively. For the main term, we will use the following cancellation:

$$\mathbb{E}\left[\int_{\tau^h_u+h}^{\tau^h_t}\Big(b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s}) - b_h(s, X^h_{\tau^h_s})\Big)\cdot\nabla f(s, X^h_{\tau^h_s})\,\mathrm{d}s \,\middle|\, \mathcal{F}_{\tau^h_u+h}\right] = 0.$$

For the other two terms, we use that $|b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s}) - b_h(s, X^h_{\tau^h_s})| \le |b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s})| + |b_h(s, X^h_{\tau^h_s})| \lesssim h^{-\frac{d}{\alpha p}-\frac{1}{q}}$ and the inequalities $\tau^h_u + h - u \le h$ and $t - \tau^h_t \le h$ to write

$$\begin{aligned}
|\Delta_2| &\lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\mathbb{E}\left[\int_{[u,\tau^h_u+h]\cup[\tau^h_t,t]}\Big(|b_h(U_{\lfloor s/h\rfloor}, X^h_{\tau^h_s})| + |b_h(s, X^h_{\tau^h_s})|\Big)\,\mathrm{d}s\right] \\
&\lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}h^{1-\frac{d}{\alpha p}-\frac{1}{q}}.
\end{aligned}$$

In the same way, $|\bar{\Delta}_2| \lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}h^{\frac{1}{\alpha}}$.

When $p = q = \infty$, Δ_3 vanishes. Otherwise, applying (5.2.15) with $\lambda = \eta$ where $\eta > 1$ is such that (5.4.1) holds, we get

$$|\Delta_3| \leq \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} h^{(\frac{d}{\alpha p}+\frac{1}{q})(\eta-1)} \mathbb{E}\left[\int_u^t |b(s, X^h_{\tau^h_s})|^\eta \, \mathrm{d}s\right] \lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} h^{(\frac{d}{\alpha p}+\frac{1}{q})(\eta-1)}.$$

Since $|b - \bar{b}^h| \leq |b|^\eta B^{1-\eta} h^{(1-\frac{1}{\alpha})(\eta-1)}$, we obtain in the same way that $|\bar{\Delta}_3| \lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} h^{(1-\frac{1}{\alpha})(\eta-1)}$. For Δ_4, we have

$$\begin{aligned}
&\left|\mathbb{E}_{\mathcal{F}_{\tau^h_u - h}}\left[\int_u^t \Big(b(s, X^h_{\tau^h_s}) \cdot \nabla f(s, X^h_{\tau^h_s}) - b(s, X^h_s) \nabla f(s, X^h_s)\Big) \, \mathrm{d}s\right]\right| \\
&\leq \int_u^t \int |b(s,z) \cdot \nabla f(z)| \left|\Gamma^h(\tau^h_u - h, X^h_{\tau^h_u - h}, \tau^h_s, z) - \Gamma^h(\tau^h_u - h, X^h_{\tau^h_u - h}, s, z)\right| \mathrm{d}z \, \mathrm{d}s.
\end{aligned}$$

Assuming w.l.o.g. that h is small enough to have $\tau^h_u - h \geq s_p$, we deduce that:

$$|\Delta_4| \lesssim \|\psi\|_{L^\infty} \int_u^t \int \int |b(s,z) \cdot \nabla f(z)| \left|\Gamma^h(\tau^h_u - h, y, \tau^h_s, z) - \Gamma^h(\tau^h_u - h, y, s, z)\right| \Gamma^h(0, x, \tau^h_u - h, y) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}s.$$

Then, we use the Hölder regularity (5.1.26) of Γ^h in the forward time variable:

$$|\Delta_4| \lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} \int_u^t \int \int |b(s,z)| \frac{(s - \tau^h_s)^{\frac{\gamma}{\alpha}}}{(\tau^h_s - \tau^h_u + h)^{\frac{\gamma}{\alpha}}} p_\alpha(s - \tau^h_u + h, z - y) \Gamma^h(0, x, \tau^h_u - h, y) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}s.$$

Since $s - \tau^h_s \leq h$ and $\tau^h_s - \tau^h_u + h > s - u$, we get, using (5.1.24), Hölder's inequality and (5.2.4)

$$\begin{aligned}
|\Delta_4| &\lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} h^{\frac{\gamma}{\alpha}} \int_u^t \int \int \frac{|b(s,z)|}{(s-u)^{\frac{\gamma}{\alpha}}} p_\alpha(s - \tau^h_u + h, z - y) p_\alpha(\tau^h_u - h, y - x) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} h^{\frac{\gamma}{\alpha}} \int_u^t \int \frac{|b(s,z)|}{(s-u)^{\frac{\gamma}{\alpha}}} p_\alpha(s, z - x) \, \mathrm{d}z \, \mathrm{d}s \\
&\lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} h^{\frac{\gamma}{\alpha}} \int_u^t \frac{\|b(s,\cdot)\|_{L^p}}{(s-u)^{\frac{\gamma}{\alpha}} s^{\frac{d}{\alpha p}}} \, \mathrm{d}s.
\end{aligned}$$

Finally, using Hölder's inequality in time and $\frac{1}{q'} - \frac{\gamma}{\alpha} - \frac{d}{\alpha p} = \frac{1}{\alpha}$, we obtain

$$|\Delta_4| \lesssim \|\psi\|_{L^\infty} \|\nabla f\|_{L^\infty} \|b\|_{L^q - L^p} h^{\frac{\gamma}{\alpha}} (t-u)^{\frac{1}{\alpha}}.$$

The same estimation holds for $|\bar{\Delta}_4|$. Putting together the previous estimates on $(\Delta_i)_{i \in \{1,\cdots,4\}}$ in (5.4.6) we obtain $\lim_{h\to 0} \mathbb{E}\left[F(X^h)\right] = 0$. In the same way, $\lim_{h\to 0} \mathbb{E}\left[F(\bar{X}^h)\right] = 0$

P solves the martingale problem.

In this paragraph, we only consider the case when P is the limit of the laws of the schemes X^h since the argument is exactly the same when P is the limit of the laws of the schemes $\bar{X}^h$. The lack of continuity of the functional F on $\mathcal{D}([0,T], \mathbb{R}^d)$ prevents from deducing immediately that $\int_{\mathcal{D}([0,T],\mathbb{R}^d)} F(\xi) P(\mathrm{d}\xi) = 0$. Let us first suppose that $p < \infty$ and set $\tilde{q} = q \mathbb{1}_{q<\infty} + \frac{\alpha p + 1}{(\alpha-1)p - d} \mathbb{1}_{q<\infty}$. We have $\frac{d}{p} + \frac{\alpha}{\tilde{q}} < \alpha - 1$. We introduce for $\varepsilon \in (0,1]$, a smooth and bounded function b_ε such that $\lim_{\varepsilon\to 0} \|b_\varepsilon - b\|_{L^{\tilde{q}} - L^p} = 0$. The functional F_ε defined like F in (5.4.5), but with b_ε replacing b is bounded. According to (5.4.4), for fixed $\varepsilon \in (0,1]$, P gives full weight to continuity points of F_ε and since $\lim_{h\to 0} \mathbb{E}[F(X^h)] = 0$, we have

$$\int_{\mathcal{D}([0,T],\mathbb{R}^d)} F_\varepsilon(\xi) P(\mathrm{d}\xi) = \lim_{h\to 0} \mathbb{E}[F_\varepsilon(X^h)] = \lim_{h\to 0} \mathbb{E}[F_\varepsilon(X^h) - F(X^h)].$$

We deduce that

$$\left|\int_{\mathcal{D}([0,T],\mathbb{R}^d)} F(\xi)P(\mathrm{d}\xi)\right| \le \limsup_{\varepsilon\to 0}\int_{\mathcal{D}([0,T],\mathbb{R}^d)} |F(\xi)-F_\varepsilon(\xi)|P(\mathrm{d}\xi) + \limsup_{\varepsilon\to 0}\limsup_{h\to 0}\mathbb{E}[|F_\varepsilon(X^h)-F(X^h)|].$$

One has, using (5.1.24), then Hölder's inequality in space together with (5.2.4) applied with $(\ell,\delta)=(p,0)$ and last Hölder's inequality in time,

$$\begin{aligned}
\mathbb{E}[|F_\varepsilon(X^h)-F(X^h)|] &\le \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\int_u^t \mathbb{E}[|b_\varepsilon(s,X_s^h)-b(s,X_s^h)|]\,\mathrm{d}s\\
&\lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\int_u^t\int |b_\varepsilon(s,y)-b(s,y)|p_\alpha(s,y-x)\,\mathrm{d}y\,\mathrm{d}s\\
&\lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\int_u^t \frac{\|b_\varepsilon(s,.)-b(s,.)\|_{L^p}}{s^{\frac{d}{\alpha p}}}\,\mathrm{d}s\\
&\lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\|b_\varepsilon-b\|_{L^{\tilde q}-L^p}t^{1-\left(\frac{1}{\tilde q}+\frac{d}{\alpha p}\right)}.
\end{aligned}$$

Since the same estimation holds for $\int_{\mathcal{D}([0,T],\mathbb{R}^d)}|F(\xi)-F_\varepsilon(\xi)|P(\mathrm{d}\xi)$, because the heat kernel estimates hold as well for the limit point, we conclude that $\int_{\mathcal{D}([0,T],\mathbb{R}^d)}F(\xi)P(\mathrm{d}\xi)=0$. Taking $f,\psi,u,s_1,\dots,s_p,t$ in countable dense subsets, we deduce that P satisfies (iii) in Definition 5.1.

Let us now deal with the case $p=\infty$. We set $(\tilde p,\tilde q)=(\frac{dq+1}{(\alpha-1)q-\alpha},q)\mathbb{1}_{q<\infty}+(\frac{3d}{\alpha-1},\frac{3\alpha}{\alpha-1})\mathbb{1}_{q=\infty}$. We have $\frac{d}{\tilde p}+\frac{\alpha}{\tilde q}<\alpha-1$. We introduce for $\varepsilon\in(0,1]$, a smooth and bounded function b_ε such that $\|b_\varepsilon\|_{L^q-L^\infty}\le 2\|b\|_{L^q-L^\infty}$ and, for each $K\in\mathbb{N}^*$, setting $b_\varepsilon^K(t,x)=\mathbb{1}_{[-K,K]^d}(x)b_\varepsilon(t,x)$ and $b^K(t,x)=\mathbb{1}_{[-K,K]^d}(x)b(t,x)$, we have $\lim_{\varepsilon\to 0}\|b_\varepsilon^K-b^K\|_{L^{\tilde q}-L^{\tilde p}}=0$. The above reasoning when $p<\infty$ remains valid once we now bound $\mathbb{E}[|F_\varepsilon(X^h)-F(X^h)|]$ from above by

$$\begin{aligned}
&\|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\int_u^t \mathbb{E}[|b_\varepsilon^K(s,X_s^h)-b^K(s,X_s^h)|+(\|b_\varepsilon(s,\cdot)\|_{L^\infty}+\|b(s,\cdot)\|_{L^\infty})\mathbb{1}_{|X_s^h|\ge K}]\,\mathrm{d}s\\
&\lesssim \|\psi\|_{L^\infty}\|\nabla f\|_{L^\infty}\left(\|b_\varepsilon^K-b^K\|_{L^{\tilde q}-L^{\tilde p}}t^{1-\left(\frac{1}{\tilde q}+\frac{d}{\alpha\tilde p}\right)}+\|b\|_{L^q-L^\infty}\left(\int_u^t(\mathbb{P}(|X_s^h|\ge K))^{q'}\,\mathrm{d}s\right)^{1/q'}\right).
\end{aligned}$$

According to (5.4.3), $\int_u^t(\mathbb{P}(|X_s^h|\ge K))^{q'}\,\mathrm{d}s$ can be made arbitrarily small uniformly in h for K large enough while for fixed K, $\|b_\varepsilon^K-b^K\|_{L^{\tilde q}-L^{\tilde p}}$ goes to 0 with ε. This concludes the proof.

5.4.4 Uniqueness of the solution to the martingale problem

For this paragraph, we assume $p,q<\infty$ (otherwise, we can proceed in a similar way to the previous paragraph to mollify the drift). Let $(b_m)_{m\in\mathbb{N}}$ denote a sequence of bounded smooth approximating functions s.t. $\|b-b_m\|_{L^q-L^p}\to 0$ as $m\to\infty$. We study the mollified equation

$$(\partial_s+\mathcal{L}^\alpha+b_m\cdot\nabla)u_m(s,x)=f(s,x),\qquad (s,x)\in[0,t)\times\mathbb{R}^d, u_m(t,\cdot)=0. \tag{5.4.7}$$

It is well known that for a smooth compactly supported f, (5.4.7) has a unique smooth bounded classical solution (see [MP14]). Furthermore, the following Schauder estimates (whose proofs are postponed to Appendix 5.5.5) hold:

Lemma 5.5 (Schauder). *Let $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be $\mathcal{C}^{1,2}$ with compact support and $(u_m)_{m\in\mathbb{N}}$ denote the sequence of classical solutions to the mollified PDEs* (5.4.7). *Then, for all $\xi\in[0,(\gamma+1)/\alpha)$, for all $0\le s\le s'\le t$, for all $x\in\mathbb{R}^d$, and for all $m\in\mathbb{N}$,*

$$\|\nabla u_m\|_{L^\infty}\lesssim\|f\|_{L^\infty}, \tag{5.4.8}$$

$$|u_m(s',x)-u_m(s,x)|\lesssim|s'-s|^\xi\|f\|_{L^\infty}. \tag{5.4.9}$$

Let P^1 and P^2 be solutions of the martingale problem associated with $b\cdot\nabla+\mathcal{L}^\alpha$ and $x\in\mathbb{R}^d$ in the sense of Definition 5.1. Let f be a smooth bounded function. For all $m\in\mathbb{N}$, denote $u_m\in\mathcal{C}^1([0,T],\mathcal{C}^2(\mathbb{R}^d,\mathbb{R}))$ the classical solution to the Cauchy problem associated with (5.4.7) with source term f. For $i\in\{1,2\}$,

$$\left\{M_s^{u_m}=u_m(s,\xi_s)-u_m(0,x)-\int_0^s(\partial_r+\mathcal{L}^\alpha+b\cdot\nabla)u_m(r,\xi_r)\,\mathrm{d}r\right\}_{0\le s\le t} \tag{5.4.10}$$

is a P^i-martingale. Equations (5.4.9) and (5.4.8) allow us to apply the Ascoli-Arzelà theorem to (u_m): let $(u_{m_k})_k$ be a subsequence of $(u_m)_m$ which converges uniformly on every compact subset of $[0,t]\times\mathbb{R}^d$ to some u_∞. Now, taking $\rho^i\in L^{q'}((0,T],L^{p'}(\mathbb{R}^d))$ such that $P_t^i(\mathrm{d}y)=\rho^i(t,y)\,\mathrm{d}y$, taking expectations in (5.4.10) and using the Fubini theorem, we have, when $s\to t$,

$$\mathbb{E}^{P^i}\left[\int_0^t f(r,\xi_r)\,\mathrm{d}r\right]=-u_{m_k}(0,x)+\int_0^t\int(b_{m_k}-b)(r,z)\cdot\nabla u_{m_k}(r,z)\rho^i(r,z)\,\mathrm{d}z\,\mathrm{d}r. \tag{5.4.11}$$

Since $\rho^i\in L^{q'}((0,T],L^{p'}(\mathbb{R}^d))$, using a Hölder inequality in space and then one in time along with the fact that $\|\nabla u_{m_k}\|_{L^\infty}$ is bounded uniformly in k (from Equation (5.4.8)), we obtain

$$\left|\int_0^t\int(b_{m_k}-b)(r,z)\cdot\nabla u_{m_k}(r,z)\rho^i(r,z)\,\mathrm{d}z\,\mathrm{d}r\right|\lesssim\|\nabla u_{m_k}\|_{L^\infty}\|b-b_{m_k}\|_{L^q-L^p}\|\rho^i\|_{L^{q'}-L^{p'}}\underset{k\to\infty}{\longrightarrow}0.$$

Thus, taking the limit as k goes to ∞ in (5.4.11), we obtain

$$\mathbb{E}^{P^1}\left[\int_0^t f(r,\xi_r)\,\mathrm{d}r\right]=-u_\infty(0,x)=\mathbb{E}^{P^2}\left[\int_0^t f(r,\xi_r)\,\mathrm{d}r\right], \tag{5.4.12}$$

which readily gives $P^1=P^2$ (see e.g. Theorem 4.2 in [EK86]).

5.5 Proof of the technical lemmas involving the stable density

5.5.1 Proof of Lemma 5.1 (Stable Sensitivities)

Item (5.2.1) directly follows from Section 2 in [Kol00]. Let us prove (5.2.2).

- **Diagonal case:** $|x-x'|\le u^{1/\alpha}$. Since we are looking at a small perturbation in the space variable, it makes sense to use a Taylor expansion:

$$\begin{aligned}\left|\nabla_x^\zeta p_\alpha(u,x)-\nabla_x^\zeta p_\alpha(u,x')\right|&=\left|\int_0^1\nabla_x\nabla_x^\zeta p_\alpha(u,x'+(x-x')\lambda)\cdot(x-x')\,\mathrm{d}\lambda\right|\\&\lesssim\frac{|x-x'|}{u^{\frac{1+|\zeta|}{\alpha}}}\int_0^1\bar p_\alpha(u,x'+(x-x')\lambda)\,\mathrm{d}\lambda,\end{aligned}$$

  using (5.2.1) and $p_\alpha\asymp\bar p_\alpha$ (see (5.1.16) and (5.1.17)) for the last inequality. Up to a modification of the underlying constant,

$$\begin{aligned}\bar p_\alpha(u,x'+(x-x')\lambda)&\lesssim u^{-\frac{d}{\alpha}}\left(2+\frac{|x'+\lambda(x-x')|}{u^{\frac{1}{\alpha}}}\right)^{-d-\alpha}\lesssim u^{-\frac{d}{\alpha}}\left(2-\frac{|x-x'|}{u^{\frac{1}{\alpha}}}+\frac{|x'|}{u^{\frac{1}{\alpha}}}\right)^{-d-\alpha}\\&\lesssim u^{-\frac{d}{\alpha}}\left(1+\frac{|x'|}{u^{\frac{1}{\alpha}}}\right)^{-d-\alpha}\lesssim\bar p_\alpha(u,x').\end{aligned}$$

  We conclude the proof in the diagonal case noting that for all $\theta\in(0,1]$, $\frac{|x-x'|}{u^{\frac{1}{\alpha}}}\le\left(\frac{|x-x'|}{u^{\frac{1}{\alpha}}}\right)^\theta$.

- **Off-diagonal case:** $|x-x'|\ge u^{1/\alpha}$. In this case, a Taylor expansion in space is not relevant. We simply use the fact that $1=\frac{|x-x'|^\theta}{u^{\frac{\theta}{\alpha}}}\wedge 1$ and (5.2.1):

$$\begin{aligned}\left|\nabla_x^\zeta p_\alpha(u,x)-\nabla_x^\zeta p_\alpha(u,x')\right|&\le\left(\frac{|x-x'|^\theta}{u^{\frac{\theta}{\alpha}}}\wedge 1\right)\left(\left|\nabla_x^\zeta p_\alpha(u,x)\right|+\left|\nabla_x^\zeta p_\alpha(u,x')\right|\right)\\&\lesssim\left(\frac{|x-x'|^\theta}{u^{\frac{\theta}{\alpha}}}\wedge 1\right)\frac{1}{u^{\frac{|\zeta|}{\alpha}}}\left(p_\alpha(u,x)+p_\alpha(u,x')\right).\end{aligned}$$

This concludes the proof of (5.2.2).

Let us now prove (5.2.3). Let $0 < u \le u' \le T$. Assume first $|u - u'| \le \frac{u}{2}$.

$$\begin{aligned}
\left|\nabla_x^\zeta p_\alpha(u,x) - \nabla_x^\zeta p_\alpha(u',x)\right| &= \left|\int_0^1 \partial_t \nabla_x^\zeta p_\alpha(u+(u'-u)\lambda, x)(u'-u)\,\mathrm{d}\lambda\right| \\
&\lesssim \int_0^1 \frac{1}{(u+(u'-u)\lambda)^{1+\frac{|\zeta|}{\alpha}}} \bar{p}_\alpha(u+(u'-u)\lambda, x)|u'-u|\,\mathrm{d}\lambda \\
&\lesssim \frac{|u-u'|}{u^{1+\frac{|\zeta|}{\alpha}}} \int_0^1 \bar{p}_\alpha(u+(u'-u)\lambda, x)\,\mathrm{d}\lambda,
\end{aligned}$$

recalling that $u' \ge u$ for the last inequality. We now discuss in function of the position of the spatial variable x w.r.t. the current time u.

- **Diagonal case:** $|x| \le u^{1/\alpha}$. Then,

$$\bar{p}_\alpha(u+(u'-u)\lambda, x) \lesssim (u+(u'-u)\lambda)^{-\frac{d}{\alpha}} \lesssim u^{-\frac{d}{\alpha}} \asymp \bar{p}_\alpha(u,x) \asymp p_\alpha(u,x).$$

- **Off-diagonal case:** $|x| \ge u^{1/\alpha}$.

$$\bar{p}_\alpha(u+(u'-u)\lambda, x) \lesssim \frac{u+(u'-u)\lambda}{|x|^{d+\alpha}} \lesssim \frac{u}{|x|^{d+\alpha}} \asymp \bar{p}_\alpha(u,x) \asymp p_\alpha(u,x).$$

Note that the condition $|u'-u| \le \frac{u}{2}$ is actually needed only for the second above inequality. Namely, it ensures that the term $\lambda(u'-u)$ has the same magnitude than u (otherwise the previous expansions are useless and the estimation is direct as discussed below).

In turn, we obtain

$$\left|\nabla_x^\zeta p_\alpha(u,x) - \nabla_x^\zeta p_\alpha(u',x)\right| \lesssim \frac{|u-u'|}{u^{1+\frac{|\zeta|}{\alpha}}} \bar{p}_\alpha(u,z) \lesssim \frac{|u-u'|^\theta}{u^{\theta+\frac{|\zeta|}{\alpha}}} p_\alpha(u,z),$$

for all $\theta \in (0,1]$.

In the case $|u-u'| \ge \frac{u}{2}$, we simply write using (5.2.1)

$$\begin{aligned}
\left|\nabla_x^\zeta p_\alpha(u,x) - \nabla_x^\zeta p_\alpha(u',x)\right| &\le \left(2\frac{|u-u'|}{u}\right)^\theta \left(|\nabla_x^\zeta p_\alpha(u,x)| + |\nabla_x^\zeta p_\alpha(u',x)|\right) \\
&\lesssim \frac{|u-u'|^\theta}{u^{\theta+\frac{|\zeta|}{\alpha}}} \left(p_\alpha(u,x) + p_\alpha(u',x)\right),
\end{aligned}$$

which concludes the proof of (5.2.3).

Let us now prove (5.2.4). Using $\bar{p}_\alpha \asymp p_\alpha$, we can write

$$\|p_\alpha(u,\cdot)|\cdot|^\zeta\|_{L^{\ell'}}^{\ell'} = \int_{\mathbb{R}^d} p_\alpha(u,y)^{\ell'}|y|^{\zeta\ell'}\,\mathrm{d}y \lesssim \int_{\mathbb{R}^d} \frac{1}{u^{\frac{d\ell'}{\alpha}}} \times \frac{1}{\left(1+\frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{\ell'(d+\alpha)}}|y|^{\zeta\ell'}\,\mathrm{d}y.$$

Set $z = yu^{-\frac{1}{\alpha}}$:

$$\|p_\alpha(u,\cdot)|\cdot|^\zeta\|_{L^{\ell'}}^{\ell'} \lesssim u^{\frac{d}{\alpha}(1-\ell')+\frac{\zeta\ell'}{\alpha}} \int_{\mathbb{R}^d} \frac{1}{(1+|z|)^{\ell'(d+\alpha)}}|z|^{\zeta\ell'}\,\mathrm{d}z,$$

which converges whenever $\ell'(\zeta-d-\alpha)+d-1 < -1 \iff \zeta < d+\alpha-\frac{d}{\ell'}$, in which case we obtain

$$\|p_\alpha(u,\cdot)|\cdot|^\zeta\|_{L^{\ell'}} \lesssim u^{-\frac{d}{\alpha\ell}+\frac{\zeta}{\alpha}}.$$

Let us now prove the convolution part (5.2.5). Denote

$$\begin{aligned}\mathfrak{I} &:= \|\bar p_\alpha(t-u,\cdot-y)\bar p_\alpha(u-s,x-\cdot)\|_{L^{\ell'}}^{\ell'}\\ &\lesssim \int \frac{1}{(t-u)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{\left(1+\frac{|z-y|}{(t-u)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}} \times \frac{1}{(u-s)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{\left(1+\frac{|x-z|}{(u-s)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}}\,\mathrm{d}z. \end{aligned}\tag{5.5.1}$$

We now discuss in function of the magnitude of the distance $|x-y|$ w.r.t. to the global time scale $t-s$.

- **Diagonal case:** $|x-y| < (t-s)^{1/\alpha}$
  In this case, either $(t-u) \geq \frac{1}{2}(t-s)$ or $(u-s) \geq \frac{1}{2}(t-s)$, we can then use the global diagonal bound in (5.5.1) for the corresponding density.
  - If $(t-u) \geq \frac{1}{2}(t-s)$,

    $$\mathfrak{I} \lesssim \frac{1}{(t-u)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{(u-s)^{\frac{d}{\alpha}(\ell'-1)}} \int \frac{1}{(u-s)^{\frac{d}{\alpha}}} \times \frac{1}{\left(1+\frac{|x-z|}{(u-s)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}}\,\mathrm{d}z \lesssim \frac{1}{(t-u)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{(u-s)^{\frac{d}{\alpha}(\ell'-1)}}.$$

    Since $(t-u) \geq \frac{1}{2}(t-s)$, $\frac{1}{(t-u)^{\frac{d}{\alpha}}} \lesssim \frac{1}{(t-s)^{\frac{d}{\alpha}}} \asymp \bar p_\alpha(t-s,y-x)$, and

    $$\mathfrak{I} \lesssim \bar p_\alpha(t-s,x-y)^{\ell'} \frac{1}{(u-s)^{\frac{d}{\alpha}(\ell'-1)}}.$$

  - If $(u-s) \geq \frac{1}{2}(t-s)$, we readily obtain by symmetry

    $$\mathfrak{I} \lesssim \bar p_\alpha(t-s,x-y)^{\ell'} \frac{1}{(t-u)^{\frac{d}{\alpha}(\ell'-1)}}.$$

- **Off-diagonal case:** $|x-y| \geq (t-s)^{1/\alpha}$
  In this case, either $|x-z| \geq \frac{1}{2}|x-y|$ or $|z-y| > \frac{1}{2}|x-y|$, i.e. one of the two contributions in $\mathfrak{I}$ is in the off-diagonal regime, allowing us to use (5.2.9). In this case we split the upper-bound for $\mathfrak{I}$ in (5.5.1) as follows:

$$\mathfrak{I} \lesssim \int \frac{1}{(t-u)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{\left(1+\frac{|z-y|}{(t-u)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}} \times \frac{1}{(u-s)^{\frac{d\ell'}{\alpha}}} \times \frac{\left(\mathbb{1}_{|x-z|\geq\frac{1}{2}|x-y|} + \mathbb{1}_{|z-y|\geq\frac{1}{2}|x-y|}\right)}{\left(1+\frac{|x-z|}{(u-s)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}}\,\mathrm{d}z =: \mathfrak{I}_1 + \mathfrak{I}_2.$$

  - For $\mathfrak{I}_1$, $|x-z| \geq \frac{1}{2}|x-y| > \frac{1}{2}(t-s)^{1/\alpha}$, we get

    $$\begin{aligned}\mathfrak{I}_1 &\lesssim \frac{1}{(u-s)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{\left(1+\frac{|x-y|}{(u-s)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}} \int \frac{1}{(t-u)^{\frac{d\ell'}{\alpha}}} \times \frac{1}{\left(1+\frac{|z-y|}{(t-u)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}} \mathbb{1}_{|x-z|\geq\frac{1}{2}|x-y|}\,\mathrm{d}z\\ &\lesssim \bar p_\alpha(u-s,x-y)^{\ell'} \frac{1}{(t-u)^{\frac{d}{\alpha}(\ell'-1)}} \int \frac{1}{(t-u)^{\frac{d}{\alpha}}} \times \frac{1}{\left(1+\frac{|z-y|}{(t-u)^{\frac{1}{\alpha}}}\right)^{(d+\alpha)\ell'}} \mathbb{1}_{|x-z|\geq\frac{1}{2}|x-y|}\,\mathrm{d}z\end{aligned}$$

    Since $|x-y| > (u-s)^{1/\alpha}$, $\bar p_\alpha(u-s,x-y) \asymp \frac{u-s}{|x-y|^{d+\alpha}} \leq \frac{t-s}{|x-y|^{d+\alpha}} \asymp \bar p_\alpha(t-s,x-y)$, and

    $$\mathfrak{I}_1 \lesssim \bar p_\alpha(t-s,x-y)^{\ell'} \frac{1}{(t-u)^{\frac{d}{\alpha}(\ell'-1)}}.$$

  - For $\mathfrak{I}_2$, $|z-y| > \frac{1}{2}|x-y| > \frac{1}{2}(t-s)^{1/\alpha}$, the same computations give, when swapping the roles of $|x-z|$ and $|y-z|$,

    $$\mathfrak{I}_2 \lesssim \bar p_\alpha(t-s,x-y)^{\ell'} \frac{1}{(u-s)^{\frac{d}{\alpha}(\ell'-1)}}. \tag{5.5.2}$$

In each case, we have established that

$$\|\bar{p}_\alpha(t-u,\cdot-y)\bar{p}_\alpha(u-s,x-\cdot)\|_{L^{\ell'}} = \mathfrak{I}^{\frac{1}{\ell'}} \lesssim \left[\frac{1}{(t-u)^{\frac{d}{\alpha}\frac{\ell'-1}{\ell'}}} + \frac{1}{(u-s)^{\frac{d}{\alpha}\frac{\ell'-1}{\ell'}}}\right]\bar{p}_\alpha(t-s,x-y)$$

$$\lesssim \left[\frac{1}{(t-u)^{\frac{d}{\alpha\ell}}} + \frac{1}{(u-s)^{\frac{d}{\alpha\ell}}}\right]\bar{p}_\alpha(t-s,x-y),$$

which concludes the proof of (5.2.5).
Equation (5.2.6) then eventually follows from (5.2.5) and Hölder's inequality.

5.5.2 Proof of Lemma 5.2 (Feynman-Kac partial differential equation)

Recall that for $u>0, z\in\mathbb{R}^d$:

$$p_\alpha(u,z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp(u\Phi_{\mu,\alpha}(\zeta))\exp(-i\zeta\cdot z)\,\mathrm{d}\zeta,$$

$$\Phi_{\mu,\alpha}(\zeta) = \int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}}[\exp(i\zeta\cdot\rho\xi)-1]\,\mu(\mathrm{d}\xi)\frac{\mathrm{d}\rho}{\rho^{1+\alpha}} = -C_{\alpha,d}\int_{\mathbb{S}^{d-1}}|\zeta\cdot\xi|^\alpha\mu(\mathrm{d}\xi),\ C_{\alpha,d}>0,$$

being the Khinchin exponent associated with the operator $\mathcal{L}^\alpha$. It is thus direct to see from the non-degeneracy assumption (5.1.15) that there exists $c>0$ s.t. $\forall\zeta\in\mathbb{R}^d$, $\int_{\mathbb{S}^{d-1}}|\zeta\cdot\xi|^\alpha\mu(\mathrm{d}\xi)\geq c|\zeta|^\alpha$ so that $\exp(u\Phi_{\mu,\alpha}(\zeta))\leq\exp(-cC_{\alpha,d}u|\zeta|^\alpha)$. We deduce that p_α is smooth on $\mathbb{R}_+^*\times\mathbb{R}^d$ and

$$\partial_u p_\alpha(u,z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\Phi_{\mu,\alpha}(\zeta)\exp(u\Phi_{\mu,\alpha}(\zeta))\exp(-i\zeta\cdot z)\,\mathrm{d}\zeta.$$

Since, by symmetry of the measure μ and Fubini's theorem,

$$\Phi_{\mu,\alpha}(\zeta)\exp(u\Phi_{\mu,\alpha}(\zeta)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}}[\exp(-i\zeta\cdot\rho\xi)-1]\,\mu(\mathrm{d}\xi)\frac{\mathrm{d}\rho}{\rho^{1+\alpha}}\int_{\mathbb{R}^d}\exp(i\zeta\cdot x)p_\alpha(u,x)\,\mathrm{d}x$$

$$= \int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}(\exp(i\zeta\cdot(x-\rho\xi))-\exp(i\zeta\cdot x))\,p_\alpha(u,x)\,\mathrm{d}x\mu(\mathrm{d}\xi)\frac{\mathrm{d}\rho}{\rho^{1+\alpha}}$$

$$= \int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\exp(i\zeta\cdot x)\left(p_\alpha(u,x+\rho\xi)-p_\alpha(u,x)\right)\,\mathrm{d}x\mu(\mathrm{d}\xi)\frac{\mathrm{d}\rho}{\rho^{1+\alpha}}$$

$$= \int_{\mathbb{R}^d}\exp(i\zeta\cdot x)\mathcal{L}^\alpha p_\alpha(u,x)\,\mathrm{d}x,$$

one has $\partial_u p_\alpha(u,z) = \mathcal{L}^\alpha p_\alpha(u,z)$.

The fact that v solves the Feynman-Kac partial differential equation on $[0,t)\times\mathbb{R}^d$ is easily deduced using (5.1.16) and (5.2.1) to apply Lebesgue's and Fubini's theorems. Last, for $s\in[0,t)$,

$$|v(s,y)-\phi(y)| = \int_{\mathbb{R}^d}\left|\phi\left(y-(t-s)^{\frac{1}{\alpha}}z\right)-\phi(y)\right|(t-s)^{\frac{d}{\alpha}}p_\alpha(t-s,(t-s)^{\frac{1}{\alpha}}z)\,\mathrm{d}z$$

$$= \int_{\mathbb{R}^d}\left|\phi\left(y-(t-s)^{\frac{1}{\alpha}}z\right)-\phi(y)\right|p_\alpha(1,z)\,\mathrm{d}z,$$

and Lebesgue's theorem ensures that the right-hand side converges to 0 as s goes to t.

We refer to [Kol00] (in particular the introduction, Proposition 2.5 and Section 3) for additional details and properties about the density p_α.

5.5.3 Proof of Lemma 5.3: stable time-space convolutions with Lebesgue function

Let us first use (5.2.6) with $\ell = p$ and then Hölder's inequality in time:

$$\begin{aligned}
I_{\beta_1,\beta_2}(u,v) &:= \int_u^v \int p_\alpha(r, z-x)|b(r,z)|p_\alpha(t-r, y-z)\frac{1}{(t-r)^{\beta_1}}\frac{1}{r^{\beta_2}}\,\mathrm{d}z\,\mathrm{d}r \\
&\lesssim p_\alpha(t, y-x)\int_u^v \|b(r,\cdot)\|_{L^p}\left[\frac{1}{r^{\frac{d}{\alpha p}}} + \frac{1}{(t-r)^{\frac{d}{\alpha p}}}\right]\frac{1}{(t-r)^{\beta_1}}\frac{1}{r^{\beta_2}}\,\mathrm{d}r \\
&\lesssim p_\alpha(t, y-x)\left(\int_u^v \left[\frac{1}{r^{\frac{dq'}{\alpha p}}} + \frac{1}{(t-r)^{\frac{dq'}{\alpha p}}}\right]\frac{1}{(t-r)^{q'\beta_1}}\frac{1}{r^{q'\beta_2}}\,\mathrm{d}r\right)^{\frac{1}{q'}} \\
&=: p_\alpha(t, y-x)S_{\beta_1,\beta_2}(u,v).
\end{aligned}$$

Set $\lambda = \frac{r-u}{v-u} \iff r = u + \lambda(v-u)$, then

$$\begin{aligned}
&S_{\beta_1,\beta_2}(u,v)^{q'} \\
\lesssim &(v-u)\int_0^1 \left[\frac{1}{(u+\lambda(v-u))^{\frac{dq'}{\alpha p}}} + \frac{1}{(t-u-\lambda(v-u))^{\frac{dq'}{\alpha p}}}\right]\frac{1}{(t-u-\lambda(v-u))^{q'\beta_1}}\frac{1}{(u+\lambda(v-u))^{q'\beta_2}}\,\mathrm{d}\lambda.
\end{aligned}$$

Assume first that $q'\left(\frac{d}{\alpha p} + \beta_i\right) < 1,\ i \in \{1,2\}$ (integrable case). Then,

$$\begin{aligned}
&S_{\beta_1,\beta_2}(u,v)^{q'} \\
\lesssim &(v-u)^{1-\frac{dq'}{\alpha p} - q'(\beta_1+\beta_2)}\int_0^1 \left[\frac{1}{\lambda^{\frac{dq'}{\alpha p}}} + \frac{1}{(1-\lambda)^{\frac{dq'}{\alpha p}}}\right]\frac{1}{(1-\lambda)^{q'\beta_1}}\frac{1}{\lambda^{q'\beta_2}}\,\mathrm{d}\lambda \lesssim (v-u)^{1-\frac{dq'}{\alpha p} - q'(\beta_1+\beta_2)},
\end{aligned}$$

and

$$I_{\beta_1,\beta_2}(u,v) \lesssim p_\alpha(t, y-x)(v-u)^{1-\frac{1}{q}-\frac{d}{\alpha p}-(\beta_1+\beta_2)}$$

which, recalling (5.1.20), gives (5.2.8).

Let us now consider the case $q'\left(\frac{d}{\alpha p} + \beta_1\right) > 1, q'\left(\frac{d}{\alpha p} + \beta_2\right) < 1$ (singular case) with $v < t$. We then write:

$$\begin{aligned}
&S_{\beta_1,\beta_2}(u,v)^{q'} \\
\lesssim &(v-u)^{1-q'(\frac{d}{\alpha p}+\beta_1+\beta_2)}\int_0^1 \left[\frac{1}{\lambda^{q'(\frac{d}{\alpha p}+\beta_2)}(\frac{t-u}{v-u}-\lambda)^{q'\beta_1}} + \frac{1}{(\frac{t-u}{v-u}-\lambda)^{q'(\frac{d}{\alpha p}+\beta_1)}\lambda^{q'\beta_2}}\right]\mathrm{d}\lambda \\
\lesssim &(v-u)^{1-q'(\frac{d}{\alpha p}+\beta_1+\beta_2)}\Big(\int_0^{\frac{1}{2}} \frac{1}{\lambda^{q'(\frac{d}{\alpha p}+\beta_2)}}\,\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 \frac{1}{(\frac{t-u}{v-u}-\lambda)^{q'(\frac{d}{\alpha p}+\beta_1)}}\,\mathrm{d}\lambda\Big) \\
\lesssim &(v-u)^{1-q'(\frac{d}{\alpha p}+\beta_1+\beta_2)}\Big(1 + (\frac{t-u}{v-u}-1)^{1-q'(\frac{d}{\alpha p}+\beta_1)}\Big) \\
\lesssim &(v-u)^{1-q'(\frac{d}{\alpha p}+\beta_1+\beta_2)} + (v-u)^{-q'\beta_2}(t-v)^{1-q'(\frac{d}{\alpha p}+\beta_1)}.
\end{aligned}$$

Hence, in the divergent case we have established

$$I_{\beta_1,\beta_2}(u,v) \lesssim p_\alpha(t, y-x)\Big((v-u)^{1-\frac{1}{q}-\frac{d}{\alpha p}-(\beta_1+\beta_2)} + (v-u)^{-\beta_2}(t-v)^{1-\frac{1}{q}-(\frac{d}{\alpha p}+\beta_1)}\Big),$$

which precisely gives (5.2.7). This concludes the proof of Lemma 5.3.

5.5.4 Proof of Lemma 5.4 (About the cutoff on a one-step transition)

Using the fact that $p_\alpha \asymp \bar{p}_\alpha$ and $|y - sb_h(r,x)| \geq |y| - s|b_h(r,x)|$, we get for $0 \leq s \leq \min(u,h)$,

$$\begin{aligned}
\bar{p}_\alpha(u, y - sb_h(r,x)) &\lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(2 + \frac{|y - sb_h(r,x)|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(2 - \frac{s}{u^{\frac{1}{\alpha}}}|b_h(r,x)| + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \\
&\lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(2 - s^{1-\frac{1}{\alpha}}h^{-\frac{d}{\alpha p}-\frac{1}{q}} + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(2 - h^{\frac{\gamma}{\alpha}} + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \\
&\lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(1 + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \lesssim \bar{p}_\alpha(u,y), \qquad (5.5.3)
\end{aligned}$$

provided $h < 1$ for the last inequality, which we can assume w.l.o.g.
In the case of $\bar{b}_h$, we derive similarly,

$$\begin{aligned}
\bar{p}_\alpha(u, y - s\bar{b}_h(r,x)) &\lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(2 - \frac{s}{u^{\frac{1}{\alpha}}}|\bar{b}_h(r,x)| + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(2 - h^0 + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \\
&\lesssim \frac{1}{u^{\frac{d}{\alpha}}}\left(1 + \frac{|y|}{u^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} = \bar{p}_\alpha(u,y).
\end{aligned}$$

This proves (5.2.11) for $|\zeta| = 0$. For $0 < |\zeta| \leq 1$, one simply needs to apply (5.2.1) beforehands. For the proof of (5.2.12), it is enough to apply (5.2.2) to $\left|\nabla^\zeta p_\alpha(u, y - s\mathfrak{b}_h(r,x)) - \nabla^\zeta p_\alpha(u, y' - s\mathfrak{b}_h(r,x))\right|$, where $\mathfrak{b}_h \in \{b_h, \bar{b}_h\}$, which yields

$$\begin{aligned}
&\left|\nabla^\zeta p_\alpha(u, y - s\mathfrak{b}_h(r,x)) - \nabla^\zeta p_\alpha(u, y' - s\mathfrak{b}_h(r,x))\right| \\
&\qquad \lesssim \left(\frac{|y-y'|^\delta}{u^{\frac{\delta}{\alpha}}} \wedge 1\right)\frac{1}{u^{\frac{|\zeta|}{\alpha}}}\left(p_\alpha(u, y - s\mathfrak{b}_h(r,x)) + p_\alpha(u, y' - s\mathfrak{b}_h(r,x))\right),
\end{aligned}$$

for all *$\delta \in (0,1]$ and then use (5.2.11) to get rid of the drift in the previous equation.

5.5.5 Proof of Lemma 5.5: Schauder estimates for the mollified PDE (5.4.7)

Let $m \in \mathbb{N}$ and u_m denote the classical solution to (5.4.7). For $s \in (0,t], x \in \mathbb{R}^d$, computing $u_m(t,x) + (Z_t - Z_s) - u_m(s,x)$ by Itô's formula and taking expectations, we obtain

$$\begin{aligned}
u_m(s,x) &= -\int_s^t \int f(r,y) p_\alpha(r-s, y-x)\,\mathrm{d}y\,\mathrm{d}r \\
&\quad + \int_s^t \int b_m(r,y)\cdot\nabla u_m(r,y) p_\alpha(r-s, y-x)\,\mathrm{d}y\,\mathrm{d}r \\
&=: I_1(s,x) + I_2(s,x). \qquad (5.5.4)
\end{aligned}$$

Let us first prove the gradient bound (5.4.8). For I_1, using that f is bounded on $[0,T]\times\mathbb{R}^d$ along with (5.2.1), we get

$$\begin{aligned}
|\nabla I_1(s,x)| &\leq \int_s^t \int |f(r,y)||\nabla_x p_\alpha(r-s, y-x)|\,\mathrm{d}y\,\mathrm{d}r \\
&\lesssim \|f\|_{L^\infty - L^\infty} \int_s^t \int p_\alpha(r-s, y-x)\frac{1}{(r-s)^{\frac{1}{\alpha}}}\,\mathrm{d}y\,\mathrm{d}r \\
&\lesssim \|f\|_{L^\infty - L^\infty} \int_s^t \frac{1}{(r-s)^{\frac{1}{\alpha}}}\,\mathrm{d}r \\
&\lesssim \|f\|_{L^\infty - L^\infty}.
\end{aligned}$$

For I_2, let us first note that due to standard Schauder estimates (see [MP14]), we already know that $\nabla u_m(r,\cdot)$ is bounded (although not necessarily uniformly in m) for all $r \in [0,t]$. We can thus write, using a Hölder inequality, then (5.2.4), and finally a Hölder inequality in time,

$$\begin{aligned}
|\nabla I_2(s,x)| &\leq \int_s^t \|\nabla u_m(r,\cdot)\|_{L^\infty} \int |b_m(r,y)| p_\alpha(r-s,y-x)\,\mathrm{d}y\,\mathrm{d}r \\
&\leq \int_s^t \|\nabla u_m(r,\cdot)\|_{L^\infty} \|b_m(r,\cdot)\|_{L^p} \frac{1}{(r-s)^{\frac{d}{\alpha p}}}\,\mathrm{d}r \\
&\leq \|b_m\|_{L^q-L^p} \left(\int_s^t \|\nabla u_m(r,\cdot)\|_{L^\infty}^{q'} \frac{1}{(r-s)^{\frac{dq'}{\alpha p}}}\,\mathrm{d}r \right)^{\frac{1}{q'}}.
\end{aligned}$$

Gathering the previous estimates, we have

$$\|\nabla u_m(s,\cdot)\|_{L^\infty}^{q'} \lesssim \|f\|_{L^\infty-L^\infty}^{q'} + \|b_m\|_{L^q-L^p}^{q'} \int_s^t \|\nabla u_m(r,\cdot)\|_{L^\infty}^{q'} \frac{1}{(r-s)^{\frac{dq'}{\alpha p}}}\,\mathrm{d}r.$$

Since $\frac{dq'}{\alpha p} < 1$, using Lemma 2.2 and Example 2.4 [Zha10], we deduce (5.4.8).
Let us now prove (5.4.9). Using the previous notations, we can write

$$|u_m(s',x) - u_m(s,x)| \leq |I_1(s',x) - I_1(s,x)| + |I_2(s',x) - I_2(s,x)|.$$

For the first term, using (5.2.3) with $\theta = \xi$, for any $\xi \in [0,(\gamma+1)/\alpha)$ and $\zeta = 0$, we readily have

$$|I_1(s',x) - I_1(s,x)| \lesssim |s'-s|^\xi \|f\|_{L^\infty-L^\infty}.$$

For the second term, using (5.2.3) with $\theta = \xi$ and $\zeta = 0$, as well as (5.2.1), we can write

$$\begin{aligned}
|I_2(s',x) - I_2(s,x)| \lesssim& \int_{s'}^t \int |b_m(r,y)\cdot\nabla u_m(r,y)| \frac{|s'-s|^\xi}{(r-s')^\xi} (p_\alpha(r-s',y-x) + p_\alpha(r-s,y-x))\,\mathrm{d}y\,\mathrm{d}r \\
&+ \int_s^{s'} \int |b_m(r,y)\cdot\nabla u_m(r,y)| p_\alpha(r-s,y-x)\,\mathrm{d}y\,\mathrm{d}r.
\end{aligned}$$

Using then a Hölder inequality in space, (5.2.4), a Hölder inequality in time and the previously established boundedness of ∇u_m, we get

$$\begin{aligned}
|I_2(s',x) - I_2(s,x)| \lesssim& \|b_m\|_{L^q-L^p} \|\nabla u_m\|_{L^\infty} |s'-s|^\xi \left(\int_{s'}^t \frac{1}{(r-s')^{q'\xi}} \left[\frac{1}{(r-s)^{\frac{dq'}{\alpha p}}} + \frac{1}{(r-s')^{\frac{dq'}{\alpha p}}} \right] \mathrm{d}r \right)^{\frac{1}{q'}} \\
&+ \|b_m\|_{L^q-L^p} \|\nabla u_m\|_{L^\infty} \left(\int_s^{s'} \frac{1}{(r-s)^{\frac{dq'}{\alpha p}}}\,\mathrm{d}r \right)^{\frac{1}{q'}}.
\end{aligned}$$

Notice now that

$$q'\left(\xi + \frac{d}{\alpha p}\right) < q'\left(\frac{\gamma}{\alpha} + \frac{1}{\alpha} + \frac{d}{\alpha p}\right) = 1,$$

which concludes the proof of (5.4.9).

Chapter 6

Weak Error on the densities for the Euler scheme of stable additive SDEs with Besov drift

This chapter is based on a work in progress with Stéphane Menozzi and Elena Issoglio. Therein, we are interested in the Euler-Maruyama dicretization of the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d,$$

where Z_t is a symmetric isotropic d-dimensional α-stable process, $\alpha \in (1,2)$ and the drift $b \in L^r\left([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d)\right)$, $\beta < 0$ is distributional and the parameters satisfy some constraints which guarantee weak-well posedness. Defining an appropriate Euler scheme, we show that, denoting $\gamma := \alpha + 2\beta - d/p - \alpha/r - 1 > 0$, the weak error on densities related to this discretization converges at the rate $(\gamma - \varepsilon)/\alpha$ for any $\varepsilon \in (0, \gamma)$.

6.1 Introduction

For a fixed finite time horizon $T > 0$, we are interested in the Euler-Maruyama dicretization of the *formal* SDE

$$X_t = x + \int_0^t b(s, X_s)\,\mathrm{d}s + \mathrm{d}Z_s, \qquad \forall t \in [0,T], \tag{6.1.1}$$

where Z_t is a symmetric isotropic d-dimensional α-stable process, $\alpha \in (1,2)$ and $b \in L^r\left([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d)\right)$, $\beta < 0$. In this pure-jump setting, it was established in [CdRM22a] that well-posedness of the generalized martingale problem holds for the generator formally associated with (6.1.1) under the condition

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2\right) \qquad \beta \in \left(\frac{1-\alpha+\frac{d}{p}+\frac{\alpha}{r}}{2}, 0\right), \tag{6.1.2}$$

which we assume to hold throughout this paper. Density estimates on the time marginals of (6.1.1) were obtained in [Fit23].

The goal of this paper is to prove a convergence rate for the weak error on densities associated with an *appropriate* Euler scheme for (6.1.1). The proof consists in approaching $b(\cdot,\cdot)$ with a sequence $(\mathfrak{b}(\cdot,\cdot,h))_{h\geq 0}$ of bounded Hölder functions, where the mollification parameter h is also the time step of the scheme.

6.1.1 Definition of the scheme

To introduce the scheme associated with the formal previous SDE (6.1.1), one first needs to recall that the precise meaning to be given to the SDE, following [CdRJM22] in the pure-jump setting, inspired by [DD16] in the Brownian setting, is:

$$X_t = x + \int_0^t \mathfrak{b}(s, X_s, \mathrm{d}s) + Z_t, \tag{6.1.3}$$

where for all $(s,z) \in [0,T] \times \mathbb{R}^d, h > 0$,

$$\mathfrak{b}(s,z,h) := \int_s^{s+h} \int b(u,y) p_\alpha(u-s, z-y)\,\mathrm{d}y\,\mathrm{d}u = \int_s^{s+h} P^\alpha_{u-s} b(u,z)\,\mathrm{d}u, \tag{6.1.4}$$

$p_\alpha(v,\cdot)$ denoting the density of the α-stable driving noise $(Z_v)_{v\geq 0}$ at time v and P^α the associated semi-group. The integral in (6.1.3) is intended as a Young integral obtained by passing to the limit in a suitable procedure aimed at reconstructing the drift (see again [CdRJM22]). The resulting drift in (6.1.3) is, *per se*, a Dirichlet process (as it had already been indicated in the literature, see e.g. [ABM20] and references therein). Importantly, the dynamics in (6.1.3) also naturally provides a corresponding approximation scheme to be analyzed. Note that, in order to give a precise meaning to the integral appearing in (6.1.3), we need the following condition:

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2 \right) \qquad \beta \in \left(\frac{1-\alpha+\frac{2d}{p}+\frac{2\alpha}{r}}{2}, 0 \right), \tag{6.1.5}$$

which is more stringent than (6.1.2). Interestingly enough, this condition does not appear elsewhere in the present work since we only consider the time marginals of the process.

We will use a discretization scheme with n time steps over $[0,T]$, with constant step size $h := T/n$. For the rest of this paper, we denote, $\forall k \in \{0,...,n\}, t_k := kh$ and $\forall s > 0, \tau_s^h := h\lfloor \frac{s}{h} \rfloor \in (s-h, s]$, which is the last grid point before time s. Namely, if $s \in [t_k, t_{k+1}), \tau_s^h = t_k$.

We can now define the related Euler scheme X^h, starting from $X_0^h = x$, on the time grid as

$$X^h_{t_{i+1}} = X^h_{t_i} + \mathfrak{b}(t_i, X^h_{t_i}, h) + Z_{t_{i+1}} - Z_{t_i}. \tag{6.1.6}$$

We have precisely plugged the expression (6.1.4), which served to define the limit dynamics (6.1.3) for the SDE, with a time argument corresponding to the chosen time step.

Set now for $(s,z) \in [0,T]\backslash(kh)_{k\in\{0,\cdots,n\}} \times \mathbb{R}^d$,

$$\mathfrak{b}_h(s,z) := P^\alpha_{s-\tau^h_s} b(s,z). \tag{6.1.7}$$

Observe from that definition that, on any time step, the drift also writes as

$$\mathfrak{b}(t_i, X^h_{t_i}, h) = \mathbb{E}[\mathfrak{b}_h(U_i, X^h_{t_i}) | X^h_{t_i}] h = \int_{t_i}^{t_{i+1}} \mathfrak{b}_h(u, X^h_{t_i})\,\mathrm{d}u, \tag{6.1.8}$$

where the $(U_k)_{k\in\mathbb{N}}$ are independent random variables, independent as well from the driving noise, s.t. $U_k \overset{(\text{law})}{=} \mathcal{U}([t_k, t_{k+1}])$, i.e. U_k is uniform on the time interval $[t_k, t_{k+1}]$.

From a practical viewpoint, the above time and spatial expectations will anyhow have to be approximated if one was to fully implement this discretization. These computations are however case-dependent. We mention that a usual way to spare one of these approximations consists in randomizing the time, namely this amounts to consider $\tilde{\mathfrak{b}}(t_i, X^h_{t_i}, h) := \mathfrak{b}_h(U_i, X^h_{\tau^h_s})$. This approach was successfully carried out for Lebesgue drifts (see [JM24b], [FJM24]) and also allowed in the spatial Hölder setting to achieve the somehow expected convergence rates without any requirements on the time regularity (see [FM24]).

Anyhow, in the current singular setting it seems difficult to benefit from such an effect in the sense that without any additional time integration we do not have controls on the approximate drift norm. This can be seen e.g. in (6.2.5) below or in the proof of the sensitivity analysis involving the local transitions (see proof of control (6.3.11)).

The representation (6.1.4) naturally suggests to extend the dynamics of the scheme in continuous time as follows

$$X_t^h = X_{\tau_t^h}^h + \mathfrak{b}(\tau_t^h, X_{\tau_t^h}^h, t - \tau_t^h) + Z_t - Z_{\tau_t^h} = X_{\tau_t^h}^h + \int_{\tau_t^h}^t \mathfrak{b}_h(s, X_{\tau_t^h}^h)\,\mathrm{d}s + Z_t - Z_{\tau_t^h}, \tag{6.1.9}$$

which gives an extension in integral form which is similar to the dynamics of Euler schemes involving non-singular drifts, i.e. it is an Itô type process and the approximate drift appears through a usual time integral.

6.1.2 Euler Scheme - State of the art

Estimates on the weak error involving a suitably smooth test function have been studied for a long time. In the Brownian setting, we can mention, the seminal works of [TT90] (smooth coefficients and test function) and [MP91] (non-degeneracy and Hölder coefficients). Going to density (i.e. taking a dirac mass as test function) requires some additional non-degeneracy of the noise. We can refer to see [**?**] in a suitable Hörmander setting and [KM02] in a non-degenerate case, which deal with smooth coefficients as well as [KM10] for stable-driven SDEs with smooth coefficients. The key tool to derive these results resides in studying the smoothness of the backward Kolmogorov equation with the corresponding test function.

When the coefficients are smooth, the aforementioned works prove that the weak error rate of the Euler scheme is of order 1 with respect to the time step h. When the drift and the (possibly non-trivial) diffusion coefficients are η-Hölder, in the brownian setting, the expected rate falls to $h^{\frac{\eta}{2}}$ (see [MP91] for smooth test functions and [KM17] for the error on densities). To the best of the authors' knowledge, this has not yet been proven in the pure-jump setting $\alpha < 2$, but an associated rate of order η/α would be expected.

For a more detailed review of those topics, we refer to the introduction of [FM24].

Recently, a series of works considered the Euler approximation of SDEs with stable additive noise and low-regularity drifts and a randomization in the time variable for the scheme. The first work in this direction goes back to [BJ22], which addressed the case of a Brownian SDE with bounded drift, achieving a convergence rate of order $1/2$ up to a logarithmic factor for the total variation error. The ideas introduced therein have been generalised in [JM24a] in order to handle Lebesgue drifts in $L_t^q - L_x^p$ under the Krylov-Röckner condition $2/q + d/p < 1$. The achieved rate on densities then writes $(1 - 2/q - d/p)/2$, which corresponds to the margin in the Krylov-Röckner condition multiplied by the self-similarity index of the noise. This has been extended to the strictly stable case in [FJM24] under the condition $\alpha/q + d/p < \alpha - 1$, achieving the rate $(\alpha - 1 - \alpha/q - d/p)/\alpha$, although in this setting, the above condition only ensures *weak* well-posedness of the underlying SDE (see [XZ20] for the conditions leading to strong well-posedness). Eventually, in a Hölder setting, it was derived in [FM24] that for bounded η-Hölder (in space) coefficients with $\alpha \in (1, 2]$, the convergence rate writes $(\alpha + \eta - 1)/\alpha$. Keeping in mind that weak well-posedness holds for $\alpha + \eta - 1 > 0$ (see [CZZ21]), this rate again corresponds to the margin multiplied by the self-similarity index of the noise. All those works rely on first deriving heat kernel estimates for the diffusion and the scheme in order to bypass the lack of regularity of the drift. In the present work, we manage to apply this approach to handle Besov drifts through the previously introduced scheme, achieving, up to some $\varepsilon > 0$ (which is intrisic to the dsitributional setting), a rate which corresponds to the margin appearing in the condition required for weak well-posedness multiplied by the self-similarity index of the noise, thus emphasizing the robustness of the approach.

Nevertheless, another approach has proven fruitful when handling SDEs with (possibly fractional) brownian noise, mainly to derive strong error rates, leading, surprisingly, to better convergence rates using the stochastic sewing lemma (see [Lê20]). In the Krylov-Röckner setting, the strong error rate derived in [LL21] is $1/2$ up to a logarithmic factor. For the weak error, an approach involving the stochastic sewing lemma has been successfully applied by [Hol24] for Hölder drifts, leading to the rate $(\eta + 1)/2$, up to some $\varepsilon > 0$ (which is intrisic to the sewing lemma), thus achieving similar rates to those discussed in the previous paragraph in an analog setting.

Let us also mention [?] in a brownian scalar setting, who derived a convergence rate for the strong error with a distributional drift. The results of the current paper are, to the best of our knowledge, the first ones concerning multi-dimensional SDEs with distributional drifts in the strictly stable setting.

6.1.3 Driving noise and related density properties

Let us denote by $\mathcal{L}^\alpha$ the generator of the driving noise Z. When $\alpha \in (1,2)$, in whole generality, the generator of a symmetric stable process writes, $\forall \phi \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$ (smooth compactly supported functions),

$$\begin{aligned}\mathcal{L}^\alpha \phi(x) &= \text{p.v.} \int_{\mathbb{R}^d} [\phi(x+z)-\phi(x)]\, \nu(\,\mathrm{d}z)\\ &= \text{p.v.} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} [\phi(x+\rho\xi)-\phi(x)]\, \mu(\,\mathrm{d}\xi) \frac{\mathrm{d}\rho}{\rho^{1+\alpha}}\end{aligned}$$

(see [Sat99] for the polar decomposition of the stable Lévy measure) where μ is a symmetric measure on the unit sphere $\mathbb{S}^{d-1}$. We will here restrict to the case where $\mu = m$ the Lebesgue measure on the sphere but it is very likely that the analysis below can be extended to the case where μ is symmetric and $\exists \kappa \geq 1 : \forall \lambda \in \mathbb{R}^d$,

$$C^{-1} m(\,\mathrm{d}\xi) \leq \mu(\,\mathrm{d}\xi) \leq C m(\,\mathrm{d}\xi),$$

i.e. it is equivalent to the Lebesgue measure on the sphere. Indeed, in that setting Watanabe (see [Wat07], Theorem 1.5) and Kolokoltsov ([Kol00], Propositions 2.1–2.5) showed that if $C^{-1}m(\,\mathrm{d}\xi) \leq \mu(\,\mathrm{d}\xi) \leq Cm(\,\mathrm{d}\xi)$, the following estimates hold: denoting $p_\alpha(v,\cdot)$ the density of the noise at time v, there exists a constant C depending only on α, d, s.t. $\forall v \in \mathbb{R}_+^*, z \in \mathbb{R}^d$,

$$C^{-1} v^{-\frac{d}{\alpha}} \left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \leq p_\alpha(v,z) \leq C v^{-\frac{d}{\alpha}} \left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}. \tag{6.1.10}$$

Note that, additionally to the previous non-degeneracy condition, in order to have the estimates on the derivatives of p_α appearing in Lemma 6.2, some smoothness is required on the Lebesgue density of μ.

On the other hand let us mention that the sole non-degeneracy condition

$$\kappa^{-1}|\lambda|^\alpha \leq \int_{\mathbb{S}^{d-1}} |\lambda \cdot \xi|^\alpha \mu(\,\mathrm{d}\xi) \leq \kappa |\lambda|^\alpha,$$

does not allow to derive *global* heat kernel estimates for the noise density. In [Wat07], Watanabe investigates the behavior of the density of an α-stable process in terms of properties fulfilled by the support of its spectral measure μ. From this work, we know that whenever the measure μ is not equivalent to the Lebesgue measure m on the unit sphere, accurate estimates on the density of the stable process can be delicate to obtain.

Let us now introduce

$$\bar{p}_\alpha(v,z) := C_\alpha v^{-\frac{d}{\alpha}} \left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)} \qquad v>0, z \in \mathbb{R}^d, \tag{6.1.11}$$

where C_α is chosen so that $\forall v>0, \int \bar{p}_\alpha(v,y)\,\mathrm{d}y = 1$. Observe as well that from the definition in (6.1.11) we readily have the following important properties:

Lemma 6.1 (Convolution properties and spatial moment for $\bar{p}_\alpha$)**.**

- *(Approximate) convolution property: there exists* $\mathfrak{c} \geq 1$ *s.t. for all* $(u,v) \in (\mathbb{R}_+^*)^2$, $(x,y) \in (\mathbb{R}^d)^2$,

$$\int_{\mathbb{R}^d} \bar{p}_\alpha(u,z-x)\bar{p}_\alpha(v,y-z)\,\mathrm{d}z \leq \mathfrak{c}\bar{p}_\alpha(u+v,y-x). \tag{6.1.12}$$

- *Time-scale for the spatial moments: for all* $0 \le \delta < \alpha, \alpha \in (1,2)$ *and for all* $\delta \ge 0$ *if* $\alpha = 2$*, there exists* $C_{\alpha,\delta}$ *s.t.*

$$\int_{\mathbb{R}^d} |z|^\delta \bar{p}_\alpha(v,z)\,\mathrm{d}z \le C_{\alpha,\delta} v^{\frac{\delta}{\alpha}}. \tag{6.1.13}$$

From now on, we will use assume w.l.o.g that $0 < T < 1$. The results below can be extended to an abitrary fixed $T > 0$ through a simple iteration procedure. For two quantities A and B the symbol $A \lesssim B$ whenever there exists a constant $C := C(d,b,\alpha)$ s.t. $A \le CB$. Namely,

$$A \lesssim B \Longleftrightarrow \exists C := C(d,b,\alpha),\ A \le CB. \tag{6.1.14}$$

We will also use the notation $A \asymp B$ whenever $A \lesssim B$ and $B \lesssim A$. Also, for any $\ell \in [1,\infty]$, ℓ' will always denote its conjugate exponent, i.e. $\frac{1}{\ell} + \frac{1}{\ell'} = 1$.

Lemma 6.2 (Stable sensitivities - Estimates on the α-stable kernel)**.** *For each multi-index* ζ *with length* $|\zeta| \le 2$*, and for all* $0 < u \le u' < +\infty$*,* $(z,z') \in (\mathbb{R}^d)^2$*,*

- *Spatial derivatives: for all* $\delta \in \{0,1\}$*,*

$$\left|\partial_u^\delta \nabla_z^\zeta p_\alpha(u,z)\right| \lesssim \frac{1}{u^{\delta + \frac{|\zeta|}{\alpha}}} \bar{p}_\alpha(u,z). \tag{6.1.15}$$

- *Time Hölder regularity: for all* $\theta \in [0,1]$*,*

$$\left|\partial_u^\delta \nabla_z^\zeta p_\alpha(u,z) - \partial_u^\delta \nabla_z^\zeta p_\alpha(u',z)\right| \lesssim \frac{|u-u'|^\theta}{u^{\delta+\theta+\frac{|\zeta|}{\alpha}}} \left(\bar{p}_\alpha(u,z) + \bar{p}_\alpha(u',z)\right). \tag{6.1.16}$$

- *Spatial Hölder regularity: for all* $\theta \in [0,1]$*,*

$$\left|\partial_u^\delta \nabla_z^\zeta p_\alpha(u,z) - \partial_u^\delta \nabla_z^\zeta p_\alpha(u,z')\right| \lesssim \left(\frac{|z-z'|^\theta}{u^{\frac{\theta}{\alpha}}} \wedge 1\right) \frac{1}{u^{\delta+\frac{|\zeta|}{\alpha}}} \left(\bar{p}_\alpha(u,z) + \bar{p}_\alpha(u,z')\right). \tag{6.1.17}$$

- *Convolution*
  $\forall x, y \in (\mathbb{R}^d)^2$*,* $\forall 0 \le s \le u \le t$*,* $\forall \ell \ge 1$*,*

$$\|\bar{p}_\alpha(t-u,\cdot - y)\bar{p}_\alpha(u-s, x - \cdot)\|_{L^{\ell'}} \lesssim \left[\frac{1}{(t-u)^{\frac{d}{\alpha\ell}}} + \frac{1}{(u-s)^{\frac{d}{\alpha\ell}}}\right] \bar{p}_\alpha(t-s, x-y). \tag{6.1.18}$$

- *Besov norm*
  For all $\vartheta \in \mathbb{R}_+$*,* $(\ell, m) \in [1,+\infty]^2$

$$\|p_\alpha(t,\cdot)\|_{\mathbb{B}^\vartheta_{\ell,m}} \lesssim t^{-\frac{\vartheta}{\alpha} - \frac{d}{\alpha\ell'}} \tag{6.1.19}$$

The controls of Lemma 6.2 are somehow standard. A proof can be found e.g. in [FJM24]. Importantly, note that those controls are valid both for $\bar{p}_\alpha$ and p_α.

6.1.4 Main results

We first give the following bounds concerning the densities of the SDE (6.1.1) and its associated scheme (6.1.9), assuming (6.1.2) holds. Let us state the following controls:

Proposition 6.1 (Heat kernel estimates for the densities)**.**

- *Heat kernel bound for the density of the Euler scheme: for all $\rho \in (-\beta, \gamma - \beta)$ there exists $C = C(d,b,\alpha)$ such that for all $(x,y,y') \in (\mathbb{R}^d)^3$, $t>0$,*

$$\Gamma^h(0,x,t,y) \leq C\bar{p}_\alpha(t, y-x), \tag{6.1.20}$$

$$|\Gamma^h(0,x,t,y') - \Gamma^h(0,x,t,y)| \leq C\frac{|y-y'|^\rho}{t^{\frac{\rho}{\alpha}}}\bar{p}_\alpha(t,y-x). \tag{6.1.21}$$

Consequently, in terms of Besov spaces (see Subsection 6.2.1 below for a precise definition)

$$\left\| \frac{\Gamma^h(0,x,t,\cdot)}{\bar{p}_\alpha(t,\cdot - x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \leq C(1 + t^{-\frac{\rho}{\alpha}}), \tag{6.1.22}$$

- *Heat kernel and Sensitivity bounds for the density of the SDE: for all $\rho \in (-\beta, \gamma - \beta)$, there exists $C = C(d,b,\alpha)$ such that for all $(x,y,y') \in (\mathbb{R}^d)^3$, $t \in (0,T]$,*

$$\Gamma(0,x,t,y) \leq C\bar{p}_\alpha(t,y-x), \tag{6.1.23}$$

$$|\Gamma(0,x,t,y') - \Gamma(0,x,t,y)| \leq C\frac{|y-y'|^\rho}{t^{\frac{\rho}{\alpha}}}\bar{p}_\alpha(t,y-x). \tag{6.1.24}$$

Consequently, in terms of Besov spaces,

$$\left\| \frac{\Gamma(0,x,t,\cdot)}{\bar{p}_\alpha(t,\cdot - x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \leq C(1 + t^{-\frac{\rho}{\alpha}}). \tag{6.1.25}$$

Moreover, it holds that for all $\varepsilon > 0, t' \in (t,T]$ such that $|t-t'| \leq t/2$,

$$\left\| \frac{\Gamma(0,x,t,\cdot) - \Gamma(0,x,t',\cdot)}{\bar{p}_\alpha(t',\cdot - x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \leq C\frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon+\rho}{\alpha}}}. \tag{6.1.26}$$

The bound (6.1.25) was obtained in [Fit23], (6.1.26) would follow from the same lines but is detailed for self-containedness in Appendix 6.6. The main result of the paper is the following theorem:

Theorem 6.1 (Convergence Rate for the stable-driven Euler scheme with Besov drift)**.** *Denoting by Γ and Γ^h the respective densities of the SDE* (6.1.1) *and its Euler scheme defined in* (6.1.9)*, for all $\varepsilon > 0, \rho > -\beta$ there exists a constant $C := C(d,b,\alpha,T,\varepsilon,\rho) < \infty$ s.t. for all $h = T/n$ with $n \in \mathbb{N}^*$, and all $t \in (0,T]$, $x,y \in \mathbb{R}^d$,*

$$|\Gamma^h(0,x,t,y) - \Gamma(0,x,t,y)| \leq Ch^{\frac{\gamma-\varepsilon}{\alpha}}p_\alpha(t,y-x), \tag{6.1.27}$$

where $\gamma = 2\beta - \frac{d}{p} - \frac{\alpha}{r} + \alpha - 1 > 0$ is the "gap to singularity" in the Besov case.

6.2 About Besov spaces and related controls on the mollified drift

6.2.1 Definition and related properties

We first recall that denoting by $\mathcal{S}'(\mathbb{R}^d)$ the dual space of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, for $\ell, m \in (0,+\infty]$, $\vartheta \in \mathbb{R}$, the Besov space $\mathbb{B}^\vartheta_{\ell,m}$ can be characterized with

$$\mathbb{B}^\vartheta_{\ell,m} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f\|_{\mathbb{B}^\vartheta_{\ell,m}} := \|\mathcal{F}^{-1}(\phi\mathcal{F}(f))\|_{L^\ell} + \mathcal{T}^\vartheta_{\ell,m}(f) < \infty \right\},$$

$$\mathcal{T}^\vartheta_{\ell,m}(f) := \begin{cases} \left(\displaystyle\int_0^1 \frac{\mathrm{d}v}{v} v^{(n-\vartheta/\alpha)m} \|\partial_v^n \tilde{p}_\alpha(v,\cdot) \star f\|^m_{L^\ell} \right)^{\frac{1}{m}} & \text{for } 1 \leq m < \infty, \\ \sup\limits_{v \in (0,1]} \left\{ v^{(n-\vartheta/\alpha)} \|\partial_v^n \tilde{p}_\alpha(v,\cdot) \star f\|_{L^\ell} \right\} & \text{for } m = \infty, \end{cases} \tag{6.2.1}$$

with $\star$ denoting the spatial convolution, n being any non-negative integer (strictly) greater than ϑ/α, the function ϕ being a $\mathcal{C}_0^\infty$-function (infinitely differentiable function with compact support) such that $\phi(0) \neq 0$, and $\tilde{p}_\alpha(v,\cdot)$ denoting the density function at time v of the d-dimensional isotropic stable process.

For our analysis we will rely on the following important inequalities:

- Product rule: for all $\vartheta \in \mathbb{R}$, $(\ell, m) \in [1, +\infty]^2$ and $\rho > \max\Big(\vartheta, -\vartheta\Big)$, $\forall (f,g) \in \mathbb{B}^\rho_{\infty,\infty} \times \mathbb{B}^\vartheta_{\ell,m}$,

$$\|f \cdot g\|_{\mathbb{B}^\vartheta_{\ell,m}} \leq \|f\|_{\mathbb{B}^\rho_{\infty,\infty}} \|g\|_{\mathbb{B}^\vartheta_{\ell,m}}. \tag{6.2.2}$$

  See Theorem 4.37 in [Saw18] for a proof.

- Duality inequality: for all $\vartheta \in \mathbb{R}$, $(\ell, m) \in [1, +\infty]^2$, with m' and ℓ' respective conjugates of m and ℓ, and $(f,g) \in \mathbb{B}^\vartheta_{\ell,m} \times \mathbb{B}^{-\vartheta}_{\ell',m'}$,

$$\left|\int f(y)g(y)\mathrm{d}y\right| \leq \|f\|_{\mathbb{B}^\vartheta_{\ell,m}} \|g\|_{\mathbb{B}^{-\vartheta}_{\ell',m'}}. \tag{6.2.3}$$

  See Proposition 6.6 in [LR02] for a proof.

- Young inequality: for all $\vartheta \in \mathbb{R}$, $(\ell, m) \in [1, +\infty]^2$, for any $\delta \in \mathbb{R}$ and for $(\ell_1, \ell_2) \in [1, \infty]^2$ and $(m_1, m_2) \in (0, \infty]^2$ such that

$$1 + \frac{1}{\ell} = \frac{1}{\ell_1} + \frac{1}{\ell_2} \qquad \text{and} \qquad \frac{1}{m} \leq \frac{1}{m_1} + \frac{1}{m_2},$$

  there exists C such that, for $f \in \mathbb{B}^{\vartheta-\delta}_{\ell_1,m_1}$ and $g \in \mathbb{B}^\delta_{\ell_2,m_2}$,

$$\|f \star g\|_{\mathbb{B}^\vartheta_{\ell,m}} \leq C\|f\|_{\mathbb{B}^{\vartheta-\delta}_{\ell_1,m_1}} \|g\|_{\mathbb{B}^\delta_{\ell_2,m_2}}. \tag{6.2.4}$$

  See Theorem 2.2 in [KS21] for a proof (or [Saw18]).

6.2.2 Controls for the mollified drift

Let us now state some important properties of the chosen approximate drift.

Lemma 6.3. *[Useful bounds for $\mathfrak{b}_h$] There exists $C \geq 1$ s.t. for all $h > 0$ and all $(s,z) \in [0,T] \times \mathbb{R}^d$, $s \neq \tau^h_s$,*

- *Pointwise control*

$$|\mathfrak{b}_h(s,z)| \leq C(s - \tau^h_s)^{-\frac{d}{\alpha p} + \frac{\beta}{\alpha}} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}. \tag{6.2.5}$$

- *Time-integrated pointwise control*

$$\left|\int_{\tau^h_s}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u\right| \leq C(s - \tau^h_s)^{\frac{\gamma}{\alpha} + \frac{1-\beta}{\alpha}} \|b\|_{L^r - \mathbb{B}^\beta_{p,q}}. \tag{6.2.6}$$

- *Spatial Hölder modulus of the integrated drift*
  For all $(z,z') \in (\mathbb{R}^d)^2$, $\zeta \in [-\beta, \alpha - 1 + \beta - d/p - \alpha/r)$,

$$\left|\int_{\tau^h_s}^s \Big(\mathfrak{b}_h(u,z) - \mathfrak{b}_h(u,z')\Big)\,\mathrm{d}u\right| \leq C|z - z'|^\zeta h^{\frac{\gamma}{\alpha} + \frac{1-\beta-\zeta}{\alpha}} \|b\|_{L^r - \mathbb{B}^\beta_{p,q}}. \tag{6.2.7}$$

- *Besov norm of the mollified drift*

$$\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \leq \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}. \tag{6.2.8}$$

Proof. Let us recall the definition of $\mathfrak{b}_h$:

$$\mathfrak{b}_h(s,\cdot) := P^{\alpha}_{s-\tau^h_s} b(s,\cdot) = p_\alpha(s-\tau^h_s,\cdot) \star b(s,\cdot). \tag{6.2.9}$$

Using the duality inequality (6.2.3) and the estimate on the besov norm of p_α, (6.1.19), we immediately obtain (6.2.5). Using the same arguments for the integrand, we have

$$\begin{aligned}\left|\int_{\tau^h_s}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u\right| \le \int_{\tau^h_s}^s \|p_\alpha(u-\tau^h_s,\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \|b(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}\,\mathrm{d}u \le \int_{\tau^h_s}^s (u-\tau^h_s)^{\frac{\beta}{\alpha}-\frac{d}{p\alpha}} \|b(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}\,\mathrm{d}u \qquad (6.2.10)\\ \le \|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} \Big(\int_{\tau^h_s}^s \frac{\mathrm{d}u}{(u-\tau^h_s)^{r'(\frac{-\beta}{\alpha}+\frac{d}{p\alpha})}}\Big)^{\frac{1}{r'}} \le C\|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} (s-\tau^h_s)^{1-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}}.\end{aligned}$$

This proves (6.2.6). Similarly, for $\zeta \in [-\beta, \alpha-1+\beta-d/p-\alpha/r)$, using this time the Young inequality (6.2.4), we have

$$\begin{aligned}&\left|\int_{\tau^h_s}^s \big(\mathfrak{b}_h(u,z)-\mathfrak{b}_h(u,z')\big)\,\mathrm{d}u\right|\\ \le& |z-z'|^{\zeta} \int_{\tau^h_s}^s \big\|\mathfrak{b}_h(u,\cdot)\big\|_{\mathbb{B}^{\zeta}_{\infty,\infty}}\,\mathrm{d}u\\ \le& |z-z'|^{\zeta} \int_{\tau^h_s}^s \|p_\alpha(u-\tau^h_s,\cdot)\|_{\mathbb{B}^{\zeta-\beta}_{p',q'}} \|b(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}\,\mathrm{d}u \le |z-z'|^{\zeta}\int_{\tau^h_s}^s (u-\tau^h_s)^{\frac{\beta}{\alpha}-\frac{\zeta}{\alpha}-\frac{d}{p\alpha}} \|b(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}\,\mathrm{d}u\\ \le& |z-z'|^{\zeta}\|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} \Big(\int_{\tau^h_s}^s \frac{\mathrm{d}u}{(u-\tau^h_s)^{r'(\frac{-\beta}{\alpha}+\frac{\zeta}{\alpha}+\frac{d}{p\alpha})}}\Big)^{\frac{1}{r'}} \le C|z-z'|^{\zeta}\|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} (s-\tau^h_s)^{1-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}-\frac{\zeta}{\alpha}}.\end{aligned}$$

This proves (6.2.7). Eventually, (6.2.8) follows from the Young inequality (6.2.4) and the fact that the $\mathbb{B}^0_{1,1}$ norm of the stable kernel is uniformly bounded. □

Let us also state the following lemma, which indicates that the deviation induced by the mollified drift over a single time step can be neglected at the scale of the noise:

Lemma 6.4 (The approximate singular drift in the density of the driving noise)**.** *There exists C s.t. for $0 \le s < t \le T$ s.t. $t-s \ge s-\tau^h_s$ and $(z,z') \in (\mathbb{R}^d)^2$, for all k, $|k| \le 2$,*

$$\left|\nabla^k_z p_\alpha\left(t-s, z-\int_{\tau^h_s}^s \mathfrak{b}_h(u,z')\,\mathrm{d}u\right)\right| \le \frac{C}{(t-s)^{\frac{|k|}{\alpha}}}\bar{p}_\alpha(t-s,z). \tag{6.2.11}$$

Proof. Write from (6.1.15),

$$\begin{aligned}\left|\nabla^k_z p_\alpha\left(t-s, z-\int_{\tau^h_s}^s \mathfrak{b}_h(u,z')\,\mathrm{d}u\right)\right| \le& \frac{C}{(t-s)^{\frac{|k|}{\alpha}}}\bar{p}_\alpha\left(t-s, z-\int_{\tau^h_s}^s \mathfrak{b}_h(u,z')\,\mathrm{d}u\right)\\ \le& \frac{C}{(t-s)^{\frac{|k|}{\alpha}+\frac{d}{\alpha}}} \frac{1}{\left(2-\frac{|\int_{\tau^h_s}^s \mathfrak{b}_h(u,z')\,\mathrm{d}u|}{(t-s)^{\frac{1}{\alpha}}}+\frac{|z|}{(t-s)^{\frac{1}{\alpha}}}\right)^{d+\alpha}}\\ \le& \frac{C}{(t-s)^{\frac{|k|}{\alpha}+\frac{d}{\alpha}}} \frac{1}{\left(2-\frac{(s-\tau^h_s)^{\frac{1}{\alpha}+\frac{\gamma-\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}}}+\frac{|z|}{(t-s)^{\frac{1}{\alpha}}}\right)^{d+\alpha}}\\ \le& \frac{C}{(t-s)^{\frac{|k|}{\alpha}+\frac{d}{\alpha}}} \frac{1}{\left(1+\frac{|z|}{(t-s)^{\frac{1}{\alpha}}}\right)^{d+\alpha}} \le \frac{C}{(t-s)^{\frac{|k|}{\alpha}}}\bar{p}_\alpha(t-s,z),\end{aligned}$$

for h sufficiently small, using (6.2.6) for the last but one inequality and up to a modification of C from line to line. This proves (6.2.11). □

6.3 Tools for the proof of Theorem 6.1

6.3.1 Duhamel expansion of the densities

Proposition 6.2 (Duhamel representations for the densities of the SDE and the Euler scheme)**.** *The density $\Gamma(s,x,t,\cdot)$ of the unique weak solution to Equation (6.1.1) starting from x at time $s \in [0,T)$ admits the following Duhamel representation: for all $t \in (s,T]$, $y \in \mathbb{R}^d$,*

$$\Gamma(s,x,t,y) = p_\alpha(t-s,y-x) - \int_s^t \mathbb{E}_{s,x}\left[b(r,X_r)\cdot\nabla_y p_\alpha(t-r,y-X_r)\right] \mathrm{d}r. \tag{6.3.1}$$

Similarly, for $k \in [\![0,n-1]\!]$, $t \in (t_k,T]$, the density of X_t^h admits, conditionally to $X_{t_k}^h = x$, a transition density $\Gamma^h(t_k,x,t,\cdot)$, which enjoys a Duhamel type representation: for all $y \in \mathbb{R}^d$,

$$\Gamma^h(t_k,x,t,y) = p_\alpha(t-t_k,y-x) - \int_{t_k}^t \mathbb{E}_{t_k,x}\left[\mathfrak{b}_h(r,X_{\tau_r^h}^h)\cdot\nabla_y p_\alpha(t-r,y-X_r^h)\right] \mathrm{d}r. \tag{6.3.2}$$

Proof. It is plain to prove, from (6.2.5), that, for any $t>0$, the above scheme admits a density which we denote Γ^h. Indeed, the scheme can be viewed as the Euler scheme associated with the solution to the SDE

$$\mathrm{d}X_t^{D,h} = \mathfrak{b}_h(t,X_t^{D,h})\,\mathrm{d}t + \mathrm{d}Z_t, \tag{6.3.3}$$

which has $L^r - L^\infty$ drift, albeit with $L^r - L^\infty$ norm depending on h. As a consequence of [FJM24], it follows that Γ^h enjoys the Duhamel-type representation (6.3.2).

As for (6.3.1), it is a consequence of the mollification procedure considered in [Fit23] (see Section 4 therein).

□

6.3.2 Auxiliary estimates

Estimates for the density of stable processes

The following estimates will be needed for the error analysis below.

Lemma 6.5 (Besov estimates for $\bar{p}_\alpha$)**.** *Let $\beta < 0$.*

- $\forall 0 < s < t$, $\forall (x,y) \in (\mathbb{R}^d)^2$, $\forall \zeta \in (-\beta, 1]$, $\forall k \in \{0,1\}$,

$$\|\bar{p}_\alpha(s,x-\cdot)\nabla_y^k p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \lesssim \frac{\bar{p}_\alpha(t,x-y)}{(t-s)^{\frac{k}{\alpha}}} t^{\frac{\beta}{\alpha}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]. \tag{6.3.4}$$

- $\forall 0 < s < t$, $\forall (x,y) \in (\mathbb{R}^d)^2$,$j \in \{0,1\}$, $\forall \zeta \in (-\beta + j\gamma, 1]$, $\forall k \in \{0,1\}$,

$$\|\Gamma(0,x,s,\cdot)\nabla_y^k p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta+j\gamma}_{p',q'}} \lesssim \frac{\bar{p}_\alpha(t,x-y)}{(t-s)^{\frac{k}{\alpha}}} t^{\frac{\beta-j\gamma}{\alpha}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]. \tag{6.3.5}$$

- $\forall 0 < s < t$, $\forall (x,y,w) \in (\mathbb{R}^d)^3$, $\forall \zeta \in (-\beta, 1]$,

$$\left\|\bar{p}_\alpha(s,x-\cdot)\left[\frac{\nabla p_\alpha(t-s,w-\cdot)}{\bar{p}_\alpha(t,w-x)} - \frac{\nabla p_\alpha(t-s,y-\cdot)}{\bar{p}_\alpha(t,y-x)}\right]\right\|_{\mathbb{B}^{-\beta}_{p',q'}}$$
$$\lesssim \frac{|w-y|^\zeta}{(t-s)^{\frac{\zeta+1}{\alpha}}} t^{\frac{\beta}{\alpha}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]. \tag{6.3.6}$$

- $\forall 0<s<t$, $\forall (x,y,y')\in(\mathbb{R}^d)^3$, *s.t.* $|y'-y|\le t^{\frac{1}{\alpha}}$, $j\in\{0,1\}$, $\forall \zeta,\rho\in(-\beta+j\gamma,1]^2$, $\forall k\in\{0,1\}$,

$$\begin{aligned}&\|\Gamma(0,x,s,\cdot)(\nabla_y^k p_\alpha(t-s,y-\cdot)-\nabla_y^k p_\alpha(t-s,y'-\cdot))\|_{\mathbb{B}^{-\beta+j\gamma}_{p',q'}}\\ \lesssim&\bar p_\alpha(t,x-y)\frac{|y-y'|^\rho}{(t-s)^{\frac{k}{\alpha}+\frac{\rho}{\alpha}}}t^{\frac{\beta-j\gamma}{\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right].\end{aligned}\tag{6.3.7}$$

- $\forall h\le s\le \tau_t^h-h$, $r\in[\tau_s^h,\tau_s^h+h)$, $\forall (x,y)\in(\mathbb{R}^d)^2$, $\forall\zeta\in(-\beta,1]$, $\forall\delta\in[0,1)$,

$$\begin{aligned}&\left\|\Gamma^h(0,x,\tau_s^h,\cdot)\left[\nabla_y p_\alpha(t-s,y-\cdot)-\nabla_y p_\alpha(t-r,y-\cdot)\right]\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ \lesssim&(s-r)^\delta\frac{\bar p_\alpha(t,y-x)}{(t-s)^{\frac{1}{\alpha}+\delta}}t^{\frac{\beta}{\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right].\end{aligned}\tag{6.3.8}$$

- $\forall h\le s\le \tau_t^h-h$, $r\in[\tau_s^h,\tau_s^h+h)$, $\forall (x,y,y')\in(\mathbb{R}^d)^3$, $\forall\zeta,\rho\in(-\beta,1]^2$, $|y-y'|\le t^{\frac{1}{\alpha}}$, $\forall\delta\in[0,1),\forall\theta\in\{0,1\}$,

$$\begin{aligned}&\left\|\bar p_\alpha(\tau_s^h,x-\cdot)\left[\partial_t^\theta\nabla_y p_\alpha(t-r,y-\cdot)-\partial_t^\theta\nabla_y p_\alpha(t-r,y'-\cdot)\right]\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ \lesssim&|y-y'|^\rho\frac{\bar p_\alpha(t,y-x)}{(t-s)^{\frac{1}{\alpha}+\theta+\frac{\rho}{\alpha}}}t^{\frac{\beta}{\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right].\end{aligned}\tag{6.3.9}$$

- $\forall h\le s\le \tau_t^h-h$, $r\in(\tau_s^h,\tau_s^h+h)$, $\forall (x,y,y')\in(\mathbb{R}^d)^3$, $\forall\zeta,\rho\in(-\beta,1]^2$, $|y-y'|\le t^{\frac{1}{\alpha}}$, $\forall\delta\in[0,1),\forall\theta\in\{0,1\}$,

$$\begin{aligned}&\left\|\Gamma^h(0,x,\tau_s^h,\cdot)\left[\partial_t^\theta\nabla_y p_\alpha(t-r,y-\cdot)-\partial_t^\theta\nabla_y p_\alpha(t-r,y'-\cdot)\right]\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ \lesssim&|y-y'|^\rho\frac{\bar p_\alpha(t,y-x)}{(t-s)^{\frac{1}{\alpha}+\theta+\frac{\rho}{\alpha}}}t^{\frac{\beta}{\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right].\end{aligned}\tag{6.3.10}$$

- $\forall h\le s\le \tau_t^h-h$, $\forall (x,y)\in(\mathbb{R}^d)^2$, $\forall\zeta\in(-\beta,1]$,

$$\begin{aligned}&\left\|\Gamma^h(0,x,\tau_s^h,\cdot)\Big(\nabla_y p_\alpha(t-\tau_s^h,y-\cdot)-\nabla_y p_\alpha(t-\tau_s^h,y-(\cdot+\int_{\tau_s^h}^s\mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))\Big)\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ \lesssim&\bar p_\alpha(t,y-x)\frac{h^{\frac{\gamma-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}}t^{\frac{\beta}{\alpha}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\right].\end{aligned}\tag{6.3.11}$$

- $\forall h\le s\le \tau_t^h-h$, $\forall (x,y,y')\in(\mathbb{R}^d)^3$, $|y-y'|\le t^{\frac{1}{\alpha}}$, $\forall\zeta,\rho\in(-\beta,1]^2,\forall\lambda\in[0,1]$,

$$\begin{aligned}&\Bigg\|\Gamma^h(0,x,\tau_s^h,\cdot)\Big(\nabla_y^2 p_\alpha(t-\tau_s^h,y-(\cdot+\lambda\int_{\tau_s^h}^s\mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))-\nabla_y^2 p_\alpha(t-\tau_s^h,y'-(\cdot+\lambda\int_{\tau_s^h}^s\mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))\Big)\\ &\times\int_{\tau_s^h}^s\mathfrak{b}_h(u,\cdot)\,\mathrm{d}u)\Bigg\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ \lesssim&|y-y'|^\rho\bar p_\alpha(t,y-x)\frac{h^{\frac{\gamma-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}}{(t-\tau_s^h)^{\frac{1}{\alpha}+\frac{\rho}{\alpha}}}t^{\frac{\beta}{\alpha}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\right].\end{aligned}\tag{6.3.12}$$

The bounds (6.3.4), (6.3.6) have been proved in Lemma 3 of [Fit23]. Equation (6.3.5) relies on the same proof as (6.3.6). The other estimates are proved in Section 6.5.1, and the approach therein would also readily give the previously mentioned bounds.

6.4 Proof of Theorem 6.1

We will proceed by controlling the $\mathbb{B}^{\rho}_{\infty,\infty}$ norm of $(\Gamma^h(0,x,t,\cdot)-\Gamma(0,x,t,\cdot))/\bar{p}_\alpha(t,\cdot-x)$ using the fact that for those exponents, it writes

$$\left\|\frac{\Gamma^h(0,x,t,\cdot)-\Gamma(0,x,t,\cdot)}{\bar{p}_\alpha(t,\cdot-x)}\right\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \asymp \left\|\frac{\Gamma^h(0,x,t,\cdot)-\Gamma(0,x,t,\cdot)}{\bar{p}_\alpha(t,\cdot-x)}\right\|_{L^\infty} + \sup_{z\neq z'\in\mathbb{R}^d} \frac{\left|\frac{(\Gamma-\Gamma^h)(0,x,t,z)}{\bar{p}_\alpha(t,z-x)} - \frac{(\Gamma-\Gamma^h)(0,x,t,z')}{\bar{p}_\alpha(t,z'-x)}\right|}{|z-z'|^\rho} \tag{6.4.1}$$

We will first control de L^∞ norm in Subsection 6.4.2 and then, using the same error decomposition, we will control the Hölder modulus in 6.4.3

6.4.1 Decomposition of the error

$$\begin{aligned}
&\Gamma^h(0,x,t,y)-\Gamma(0,x,t,y)\\
&\quad= \int_0^h \mathbb{E}_{0,x}\left[b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s) - \mathfrak{b}_h(s,x)\cdot\nabla_y p_\alpha(t-s,y-X^h_s)\right] \mathrm{d}s\\
&\qquad+ \int_h^{\tau^h_t-h} \mathbb{E}_{0,x}\left[b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s) - b(s,X_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X_{\tau^h_s})\right] \mathrm{d}s\\
&\qquad+ \int_h^{\tau^h_t-h} \mathbb{E}_{0,x}\left[b(s,X_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X_{\tau^h_s}) - \mathfrak{b}_h(s,X_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X_{\tau^h_s})\right] \mathrm{d}s\\
&\qquad+ \int_h^{\tau^h_t-h} \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s,X_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X_{\tau^h_s}) - \mathfrak{b}_h(s,X^h_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X^h_{\tau^h_s})\right] \mathrm{d}s\\
&\qquad+ \int_h^{\tau^h_t-h} \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s,X^h_{\tau^h_s})\cdot\Big(\nabla_y p_\alpha(t-s,y-X^h_{\tau^h_s}) - \nabla_y p_\alpha(t-s,y-X^h_s)\Big)\right] \mathrm{d}s\\
&\qquad+ \int_{\tau^h_t-h}^{t} \mathbb{E}_{0,x}\left[b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s) - \mathfrak{b}_h(s,X^h_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X^h_s)\right] \mathrm{d}s\\
&\quad=: \big(\Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5+\Delta_6\big)(0,x,t,y).
\end{aligned} \tag{6.4.2}$$

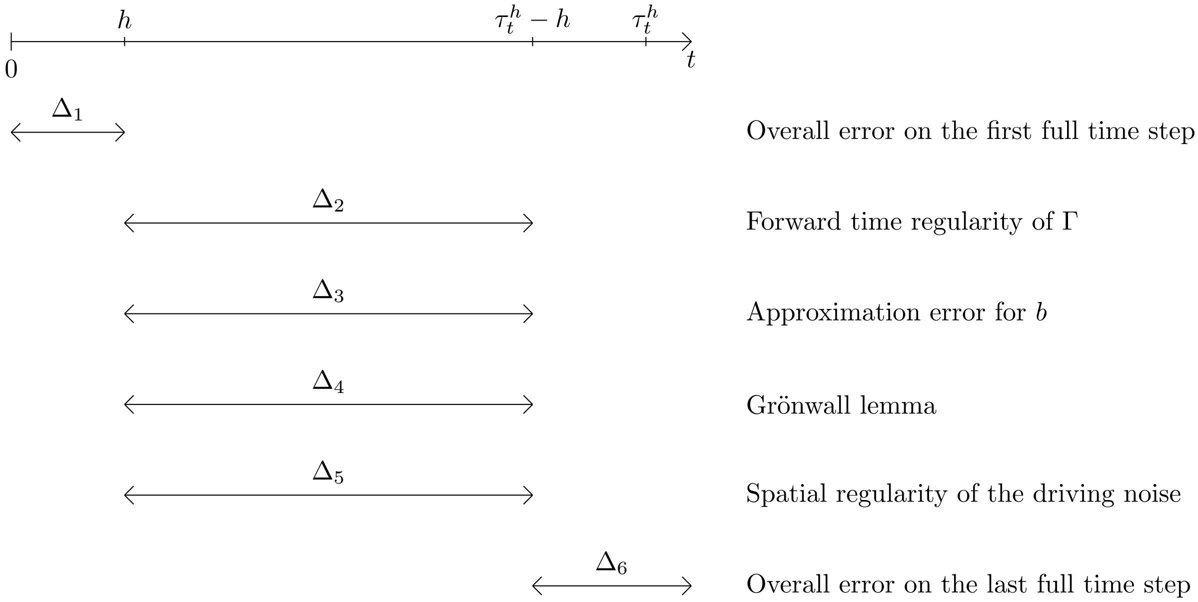


Figure 6.1: Splitting of the error

6.4.2 Control of the supremum norm

Term Δ_1: first time step. For Δ_1, we rely on the fact that we work on the first time step. Let us first expand the expectation:

$$\begin{aligned}&\Delta_1(0,x,t,y)\\ &=\int_0^h\int\Big(\Gamma(0,x,s,z)b(s,z)\cdot\nabla_y p_\alpha(t-s,y-z)-\Gamma^h(0,x,s,z)\mathfrak{b}_h(s,x)\cdot\nabla_y p_\alpha(t-s,y-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\\ &=:\Big(\Delta_{1,1}+\Delta_{1,2}\Big)(0,x,t,y).\end{aligned}$$

For $\Delta_{1,1}$, which involves the distributional b, we have to rely on the duality inequality in Besov spaces (6.2.3). Assuming w.l.o.g. that $t>2h$ so that $(t-s)\asymp t$, then using (6.3.5) (taking therein $\zeta\in(-\beta,1)$), we get

$$\begin{aligned}|\Delta_{1,1}(0,x,t,y)| &\lesssim \int_0^h \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\|\Gamma(0,x,s,\cdot)\nabla_y p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\ &\lesssim \bar{p}_\alpha(t,y-x)\int_0^h\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s\\ &\lesssim \bar{p}_\alpha(t,y-x)\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\left(\int_0^h t^{\frac{r'(\beta-1)}{\alpha}}\left[\frac{1}{s^{\frac{dr'}{\alpha p}}}+\frac{1}{t^{\frac{dr'}{\alpha p}}}\right]\frac{t^{\frac{\zeta r'}{\alpha}}}{s^{\frac{\zeta r'}{\alpha}}}\,\mathrm{d}s\right)^{\frac{1}{r'}}\\ &\lesssim \bar{p}_\alpha(t,y-x)h^{1-\frac{1}{r}}\left[h^{-\frac{d}{\alpha p}-\frac{\zeta}{\alpha}}t^{\frac{\beta-1+\zeta}{\alpha}}+h^{-\frac{\zeta}{\alpha}}t^{\frac{\beta-1+\zeta}{\alpha}-\frac{d}{\alpha p}}\right]\\ &\lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{-\frac{\beta}{\alpha}}.\end{aligned}\tag{6.4.3}$$

For $\Delta_{1,2}$, using the L^∞ bound of $\mathfrak{b}_h$, (6.2.5), (6.1.20) and (6.1.15), we get

$$\begin{aligned}
|\Delta_{1,2}(0,x,t,y)| &\lesssim \int_0^h s^{-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \frac{1}{(t-s)^{\frac{1}{\alpha}}} \int \bar{p}_\alpha(s,z-x)\bar{p}_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{1-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} t^{-\frac{1}{\alpha}} \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} h^{\frac{1}{\alpha}-\frac{\beta}{\alpha}} t^{-\frac{1}{\alpha}} \lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}, \qquad (6.4.4)
\end{aligned}$$

recalling that $h \le t$ for the last inequality. We eventually get:

$$|\Delta_1(0,x,t,y)| \lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}. \qquad (6.4.5)$$

Term Δ_2: time sensitivity of the density of the SDE. Let us turn to Δ_2. Expanding the expectation, using the product rule (6.2.2), then the duality inequality (6.2.3), the heat kernel estimate (6.1.26) and the control (6.3.4), we get for $\zeta, \rho > -\beta$,

$$\begin{aligned}
&|\Delta_2(0,x,t,y)| \qquad (6.4.6) \\
&= \left| \int_h^{\tau_t^h - h} \int [\Gamma(0,x,s,z) - \Gamma(0,x,\tau_s^h,z)] b(s,z)\cdot\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \right| \\
&\lesssim \int_h^{\tau_t^h - h} \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \left\| \frac{\Gamma(0,x,s,\cdot) - \Gamma(0,x,\tau_s^h,\cdot)}{\bar{p}_\alpha(s,\cdot - x)} \right\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|\bar{p}_\alpha(s,\cdot - x)\nabla_y p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x) \int_h^{\tau_t^h - h} \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \frac{(s-\tau_s^h)^{\frac{\gamma-\varepsilon}{\alpha}}}{s^{\frac{\gamma-\varepsilon+\rho}{\alpha}}} \frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right] \left[\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}} \right] \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}} \|b\|_{L^r(B^{\beta}_{p,q})} \Big(\int_h^{\tau_t^h - h} \frac{1}{s^{r'\frac{\gamma-\varepsilon+\rho}{\alpha}}} \frac{1}{(t-s)^{r'\frac{1}{\alpha}}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right]^{r'} \left[\frac{1}{s^{\frac{\zeta}{\alpha}}} + \frac{1}{(t-s)^{\frac{\zeta}{\alpha}}} \right]^{r'} \mathrm{d}s \Big)^{\frac{1}{r'}} \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}} \|b\|_{L^r(B^{\beta}_{p,q})} \Bigg(\int_0^t \Bigg[\frac{1}{s^{r'[\frac{\gamma-\varepsilon+\rho}{\alpha}+\frac{d}{\alpha p}+\frac{\zeta}{\alpha}]}} \frac{1}{(t-s)^{r'\frac{1}{\alpha}}} + \frac{1}{s^{r'[\frac{\gamma-\varepsilon+\rho}{\alpha}+\frac{d}{\alpha p}]}} \frac{1}{(t-s)^{r'[\frac{1}{\alpha}+\frac{\zeta}{\alpha}]}} + \\
&\qquad + \frac{1}{s^{r'[\frac{\gamma-\varepsilon+\rho}{\alpha}+\frac{\zeta}{\alpha}]}} \frac{1}{(t-s)^{r'[\frac{1}{\alpha}+\frac{\zeta}{\alpha}+\frac{d}{\alpha p}]}} + \frac{1}{s^{r'[\frac{\gamma-\varepsilon+\rho}{\alpha}]}} \frac{1}{(t-s)^{r'[\frac{1}{\alpha}+\frac{\zeta}{\alpha}+\frac{d}{\alpha p}]}} \Bigg] \mathrm{d}s \Bigg)^{\frac{1}{r'}} \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}+\frac{1}{r'}-(\frac{\gamma-\varepsilon+\rho}{\alpha}+\frac{d}{\alpha p}+\frac{\zeta}{\alpha}+\frac{1}{\alpha})} \|b\|_{L^r(B^{\beta}_{p,q})} \lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{-\frac{\beta}{\alpha}-\frac{\rho}{\alpha}+\frac{\varepsilon}{\alpha}+(\frac{1}{r'}+\frac{2\beta}{\alpha}-\frac{d}{\alpha p}-\frac{1}{\alpha})-\frac{\gamma}{\alpha}} \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{-\frac{\beta}{\alpha}-\frac{\rho}{\alpha}+\frac{\varepsilon}{\alpha}+(\frac{\alpha-1-\frac{\alpha}{r}+2\beta+\frac{d}{p}}{\alpha})-\frac{\gamma}{\alpha}} \\
&\lesssim \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\varepsilon-\beta-\rho}{\alpha}}, \qquad (6.4.7)
\end{aligned}$$

where we also used for the last but one inequality that all time singularities are integrable (Da S. a E.: sì qui bisogna pensare a ζ, ρ come $-\beta + \eta$ per η piccolo).

Note that this term is not optimally controlled: one would expect to obtain a bound in $h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}$. This is due to the fact that the heat kernel estimate (6.1.26) only allows to reach the rate $(\gamma - \varepsilon)/\alpha$ (getting rid of the ε here is doable at the expense of more involved computations, however this is not the case for other terms down the line). The extra singularity in $t^{-\frac{\rho}{\alpha}}$ is due to the use of a product rule. It would have been more natural to try to control the besov norm $\|[\Gamma(0,x,s,\cdot) - \Gamma(0,x,\tau_s^h,\cdot)]\nabla_y p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}$ directly (i.e. without normalizing by $\bar{p}_\alpha(s,\cdot - x)$). However we were not able to achieve the desired rate this way as the previous norm involves the forward regularity of Γ in space and in time *simultaneously*, thus lowering the achievable thresholds.

Term Δ_3: approximation of the singular drift. Let us turn to Δ_3. Expanding the inner expectation and using the proof of Proposition 2 in [CdRJM22] to write $\|b(s,\cdot) - P_h^{\alpha} b(s,\cdot)\|_{\mathbb{B}^{\beta-\gamma}_{p,q}} \lesssim h^{\frac{\gamma}{\alpha}} \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}$, we derive, using this time (6.2.3) and (6.3.5) (noticing for the latter that on the considered time interval, $s \asymp \tau_s^h$) for $\zeta \in (-\beta+\gamma, -2\beta+\gamma)$,

$$\begin{aligned}
|\Delta_3(0,x,t,y)| &= \left| \int_h^{\tau_t^h - h} \int \Gamma(0,x,\tau_s^h,z)\,[b(s,z) - \mathfrak{b}_h(s,z)] \cdot \nabla_z p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \right| \\
&\lesssim \int_h^t \|b(s,\cdot) - P_h^\alpha b(s,\cdot)\|_{\mathbb{B}^{\beta-\gamma}_{p,q}} \left\|\Gamma(0,x,\tau_s^h,\cdot)\nabla_y p_\alpha(t-s,y-\cdot)\right\|_{\mathbb{B}^{-\beta+\gamma}_{p',q'}}\,\mathrm{d}s \\
&\lesssim h^{\frac{\gamma}{\alpha}} \int_h^t \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \frac{\bar p_\alpha(t,x-y)}{(t-s)^{\frac{1}{\alpha}}} t^{\frac{\beta-\gamma}{\alpha}} \left[\frac{1}{(t-s)^{\frac{d}{\alpha p}}} + \frac{1}{s^{\frac{d}{\alpha p}}}\right] \left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}\right] \mathrm{d}s \\
&\lesssim \bar p_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}. \qquad (6.4.8)
\end{aligned}$$

Term Δ_4: Gronwall or circular type argument. Due to the singular drift, this term emphasizes that in order to derive the main theorem, not only will we need to control the supremum of the difference but as well a kind of Hölder modulus. Namely, for $\rho \in (-\beta, -\beta+\gamma)$,

$$\begin{aligned}
&|\Delta_4(0,x,t,y)| \\
&= \left| \int_h^{\tau_t^h - h} \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s,X_{\tau_s^h}) \cdot \nabla_y p_\alpha(t-s,y-X_{\tau_s^h}) - \mathfrak{b}_h(s,X^h_{\tau_s^h}) \cdot \nabla_y p_\alpha(t-s,y-X^h_{\tau_s^h})\right] \mathrm{d}s \right| \\
&= \left| \int_h^{\tau_t^h - h} \int_{\mathbb{R}^d} (\Gamma - \Gamma^h)(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \right| \\
&\le \int_h^{\tau_t^h - h} \|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \left\| \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)} \right\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|\bar p_\alpha(\tau_s^h,\cdot-x)\nabla_y p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s.
\end{aligned}$$

Let us now recall that:

$$\begin{aligned}
\left\| \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)} \right\|_{\mathbb{B}^{\rho}_{\infty,\infty}} &\asymp \left\| \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)} \right\|_{L^\infty} + \sup_{y\neq y'\in\mathbb{R}^d} \frac{\left| \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,y)}{\bar p_\alpha(\tau_s^h,y-x)} - \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,y')}{\bar p_\alpha(\tau_s^h,y'-x)} \right|}{|y-y'|^\rho} \\
&\asymp \left\| \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)} \right\|_{L^\infty} + \mathcal{H}_\rho\left(\frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)} \right).
\end{aligned}$$

Set now, for $s \in (h,T]$,

$$g_{h,\rho}(s) := \left\| \frac{(\Gamma-\Gamma^h)(0,x,s,\cdot)}{\bar p_\alpha(s,\cdot-x)} \right\|_{L^\infty} + s^{\frac{\rho}{\alpha}} \mathcal{H}_\rho\left(\frac{(\Gamma-\Gamma^h)(0,x,s,\cdot)}{\bar p_\alpha(s,\cdot-x)} \right), \qquad (6.4.9)$$

where we set the overall value for ε to be $\varepsilon := 2(\rho+\beta)$, which can be chossen as small as desired. Thanks to the heat kernel estimates (6.1.25) and (6.1.22), we already know that $g_{h,\rho}(s)$ is finite. We carefully mention that the additional normalization in $s^{\frac{\rho}{\alpha}}$ for the Hölder modulus is the natural one associated with the spatial ρ-Hölder modulus of continuity of the stable heat kernel. With these notations at hand we write:

$$\begin{aligned}
|\Delta_4(0,x,t,y)| &\le \sup_{s\in(h,T]} g_{h,\rho}(s)\bar p_\alpha(t,y-x) \\
&\qquad \times \int_0^t \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \frac{1}{s^{\frac{\rho}{\alpha}}} \frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right] \left[\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right] \mathrm{d}s \\
&\lesssim \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{-\beta-\rho+\gamma}{\alpha}} \bar p_\alpha(t,y-x) = \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}} \bar p_\alpha(t,y-x), \qquad (6.4.10)
\end{aligned}$$

where we again used (6.3.4) with $\zeta \in (-\beta,1)$ for the last but one inequality. The above contribution actually emphasizes that, in order to control the error on the densities we actually need to control a related Hölder modulus of continuity. Let us as well point out that for the time contribution to be small (in order to perform the circular argument on the quantity $g_{h,\rho}$) we will as well assume w.l.o.g. that $\gamma > \varepsilon/2$.

Term Δ_5: spatial sensitivities of the driving noise. Let us now turn to

$$\begin{aligned}\Delta_5(0,t,x,y) &= \int_h^{\tau_t^h-h} \mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Big(\nabla_y p_\alpha(t-s,y-X^h_{\tau_s^h})-\nabla_y p_\alpha(t-s,y-X^h_s)\Big)\Big]\,\mathrm{d}s\\ &= \int_h^{\tau_t^h-h} \mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Big(\nabla_y p_\alpha(t-s,y-X^h_{\tau_s^h})\\ &\quad -\nabla_y p_\alpha\Big(t-\tau_s^h,y-(X^h_{\tau_s^h}+\int_{\tau_s^h}^s \mathfrak{b}_h(u,X^h_{\tau_s^h})\,\mathrm{d}u)\Big)\Big)\Big]\,\mathrm{d}s,\end{aligned}$$

using the harmonicity of the stable heat kernel (or martingale property of the driving noise). Write now,

$$\begin{aligned}&|\Delta_5(0,x,t,y)|\\ &=\Big|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot(\nabla_y p_\alpha(t-s,y-z)\ -\nabla_y p_\alpha(t-\tau_s^h,y-(z+\int_{\tau_s^h}^s\mathfrak{b}_h(u,z)\,\mathrm{d}u))\Big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\Big|\\ &\le\Big|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot\big(\nabla_y p_\alpha(t-s,y-z)-\nabla_y p_\alpha(t-\tau_s^h,y-z)\big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\Big|\\ &\quad+\Big|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(r,z)\cdot\big(\nabla_y p_\alpha(t-\tau_s^h,y-z)\\ &\quad -\nabla_y p_\alpha(t-\tau_s^h,y-(z+\int_{\tau_s^h}^s\mathfrak{b}_h(u,X^h_{\tau_s^h})\,\mathrm{d}u))\Big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\Big| =: |\Delta_{51}(0,x,t,y)|+|\Delta_{52}(0,x,t,y)|.\end{aligned}$$

For Δ_{51}, using the duality inequality (6.2.3) and (6.3.8), we have

$$\begin{aligned}|\Delta_{51}(0,x,t,y)| &\le \int_h^{\tau_t^h-h}\Big|\int\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot\big(\nabla_y p_\alpha(t-\tau_s^h,y-z)-\nabla_y p_\alpha(t-s,y-z)\big)\,\mathrm{d}z\Big|\,\mathrm{d}s\\ &\lesssim \int_h^{\tau_t^h-h}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\|\Gamma^h(0,x,\tau_s^h,\cdot)\big(\nabla_y p_\alpha(t-\tau_s^h,y-\cdot)-\nabla_y p_\alpha(t-s,y-\cdot)\big)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\ &\lesssim \bar p_\alpha(t,y-x)\int_h^{\tau_t^h-h}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{(s-\tau_s^h)^{\frac{\gamma}{\alpha}}t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1+\gamma}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s\\ &\lesssim \bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{-\frac{\beta}{\alpha}}.\end{aligned}\tag{6.4.11}$$

On the other hand, using this time (6.3.11),

$$\begin{aligned}&|\Delta_{52}(0,x,t,y)|\\ &\le \int_h^{\tau_t^h-h}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\Big\|\Gamma^h(0,x,\tau_s^h,\cdot)\Big(\nabla_y p_\alpha(t-\tau_s^h,y-\cdot)-\nabla_y p_\alpha(t-\tau_s^h,y-(\cdot+\int_{\tau_s^h}^s\mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))\Big)\Big\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\ &\lesssim \bar p_\alpha(t,y-x)\int_h^{\tau_t^h-h}\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{h^{\frac{\gamma-\beta}{\alpha}}t^{\frac{\beta}{\alpha}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\left[\frac{t^{\frac{\zeta}{\alpha}}}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s\\ &\lesssim \bar p_\alpha(t,y-x)h^{\frac{\gamma-\beta}{\alpha}}t^{\frac{\gamma-\beta}{\alpha}}.\end{aligned}\tag{6.4.12}$$

From (6.4.11) and (6.4.12) we thus derive:

$$|\Delta_5(0,x,t,y)|\lesssim \bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{-\frac{\beta}{\alpha}}.\tag{6.4.13}$$

Term Δ_6: last time steps. This term is handled very much like Δ_1, in the sense that the smallness will come from each contribution and not from sensitivities, since the time-interval is itself small. Namely,

$$\begin{aligned}
&|\Delta_6(0,x,t,y)| \\
&= \left| \int_{\tau_t^h - h}^t \mathbb{E}_{0,x} \left[b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s) - \mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X_s^h)\right] \mathrm{d}s \right| \\
&\le \left| \int_{\tau_t^h - h}^t \mathbb{E}_{0,x} \left[b(s,X_s)\cdot\nabla_y p_\alpha(t-s,y-X_s)\right] \mathrm{d}s \right| + \left| \int_{\tau_t^h - h}^t \mathbb{E}_{0,x} \left[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X_s^h)\right] \mathrm{d}s \right| \\
&=: \Big(|\Delta_{6,1}| + |\Delta_{6,2}|\Big)(0,x,t,y).
\end{aligned}$$

Using the duality inequality (6.2.3) and (6.3.5), we have, for $\zeta > -\beta$ so that $2\beta - 1 + \zeta < 0$,

$$\begin{aligned}
|\Delta_{6,1}(0,x,t,y)| &\lesssim \int_{\tau_t^h - h}^t \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|\Gamma(0,x,s,\cdot)\nabla_y p_\alpha(t-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}s \\
&\lesssim \bar p_\alpha(t,y-x) \int_{\tau_t^h - h}^t \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right] \left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}} \right] \mathrm{d}s \\
&\lesssim \bar p_\alpha(t,y-x) \|b\|_{L^r-\mathbb{B}^\beta_{p,q}} t^{\frac{\beta}{\alpha}} \left(\int_{\tau_t^h - h}^t (t-s)^{-\frac{r'}{\alpha}} \left[\frac{1}{t^{\frac{dr'}{\alpha p}}} + \frac{1}{(t-s)^{\frac{dr'}{\alpha p}}} \right] \frac{t^{\frac{\zeta r'}{\alpha}}}{(t-s)^{\frac{\zeta r'}{\alpha}}} \mathrm{d}s \right)^{\frac{1}{r'}} \\
&\lesssim \bar p_\alpha(t,y-x) h^{1-\frac{1}{r}} \left[h^{-\frac{1}{\alpha}-\frac{\zeta}{\alpha}} t^{\frac{\beta - \frac{d}{p} + \zeta}{\alpha}} + h^{-\frac{1}{\alpha} - \frac{d}{\alpha p} - \frac{\zeta}{\alpha}} t^{\frac{\beta+\zeta}{\alpha}} \right] \\
&\lesssim \bar p_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}.
\end{aligned} \tag{6.4.14}$$

Write now similarly for $\Delta_{6,2}$, using the L^∞ bound of $\mathfrak{b}_h$, (6.2.5), the fact that $\Gamma^h \lesssim \bar p_\alpha$ (see Proposition 6.1) and (6.1.15), we get

$$\begin{aligned}
|\Delta_{6,2}(0,x,t,y)| &\lesssim \int_{\tau_t^h - h}^t \int \frac{1}{(t-s)^{\frac{1}{\alpha}}} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} (s-\tau_s^h)^{-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} \bar p_\alpha(s,z-x)\bar p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \\
&\lesssim \bar p_\alpha(t,y-x) \left(\int_{\tau_t^h - h}^t \frac{1}{(s-\tau_s^h)^{r'(\frac{d}{\alpha p}+\frac{\beta}{\alpha})}(t-s)^{\frac{r'}{\alpha}}} \mathrm{d}s \right)^{\frac{1}{r'}} \\
&\lesssim \bar p_\alpha(t,y-x) h^{1-\frac{1}{\alpha}-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} \lesssim \bar p_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} h^{-\frac{\beta}{\alpha}} \lesssim \bar p_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}.
\end{aligned} \tag{6.4.15}$$

From (6.4.14), (6.4.15), we eventually get:

$$|\Delta_6(0,x,t,y)| \lesssim \bar p_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}. \tag{6.4.16}$$

Putting together the controls (6.4.5), (6.4.7), (6.4.8), (6.4.10), (6.4.13), (6.4.16) we derive:

$$\frac{|(\Gamma - \Gamma^h)(0,t,x,y)|}{\bar p_\alpha(t,y-x)} \lesssim h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\varepsilon}{2\alpha}} + \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}}. \tag{6.4.17}$$

6.4.3 Control of the Hölder modulus for the error

It was seen in the control of the previous term Δ_4 that a contribution in $\sup_{s\in(0,T]} \left\| \frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}}$, $\rho > -\beta$ appeared in the r.h.s. of (6.4.10). This is particular means that, in order to make a circular/Gronwall type argument we need to control the corresponding normalized Besov norm for the error in its final variable through its decomposition in (6.4.2). To this end, we will mainly reproduce the former computations observing that we still have some margin in the time singularities. We will here focus on the ρ-Hölder modulus of continuity in the diagonal regime (for the current time considered and the final variable), since otherwise the previous controls on the supremum norm already provide the estimates. Namely, we consider for all $(y,y')\in(\mathbb{R}^d)^2$ s.t. $|y-y'|\le t^{\frac1\alpha}$ the quantity

$$\begin{aligned}
&\Big|\frac{(\Gamma-\Gamma^h)(0,t,x,y)}{\bar p_\alpha(t,y-x)}-\frac{(\Gamma-\Gamma^h)(0,t,x,y')}{\bar p_\alpha(t,y'-x)}\Big|\\
=&\Big|\frac{(\Gamma-\Gamma^h)(0,t,x,y)-(\Gamma-\Gamma^h)(0,t,x,y')}{\bar p_\alpha(t,y-x)}\Big|+\Big[\frac{1}{\bar p_\alpha(t,y'-x)}-\frac{1}{\bar p_\alpha(t,y-x)}\Big](\Gamma-\Gamma^h)(0,t,x,y')\\
\lesssim&\Big|\frac{(\Gamma-\Gamma^h)(0,t,x,y)-(\Gamma-\Gamma^h)(0,t,x,y')}{\bar p_\alpha(t,y-x)}\Big|+\frac{|y-y'|^\rho}{t^{\frac\rho\alpha}}\frac{\big|(\Gamma-\Gamma^h)(0,t,x,y')\big|}{\bar p_\alpha(t,y'-x)}\\
\lesssim&\Big|\frac{(\Gamma-\Gamma^h)(0,t,x,y)-(\Gamma-\Gamma^h)(0,t,x,y')}{\bar p_\alpha(t,y-x)}\Big|+\frac{|y-y'|^\rho}{t^{\frac\rho\alpha}}\Big\|\frac{(\Gamma-\Gamma^h)(0,t,x,\cdot)}{\bar p_\alpha(t,\cdot-x)}\Big\|_{L^\infty},
\end{aligned}\tag{6.4.18}$$

using Lemma 6.2 for the first inequality. We will thus now focus on the first term using the error expansion (6.4.2). With a slight abuse of notation from now on, for $y,y'\in\mathbb{R}^d$ we will denote for $i\in\{1,\cdots,6\}$

$$\Delta_i(0,x,t,y,y'):=\Delta_i(0,x,t,y')-\Delta_i(0,x,t,y'),$$

where the terms $\Delta_i(0,x,t,y),\Delta_i(0,x,t,y')$ are those introduced in (6.4.2).

Term Δ_1: first time step. For Δ_1, we rely on the fact that we work on the first time step. Let us first expand the expectation:

$$\begin{aligned}
&\Delta_1(0,x,t,y,y')\\
:=&\int_0^h\int\Big(\Gamma(0,x,s,z)b(s,z)\cdot\big(\nabla_y p_\alpha(t-s,y-z)-\nabla_y p_\alpha(t-s,y'-z)\big)\\
&\qquad-\Gamma^h(0,x,s,z)\mathfrak{b}_h(s,x)\cdot\big(\nabla_y p_\alpha(t-s,y-z)-\nabla_y p_\alpha(t-s,y'-z)\big)\Big)\,\mathrm{d}z\,\mathrm{d}s\\
=:&\Big(\Delta_{1,1}+\Delta_{1,2}\Big)(0,x,t,y,y').
\end{aligned}$$

For $\Delta_{1,1}$, which involves the distributional b, we have to rely on duality inequalities in Besov spaces. Assuming w.l.o.g. that $t>2h$ so that $(t-s)\asymp t$ and using (6.3.7) (taking therein $\zeta\in(-\beta,1)$), we get

$$\begin{aligned}
&|\Delta_{1,1}(0,x,t,y,y')|\\
&\lesssim\int_0^h\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\|\Gamma(0,x,s,\cdot)\Big(\nabla_y p_\alpha(t-s,y-\cdot)-\nabla_y p_\alpha(t-s,y'-\cdot)\Big)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\
&\lesssim\bar p_\alpha(t,y-x)|y-y'|^\rho\int_0^h\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{t^{\frac\beta\alpha}}{(t-s)^{\frac1\alpha+\frac\rho\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac\zeta\alpha}}{(t-s)^{\frac\zeta\alpha}}+\frac{t^{\frac\zeta\alpha}}{s^{\frac\zeta\alpha}}\right]\mathrm{d}s\\
&\lesssim\bar p_\alpha(t,y-x)|y'-y|^\rho\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\left(\int_0^h t^{\frac{r'(\beta-1-\rho)}{\alpha}}\left[\frac{1}{s^{\frac{dr'}{\alpha p}}}+\frac{1}{t^{\frac{dr'}{\alpha p}}}\right]\frac{t^{\frac{\zeta r'}{\alpha}}}{s^{\frac{\zeta r'}{\alpha}}}\,\mathrm{d}s\right)^{\frac{1}{r'}}\\
&\lesssim\bar p_\alpha(t,y-x)|y'-y|^\rho\left[h^{1-\frac1r-\frac{d}{\alpha p}-\frac\zeta\alpha}t^{\frac{\beta-1-\rho+\zeta}{\alpha}}+h^{1-\frac1r-\frac\zeta\alpha}t^{\frac{\beta-1-\rho+\zeta}{\alpha}-\frac{d}{\alpha p}}\right]\\
&\lesssim\bar p_\alpha(t,y-x)|y'-y|^\rho h^{\frac\gamma\alpha}t^{-\frac{\beta+\rho}{\alpha}}.
\end{aligned}\tag{6.4.19}$$

For $\Delta_{1,2}$, using the L^∞ bound of $\mathfrak{b}_h$, (6.2.5), (6.1.22) and (6.1.17), we get

$$\begin{aligned}
&|\Delta_{1,2}(0,x,t,y,y')|\\
&\lesssim\int_0^h s^{-\frac{d}{\alpha p}+\frac\beta\alpha}\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{|y-y'|^\rho}{(t-s)^{\frac{1+\rho}{\alpha}}}\int\bar p_\alpha(s,z-x)(\bar p_\alpha(t-s,y-z)+\bar p_\alpha(t-s,y'-z))\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim\bar p_\alpha(t,y-x)|y'-y|^\rho h^{1-\frac1r-\frac{d}{\alpha p}+\frac\beta\alpha}t^{-\frac{1+\rho}{\alpha}}\\
&\lesssim\bar p_\alpha(t,y-x)|y'-y|^\rho h^{\frac\gamma\alpha}h^{\frac1\alpha-\frac\beta\alpha}t^{-\frac{1+\rho}{\alpha}}\lesssim\bar p_\alpha(t,y-x)|y'-y|^\rho h^{\frac\gamma\alpha}t^{-\frac{\beta+\rho}{\alpha}},
\end{aligned}\tag{6.4.20}$$

recalling that $h \le t$ for the last inequality. We eventually get:

$$|\Delta_1(0,x,t,y,y')| \lesssim \bar{p}_\alpha(t,y-x)|y'-y|^\rho h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta+\rho}{\alpha}}. \tag{6.4.21}$$

Term Δ_2: time sensitivity of the density of the SDE. Let us turn to Δ_2. Expanding the expectation, using the product rule (6.2.2), the duality inequality (6.2.3) in Besov spaces, the heat kernel estimate (6.1.26) and the control (6.3.9), we get for $\zeta > -\beta$,

$$\begin{aligned}
&|\Delta_2(0,x,t,y,y')| \\
&= \left| \int_h^{\tau_t^h - h} \int [\Gamma(0,x,s,z) - \Gamma(0,x,\tau_s^h,z)] b(s,z) \cdot (\nabla_y p_\alpha(t-s,y-z) - \nabla_y p_\alpha(t-s,y'-z)) \, \mathrm{d}z \, \mathrm{d}s \right| \\
&\lesssim \int_h^{\tau_t^h - h} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \left\| \frac{\Gamma(0,x,s,\cdot) - \Gamma(0,x,\tau_s^h,\cdot)}{\bar{p}_\alpha(s,\cdot - x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \\
&\qquad \times \|\bar{p}_\alpha(s,\cdot - x)\big(\nabla_y p_\alpha(t-s,y-\cdot) - \nabla_y p_\alpha(t-s,y'-\cdot)\big)\|_{\mathbb{B}^{-\beta}_{p',q'}} \, \mathrm{d}s \\
&\lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) \int_h^{\tau_t^h - h} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \frac{(s-\tau_s^h)^{\frac{\gamma-\varepsilon}{\alpha}}}{s^{\frac{\gamma-\varepsilon+\rho}{\alpha}}} \frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}+\frac{\rho}{\alpha}}} \left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}} \right] \left[\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}} \right] \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)|y-y'|^\rho h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\varepsilon-\beta-2\rho}{\alpha}},
\end{aligned} \tag{6.4.22}$$

choosing ζ, ρ, for the last inequality such that $r'(\frac{1}{\alpha} + \frac{\rho}{\alpha} + \frac{d}{p\alpha} + \frac{\zeta}{\alpha}) < 1$ so that the integral converges for $s \to t$. For such ρ, since $\gamma - \varepsilon < 1$, the exponents of s in the previous are always integrable and the integral also converges for $s \to 0$.

Term Δ_3: approximation of the singular drift. Let us turn to Δ_3. Expanding the inner expectation and from the proof of Proposition 2 in [CdRJM22] we derive, using (6.3.7) with some $\zeta > -\beta + \gamma$,

$$\begin{aligned}
&|\Delta_3(0,x,t,y,y')| \\
&= \left| \int_h^{\tau_t^h - h} \int \Gamma(0,x,\tau_s^h,z) \, [b(s,z) - \mathfrak{b}_h(s,z)] \cdot \Big(\nabla_y p_\alpha(t-s,y-z) - \nabla_y p_\alpha(t-s,y'-z) \Big) \mathrm{d}z \, \mathrm{d}s \right| \\
&\lesssim \int_h^{\tau_t^h - h} \|b(s,\cdot) - P_h^\alpha b(s,\cdot)\|_{\mathbb{B}^{\beta-\gamma}_{p,q}} \|\Gamma(0,x,\tau_s^h,\cdot)\big(\nabla p_\alpha(t-s,y-\cdot) - \nabla p_\alpha(t-s,y'-\cdot)\big)\|_{\mathbb{B}^{-\beta+\gamma}_{p',q'}} \, \mathrm{d}s \\
&\lesssim h^{\frac{\gamma}{\alpha}} |y'-y|^\rho \int_h^{\tau_t^h - h} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \frac{\bar{p}_\alpha(t,x-y)}{(t-s)^{\frac{1}{\alpha}+\frac{\rho}{\alpha}}} t^{\frac{\beta-\gamma}{\alpha}} \left[\frac{1}{(t-s)^{\frac{d}{\alpha p}}} + \frac{1}{s^{\frac{d}{\alpha p}}} \right] \left[\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} \right] \mathrm{d}s.
\end{aligned}$$

Notice that, this time, since $\zeta > -\beta + \gamma$, the previous integral is always singular near $s = t$. Recalling that $\varepsilon = 2(\rho + \beta) > 0$ (which can be chosen as small as desired) and taking $\zeta = \gamma - \beta + \varepsilon/2$, then using a Hölder inequality in time and the fact that $t - \tau_t^h + h \asymp h$, we get

$$|\Delta_3(0,x,t,y,y')| \lesssim h^{\frac{\gamma}{\alpha}} |y'-y|^\rho t^{\frac{\zeta+\beta-\gamma}{\alpha}} (t - \tau_t^h + h)^{\frac{\gamma-2\beta-\rho-\zeta}{\alpha}} \lesssim h^{\frac{\gamma-\varepsilon}{\alpha}} |y'-y|^\rho t^{\frac{\varepsilon}{2\alpha}}. \tag{6.4.23}$$

Term Δ_4: Gronwall or circular type argument. We now need to control the Hölder norm of the term associated with the Gronwall or circular type argument. Namely, for $\rho \in (-\beta, -\beta+\gamma)$ and $y, y' \in \mathbb{R}^d$ in the global diagonal regime w.r.t. time t:

$$
\begin{aligned}
&|\Delta_4(0,x,t,y,y')|\\
:=&\Bigg|\int_h^{\tau_t^h-h}\Big\{\mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X_{\tau_s^h})-\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X^h_{\tau_s^h})\Big]\\
&-\mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y'-X_{\tau_s^h})-\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y'-X^h_{\tau_s^h})\Big]\Big\}\,\mathrm{d}s\Bigg|\\
=&\Bigg|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}(\Gamma-\Gamma^h)(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\big(\nabla_y p_\alpha(t-s,y-z)-\nabla_y p_\alpha(t-s,y'-z)\big)\,\mathrm{d}z\,\mathrm{d}s\Bigg|\\
\le&\int_h^{\tau_t^h-h}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\left\|\frac{(\Gamma-\Gamma^h)(0,x,\tau_s^h,\cdot)}{\bar p_\alpha(\tau_s^h,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}\\
&\qquad\times\|\bar p_\alpha(\tau_s^h,\cdot-x)\big(\nabla p_\alpha(t-s,y-\cdot)-\nabla p_\alpha(t-s,y'-\cdot)\big)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\
\le&|y-y'|^\rho\sup_{s\in(h,T]}g_{h,\rho}(s)\bar p_\alpha(t,y-x)\\
&\quad\times\int_0^t\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\frac{1}{s^{\frac{\rho}{\alpha}}}\frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}+\frac{\rho}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s\\
\lesssim&|y-y'|^\rho\sup_{s\in(h,T]}g_{h,\rho}(s)t^{\frac{\gamma-\rho-\frac{\varepsilon}{2}}{\alpha}}\bar p_\alpha(t,y-x),
\end{aligned}
\tag{6.4.24}
$$

keeping in mind the definition (6.4.9) and using (6.3.9) with $\zeta,\rho\in(-\beta,1)$ for the last but one inequality, with $r'(\frac{1}{\alpha}+\frac{\rho}{\alpha}+\frac{d}{p\alpha}+\frac{\zeta}{\alpha})<1$.

Term Δ_5: spatial sensitivities of the driving noise. Let us now turn to

$$
\begin{aligned}
&\Delta_5(0,t,x,y,y')\\
=&\int_h^{\tau_t^h-h}\Big\{\mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Big(\nabla_y p_\alpha(t-s,y-X^h_{\tau_s^h})-\nabla_y p_\alpha(t-s,y-X^h_s)\Big)\Big]\\
&\qquad-\mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Big(\nabla_y p_\alpha(t-s,y'-X^h_{\tau_s^h})-\nabla_y p_\alpha(t-s,y'-X^h_s)\Big)\Big]\Big\}\,\mathrm{d}s\\
=&\int_h^{\tau_t^h-h}\mathbb{E}_{0,x}\Bigg[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Bigg(\nabla_y p_\alpha(t-s,y-X^h_{\tau_s^h})-\nabla_y p_\alpha\Big(t-\tau_s^h,y-(X^h_{\tau_s^h}+\int_{\tau_s^h}^s\mathfrak{b}_h(u,X^h_{\tau_s^h})\,\mathrm{d}u)\Big)\\
&-\Big(\nabla_y p_\alpha(t-s,y'-X^h_{\tau_s^h})-\nabla_y p_\alpha\Big(t-\tau_s^h,y'-(X^h_{\tau_s^h}+\int_{\tau_s^h}^s\mathfrak{b}_h(u,X^h_{\tau_s^h})\,\mathrm{d}u)\Big)\Big)\Bigg)\Bigg]\,\mathrm{d}s,
\end{aligned}
$$

using the harmonicity of the stable heat kernel (or martingale property of the driving noise). Write now,

$$
\begin{aligned}
&|\Delta_5(0,x,t,y,y')|\\
=&\Big|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot\Big(\nabla_y p_\alpha(t-s,y-z)-\nabla_y p_\alpha(t-\tau_s^h,y-(z+\int_{\tau_s^h}^s\mathfrak{b}_h(u,z)\,\mathrm{d}u))\\
&\qquad-\Big(\nabla_y p_\alpha(t-s,y'-z)-\nabla_y p_\alpha(t-\tau_s^h,y'-(z+\int_{\tau_s^h}^s\mathfrak{b}_h(u,z)\,\mathrm{d}u))\Big)\Big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\Big|\\
\le&\Big|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot\big(\nabla_y p_\alpha(t-s,y-z)-\nabla_y p_\alpha(t-\tau_s^h,y-z)\\
&\qquad-\Big(\nabla_y p_\alpha(t-s,y'-z)-\nabla_y p_\alpha(t-\tau_s^h,y'-z)\Big)\Big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\Big|\\
&+\Big|\int_h^{\tau_t^h-h}\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(r,z)\cdot\Big(\nabla_y p_\alpha(t-\tau_s^h,y-z)\\
&\quad-\nabla_y p_\alpha(t-\tau_s^h,y-(z+\int_{\tau_s^h}^s\mathfrak{b}_h(u,z)\,\mathrm{d}u))\\
&\quad-\Big(\nabla_y p_\alpha(t-\tau_s^h,y'-z)-\nabla_y p_\alpha(t-\tau_s^h,y'-(z+\int_{\tau_s^h}^s\mathfrak{b}_h(u,z)\,\mathrm{d}u))\Big)\Big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s\Big|\\
=:&|\Delta_{51}(0,x,t,y,y')|+|\Delta_{52}(0,x,t,y,y')|.
\end{aligned}
$$

For $\rho\in(-\beta,-\beta+\gamma)$ we have:

$$
\begin{aligned}
&|\Delta_{51}(0,x,t,y,y')|\\
&\lesssim\int_h^{\tau_t^h-h}\Big|\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot\big(\nabla_y p_\alpha(t-\tau_s^h,y-z)-\nabla_y p_\alpha(t-s,y-z)\\
&\qquad-\big(\nabla_y p_\alpha(t-\tau_s^h,y'-z)-\nabla_y p_\alpha(t-s,y'-z)\big)\big)\,\mathrm{d}z\Big|\,\mathrm{d}s\\
&\lesssim h\int_h^{\tau_t^h-h}\Big|\int_0^1\mathrm{d}\lambda\int_{\mathbb{R}^d}\Gamma^h(0,x,\tau_s^h,z)\mathfrak{b}_h(s,z)\cdot\big(\partial_t\nabla_y p_\alpha(t-(\tau_s^h+\lambda(s-\tau_s^h)),y-z)\\
&\qquad-\partial_t\nabla_y p_\alpha(t-(\tau_s^h+\lambda(s-\tau_s^h)),y'-z)\big)\,\mathrm{d}z\Big|\,\mathrm{d}s\\
&\lesssim h\int_h^{\tau_t^h-h}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\\
&\quad\times\int_0^1\big\|\Gamma^h(0,x,\tau_s^h,\cdot)\big(\partial_t\nabla_y p_\alpha(t-(\tau_s^h+\lambda(s-\tau_s^h)),y-\cdot)-\partial_t\nabla_y p_\alpha(t-(\tau_s^h+\lambda(s-\tau_s^h)),y'-\cdot)\big)\big\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}\lambda\,\mathrm{d}s.
\end{aligned}
$$

From (6.3.10), we get for $\zeta,\rho\in(-\beta,1)$,

$$
\begin{aligned}
&|\Delta_{51}(0,x,t,y,y')|\\
&\lesssim|y-y'|^\rho\int_h^{\tau_t^h-h}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}h^{\frac{\gamma}{\alpha}}\frac{\bar p_\alpha(t,y-x)}{(t-s)^{\frac{1+\gamma+\rho}{\alpha}}}t^{\frac{\beta}{\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s\\
&\lesssim|y-y'|^\rho\bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{\frac{\beta}{\alpha}}\left(\int_h^{\tau_t^h-h}\left\{\frac{1}{(t-s)^{\frac{1+\gamma+\rho}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\right\}^{r'}\mathrm{d}s\right)^{\frac{1}{r'}}\\
&\lesssim|y-y'|^\rho\bar p_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{\frac{\beta+\zeta}{\alpha}}(t-\tau_t^h+h)^{\frac{-(\beta+\rho)-(\beta+\zeta)}{\alpha}}=|y-y'|^\rho\bar p_\alpha(t,y-x)h^{\frac{\gamma-\varepsilon}{\alpha}}t^{\frac{\varepsilon}{2\alpha}},
\end{aligned}\tag{6.4.25}
$$

using as well (6.2.8) for the last inequality and taking ζ such that $\varepsilon = (\beta+\rho)+(\beta+\zeta)$ (i.e. $\zeta=\rho$). On the other hand, using (6.2.8) and (6.3.12), we have

$$
\begin{aligned}
&|\Delta_{52}(0,x,t,y,y')| \\
&\lesssim \int_h^{\tau_t^h} \|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \int_0^1 \mathrm{d}\lambda \Big\| \Gamma^h(0,x,\tau_s^h,\cdot)\Big(\nabla_y^2 p_\alpha(t-\tau_s^h, y-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u)) \\
&\qquad -\nabla_y^2 p_\alpha(t-\tau_s^h, y'-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))\Big)\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u \Big\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}s \\
&\lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) t^{\frac{\beta}{\alpha}} \int_h^{\tau_t^h-h} \frac{\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} h^{\frac{\gamma-\beta}{\alpha}}}{(t-\tau_s^h)^{\frac{1+\rho}{\alpha}}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\left[\frac{t^{\frac{\zeta}{\alpha}}}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s \\
&\lesssim |y-y'|^\rho h^{\frac{\gamma-\beta}{\alpha}} \bar{p}_\alpha(t,y-x) t^{-\frac{\beta+\rho}{\alpha}} \lesssim |y-y'|^\rho h^{\frac{\gamma-\varepsilon}{\alpha}} \bar{p}_\alpha(t,y-x) t^{\frac{\frac{\varepsilon}{2}-\beta}{\alpha}}.
\end{aligned} \tag{6.4.26}
$$

From (6.4.25) and (6.4.26) we thus derive:

$$
|\Delta_5(0,x,t,y,y')| \lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\varepsilon}{2\alpha}}. \tag{6.4.27}
$$

Term Δ_6: last time steps This term is once again handled very much like Δ_1, in the sense that the smallness will come from each contribution and not from sensitivities. Namely,

$$
\begin{aligned}
&|\Delta_6(0,x,t,y,y')| \\
&= \Big|\int_{\tau_t^h-h}^t \mathbb{E}_{0,x}\Big[b(s,X_s)\cdot\Big(\nabla_y p_\alpha(t-s,y-X_s)-\nabla_y p_\alpha(t-s,y'-X_s)\Big)\Big] \\
&\quad -\mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Big(\nabla_y p_\alpha(t-s,y-X_s^h)-\nabla_y p_\alpha(t-s,y'-X_s^h)\Big)\Big]\,\mathrm{d}s\Big| \\
&\le \Big|\int_{\tau_t^h-h}^t \mathbb{E}_{0,x}\Big[b(s,X_s)\cdot\Big(\nabla_y p_\alpha(t-s,y-X_s)-\nabla_y p_\alpha(t-s,y'-X_s)\Big)\Big]\,\mathrm{d}s\Big| \\
&\quad + \Big|\int_{\tau_t^h-h}^t \mathbb{E}_{0,x}\Big[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\Big(\nabla_y p_\alpha(t-s,y-X_s^h)-\nabla_y p_\alpha(t-s,y'-X_s^h)\Big)\Big]\,\mathrm{d}s\Big| \\
&=: \Big(|\Delta_{6,1}|+|\Delta_{6,2}|\Big)(0,x,t,y,y').
\end{aligned}
$$

Using (6.3.7) (taking therein $\zeta\in(-\beta,1)$), we first get

$$
\begin{aligned}
&|\Delta_{6,1}(0,x,t,y,y')| \\
&\lesssim \int_{\tau_t^h-h}^t \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|\Gamma(0,x,s,\cdot)\Big(\nabla_y p_\alpha(t-s,y-\cdot)-\nabla_y p_\alpha(t-s,y'-\cdot)\Big)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s \\
&\lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x)\int_{\tau_t^h-h}^t \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1+\rho}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s \\
&\lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x)\|b\|_{L^r-\mathbb{B}^\beta_{p,q}} t^{\frac{\beta}{\alpha}}\left(\int_{\tau_t^h-h}^t (t-s)^{-r'\left(\frac{1+\rho}{\alpha}\right)}\left[\frac{1}{t^{\frac{dr'}{\alpha p}}}+\frac{1}{(t-s)^{\frac{dr'}{\alpha p}}}\right]\frac{t^{\frac{\zeta r'}{\alpha}}}{(t-s)^{\frac{\zeta r'}{\alpha}}}\,\mathrm{d}s\right)^{\frac{1}{r'}} \\
&\lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) h^{1-\frac{1}{r}}\left[h^{-\frac{1+\rho}{\alpha}-\frac{\zeta}{\alpha}} t^{\frac{\beta-\frac{d}{p}+\zeta}{\alpha}} + h^{-\frac{1+\rho}{\alpha}-\frac{d}{\alpha p}-\frac{\zeta}{\alpha}} t^{\frac{\beta+\zeta}{\alpha}}\right] \\
&\lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta+\rho}{\alpha}}.
\end{aligned} \tag{6.4.28}
$$

Write now similarly for $\Delta_{6,2}$, using the L^∞ bound of $\mathfrak{b}_h$, (6.2.5), the fact that $\Gamma^h \lesssim p_\alpha$ (see (6.1.20)) and (6.1.17), we get

$$
\begin{aligned}
&|\Delta_{6,2}(0,x,t,y,y')| \\
\lesssim& |y-y'|^\rho \int_{\tau_t^h-h}^t \int \frac{1}{(t-s)^{\frac{1}{\alpha}+\frac{\rho}{\alpha}}} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} (s-\tau_s^h)^{-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} \bar{p}_\alpha(s,z-x)\bar{p}_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s \\
\lesssim& |y-y'|^\rho \bar{p}_\alpha(t,y-x) \left(\int_{\tau_t^h-h}^t \frac{1}{(s-\tau_s^h)^{r'(\frac{d}{\alpha p}-\frac{\beta}{\alpha})}(t-s)^{r'\left(\frac{1}{\alpha}+\frac{\rho}{\alpha}\right)}} \frac{(t-s)^{\frac{\rho+\beta}{\alpha}}}{(t-s)^{\frac{\rho+\beta}{\alpha}}}\,\mathrm{d}s \right)^{\frac{1}{r'}} \\
\lesssim& |y-y'|^\rho (\tau_t^h-h)^{-\frac{\beta+\rho}{\alpha}} \bar{p}_\alpha(t,y-x) h^{1-\frac{1}{r}-\frac{1}{\alpha}-\frac{d}{\alpha p}+\frac{2\beta}{\alpha}} \lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta+\rho}{\alpha}}.
\end{aligned} \tag{6.4.29}
$$

From (6.4.28), (6.4.29), we eventually get:

$$
|\Delta_6(0,x,t,y,y')| \lesssim |y-y'|^\rho \bar{p}_\alpha(t,y-x) h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta+\rho}{\alpha}}. \tag{6.4.30}
$$

Putting together the controls (6.4.21), (6.4.22), (6.4.23), (6.4.24), (6.4.27), (6.4.30) we derive from (6.4.18):

$$
t^{\frac{\rho}{\alpha}} \frac{\left|(\Gamma-\Gamma^h)(0,x,t,y)-(\Gamma-\Gamma^h)(0,x,t,y')\right|}{\bar{p}_\alpha(t,y-x)|y-y'|^\rho} \lesssim h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\varepsilon}{2\alpha}} + \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}}. \tag{6.4.31}
$$

Recall now for clarity the expression of (6.4.17)

$$
\frac{|(\Gamma-\Gamma^h)(0,t,x,y)|}{\bar{p}_\alpha(t,y-x)} \lesssim h^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{\varepsilon}{2\alpha}} + \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}}.
$$

We can now use (6.4.17) and (6.4.31) to derive, recalling that we set $\varepsilon = 2(\rho+\beta)$:

$$
\begin{aligned}
&\sup_{(y,y')\in(\mathbb{R}^d)^2} \frac{|\Gamma(0,x,t,y)-\Gamma^h(0,x,t,y)|}{\bar{p}_\alpha(t,y-x)} + t^{\frac{\rho}{\alpha}} \frac{\left|(\Gamma-\Gamma^h)(0,x,t,y)-(\Gamma-\Gamma^h)(0,x,t,y')\right|}{\bar{p}_\alpha(t,y-x)|y-y'|^\rho} \\
&\lesssim h^{\frac{\gamma-\varepsilon}{\alpha}} + \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}}.
\end{aligned}
$$

which rewrites from (6.4.9) and (6.4.18):

$$
g_{h,\rho}(t) \lesssim h^{\frac{\gamma-\varepsilon}{\alpha}} + \sup_{s\in(h,T]} g_{h,\rho}(s) t^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}}.
$$

Since the exponent of t in the above was chosen to be non-negative, we can take the supremum in $t \in (h,T]$ in the above equation to derive, for a constant $C \geq 1$,

$$
\sup_{t\in(h,T]} g_{h,\rho}(t) \leq C \left(h^{\frac{\gamma-\varepsilon}{\alpha}} + \sup_{s\in(h,T]} g_{h,\rho}(s) T^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}} \right),
$$

which for T small enough (i.e. s.t. $CT^{\frac{\gamma-\frac{\varepsilon}{2}}{\alpha}} \leq 1/2$) eventually yields:

$$
\sup_{t\in(h,T]} g_{h,\rho}(t) \leq 2C h^{\frac{\gamma-\varepsilon}{\alpha}} \lesssim h^{\frac{\gamma-\varepsilon}{\alpha}}.
$$

The theorem is proved.

6.5 Proof of technical Lemmas

6.5.1 Proof of Lemma 6.5

Proofs of (6.3.7),(6.3.8)**,** (6.3.9) **and** (6.3.10)

Let us first prove (6.3.7). Denote $\mathfrak{q}^{s,t}_{x,y}(\cdot) := \Gamma(0,x,s,\cdot)\left(\nabla_y p_\alpha(t-s,y-\cdot) - \nabla_{y'} p_\alpha(t-s,y'-\cdot)\right)$, of which we will control the $\mathbb{B}^{-\beta}_{p',q'}$ norm using the thermic characterization

$$\|\mathfrak{q}^{s,t}_{x,y}\|_{\mathbb{B}^{-\beta}_{p',q'}} = \|\phi(D)\mathfrak{q}^{s,t}_{x,y}\|_{L^{p'}} + \mathcal{T}^{-\beta}_{p',q'}[\mathfrak{q}^{s,t}_{x,y}].$$

Thermic part

We assume w.l.o.g. that $t<1$ and $q<+\infty$. Let us recall the definition of the thermic part and split it in two parts:

$$\begin{aligned}\mathcal{T}^{-\beta}_{p',q'}[\mathfrak{q}^{s,t}_{x,y}]^{q'} &= \int_0^t \frac{\mathrm{d}v}{v} v^{\left(1+\frac{\beta}{\alpha}\right)q'} \|\partial_v p_\alpha(v,\cdot)\star \mathfrak{q}^{s,t}_{x,y}(\cdot)\|^{q'}_{L^{p'}} + \int_t^1 \frac{\mathrm{d}v}{v} v^{\left(1+\frac{\beta}{\alpha}\right)q'} \|\partial_v p_\alpha(v,\cdot)\star \mathfrak{q}^{s,t}_{x,y}(\cdot)\|^{q'}_{L^{p'}}\\ &=: \mathcal{T}^{-\beta,(0,t)}_{p',q'}[\mathfrak{q}^{s,t}_{x,y}]^{q'} + \mathcal{T}^{-\beta,(t,1)}_{p',q'}[\mathfrak{q}^{s,t}_{x,y}]^{q'}.\end{aligned}$$

For the upper part on $(t,1)$, using a $L^1-L^{p'}$ convolution inequality, we get

$$\mathcal{T}^{-\beta,(t,1)}_{p',q'}[\mathfrak{q}^{s,t}_{x,y}]^{q'} \lesssim \int_t^1 \frac{\mathrm{d}v}{v} v^{\left(1+\frac{\beta}{\alpha}\right)q'} \|\partial_v p_\alpha(v,\cdot)\|^{q'}_{L^1} \|\Gamma(0,x,s,\cdot)\left(\nabla_y p_\alpha(t-s,y-\cdot) - \nabla_{y'} p_\alpha(t-s,y'-\cdot)\right)\|^{q'}_{L^{p'}}$$

Using the pointwise estimate (6.1.17) (in the case $|y-y'|\le (t-s)^{\frac{1}{\alpha}}$) and (6.1.23), then (6.1.18), we have, for any $\rho\in(-\beta,1]$,

$$\|\Gamma(0,x,s,\cdot)\left(\nabla_y p_\alpha(t-s,y-\cdot) - \nabla_{y'} p_\alpha(t-s,y'-\cdot)\right)\|_{L^{p'}} \lesssim \bar{p}_\alpha(t,y-x)\frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right].$$

This yields

$$\begin{aligned}\mathcal{T}^{-\beta(t,1)}_{p',q'}[\mathfrak{q}^{s,t}_{x,y}] &\lesssim \bar{p}_\alpha(t,y-x)\frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left(\int_t^1 v^{\frac{\beta q'}{\alpha}-1}\,\mathrm{d}v\right)^{\frac{1}{q'}}\\ &\lesssim \bar{p}_\alpha(t,y-x)\frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right] t^{\frac{\beta}{\alpha}}. \qquad (6.5.1)\end{aligned}$$

For the lower part, let us write

$$\begin{aligned}\|\partial_v p_\alpha(v,\cdot)\star\mathfrak{q}^{s,t}_{x,y}\|^{p'}_{L^{p'}} &= \int\left|\int \partial_v p_\alpha(v,z-w)\mathfrak{q}^{s,t}_{x,y}(w)\,\mathrm{d}w\right|^{p'}\mathrm{d}z\\ &= \int\left|\int \partial_v p_\alpha(v,z-w)\left[\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)\right]\mathrm{d}w\right|^{p'}\mathrm{d}z, \qquad (6.5.2)\end{aligned}$$

using a cancellation argument for the last equality. Next, let us distinguish whether this difference is in diagonal or off-diagonal regime.

- **Diagonal case:** $|z-w|\le s^{\frac{1}{\alpha}}$. Let us write

$$\begin{aligned}&\mathfrak{q}^{s,t}_{x,y}(w)-\mathfrak{q}^{s,t}_{x,y}(z)\\ &=\Gamma(0,x,s,w)\big[\left(\nabla_y p_\alpha(t-s,y-w)-\nabla_{y'}p_\alpha(t-s,y'-w)\right)-\left(\nabla_y p_\alpha(t-s,y-z)-\nabla_{y'}p_\alpha(t-s,y'-z)\right)\\ &\quad-\big[\Gamma(0,x,s,z)-\Gamma(0,x,s,w)\big]\left(\nabla_y p_\alpha(t-s,y-z)-\nabla_{y'}p_\alpha(t-s,y'-z)\right)\\ &=\Gamma(0,x,s,w)\Big(\int_0^1\left[\nabla_y^2 p_\alpha(t-s,y-w-\lambda(z-w))-\nabla_{y'}^2 p_\alpha(t-s,y'-w-\lambda(z-w))\right](w-z)\,\mathrm{d}\lambda\big]\mathbb{I}_{|z-w|\le(t-s)^{\frac{1}{\alpha}}}\\ &+\big[\left(\nabla_y p_\alpha(t-s,y-w)-\nabla_{y'}p_\alpha(t-s,y'-w)\right)-\left(\nabla_y p_\alpha(t-s,y-z)-\nabla_{y'}p_\alpha(t-s,y'-z)\right)\big]\mathbb{I}_{|z-w|>(t-s)^{\frac{1}{\alpha}}}\Big)\\ &\quad-\big[\Gamma(0,x,s,z)-\Gamma(0,x,s,w)\big]\left(\nabla_y p_\alpha(t-s,y-z)-\nabla_{y'}p_\alpha(t-s,y'-z)\right).\end{aligned}$$

Using the regularity of p_α, (6.1.17) and the forward regularity of Γ, (6.1.24), we get, for any $\zeta \in (0,1]$,

$$
\begin{aligned}
&|\mathfrak{q}^{s,t}_{x,y}(w) - \mathfrak{q}^{s,t}_{x,y}(z)| \\
&\lesssim \bar{p}_\alpha(s, w-x) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{(t-s)^{\frac{\zeta}{\alpha}}} \Big(\int_0^1 [\bar{p}_\alpha(t-s, y-w-\lambda(z-w)) + \bar{p}_\alpha(t-s, y'-w-\lambda(z-w))] \, \mathrm{d}\lambda \mathbb{I}_{|z-w| \le (t-s)^{\frac{1}{\alpha}}} \\
&+ (\bar{p}_\alpha(t-s, y-w) + \bar{p}_\alpha(t-s, y'-w) + \bar{p}_\alpha(t-s, y-z) + \bar{p}_\alpha(t-s, y'-z)) \mathbb{I}_{|z-w| > (t-s)^{\frac{1}{\alpha}}} \Big) \\
&\quad + \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}} (\bar{p}_\alpha(s, z-x) + \bar{p}_\alpha(s, w-x)) (\bar{p}_\alpha(t-s, y-z) + \bar{p}_\alpha(t-s, y'-z)) \\
&\lesssim \bar{p}_\alpha(s, w-x) (\bar{p}_\alpha(t-s, y-w) + \bar{p}_\alpha(t-s, y'-w)) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{(t-s)^{\frac{\zeta}{\alpha}}} \\
&\quad + \bar{p}_\alpha(s, z-x) (p_\alpha(t-s, y-z) + p_\alpha(t-s, y'-z)) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}},
\end{aligned}
\tag{6.5.3}
$$

using the current diagonal regime to write that $\bar{p}_\alpha(t-s, y-w-\lambda(z-w)) \lesssim \bar{p}_\alpha(t-s, y-w)$ (and the same estimate for y') and $\bar{p}_\alpha(s, w-x) \lesssim \bar{p}_\alpha(s, z-x)$.

- **Off-diagonal case:** $|z-w| \ge s^{\frac{1}{\alpha}}$. Using a triangular inequality, (6.1.17), (6.1.23) and the fact that $\frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}} \ge 1$, we trivially have the following:

$$
\begin{aligned}
|\mathfrak{q}^{s,t}_{x,y}(w) - \mathfrak{q}^{s,t}_{x,y}(z)| &\lesssim \bar{p}_\alpha(s, w-x) (\bar{p}_\alpha(t-s, y-w) + \bar{p}_\alpha(t-s, y'-w)) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}} \\
&\quad + \bar{p}_\alpha(s, z-x) (\bar{p}_\alpha(t-s, y-z) + \bar{p}_\alpha(t-s, y'-z)) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}}.
\end{aligned}
\tag{6.5.4}
$$

Gathering (6.5.3) and (6.5.4), we have

$$
\begin{aligned}
|\mathfrak{q}^{s,t}_{x,y}(w) - \mathfrak{q}^{s,t}_{x,y}(z)| &\lesssim \bar{p}_\alpha(s, w-x) (\bar{p}_\alpha(t-s, y-w) + \bar{p}_\alpha(t-s, y'-w)) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \left[\frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}} + \frac{|z-w|^\zeta}{(t-s)^{\frac{\zeta}{\alpha}}} \right] \\
&\quad + \bar{p}_\alpha(s, z-x) (\bar{p}_\alpha(t-s, y-z) + \bar{p}_\alpha(t-s, y'-z)) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}}.
\end{aligned}
\tag{6.5.5}
$$

Plugging this in (6.5.2), we get

$$
\begin{aligned}
&\|\partial_v p_\alpha(v, \cdot) \star \mathfrak{q}^{s,t}_{x,y}\|^{p'}_{L^{p'}} = \int \left| \int \partial_v p_\alpha(v, z-w) \mathfrak{q}^{s,t}_{x,y}(w) \, \mathrm{d}w \right|^{p'} \mathrm{d}z \\
&\lesssim \int \Big(\int v^{-1} \bar{p}_\alpha(v, z-w) \bar{p}_\alpha(s, w-x) (\bar{p}_\alpha(t-s, y-w) + \bar{p}_\alpha(t-s, y'-w)) \\
&\qquad\qquad \times \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \left[\frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}} + \frac{|z-w|^\zeta}{(t-s)^{\frac{\zeta}{\alpha}}} \right] \mathrm{d}w \Big)^{p'} \mathrm{d}z \\
&\quad + \int \Big(\int v^{-1} \bar{p}_\alpha(v, z-w) \bar{p}_\alpha(s, z-x) (\bar{p}_\alpha(t-s, y-z) + \bar{p}_\alpha(t-s, y'-z)) \\
&\qquad\qquad \times \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \frac{|z-w|^\zeta}{s^{\frac{\zeta}{\alpha}}} \, \mathrm{d}w \Big)^{p'} \mathrm{d}z.
\end{aligned}
\tag{6.5.6}
$$

From this point, we derive a smoothing effect in v by using the moments estimate (6.1.13). It is immediate for the second term, whereas for the first one, due to the order of integration, we need to use an $L^1 - L^{p'}$

convolution inequality. This yields

$$
\begin{aligned}
&\|\partial_v p_\alpha(v,\cdot)\star \mathfrak{q}_{x,y}^{s,t}\|_{L^{p'}}^{p'}\\
&\lesssim \left(\frac{v^{-1+\frac{\zeta}{\alpha}}|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[s^{-\frac{\zeta}{\alpha}}+(t-s)^{-\frac{\zeta}{\alpha}}\right]\right)^{p'}\int \left(\bar p_\alpha(s,w-x)\left(\bar p_\alpha(t-s,y-w)+\bar p_\alpha(t-s,y'-w)\right)\right)^{p'}\,\mathrm{d}w\\
&\quad+\left(\frac{v^{-1+\frac{\zeta}{\alpha}}|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}s^{-\frac{\zeta}{\alpha}}\right)^{p'}\int \left(\bar p_\alpha(s,z-x)\left(\bar p_\alpha(t-s,y-z)+\bar p_\alpha(t-s,y'-z)\right)\right)^{p'}\,\mathrm{d}z\\
&\lesssim \left(\frac{v^{-1+\frac{\zeta}{\alpha}}|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[s^{-\frac{\zeta}{\alpha}}+(t-s)^{-\frac{\zeta}{\alpha}}\right]\right)^{p'}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]^{p'}\bar p_\alpha(t,y-x)^{p'}. \qquad (6.5.7)
\end{aligned}
$$

Going back to the definition of $\mathcal{T}_{p',q'}^{-\beta,(0,t)}$, we thus obtain, taking $\zeta>-\beta$, and recalling that we are in the regime $|y-y'|\le t^{\frac{1}{\alpha}}$

$$
\begin{aligned}
\mathcal{T}_{p',q'}^{-\beta,(0,t)} &\lesssim \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[\frac{1}{s^{\frac{\zeta}{\alpha}}}+\frac{1}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left(\bar p_\alpha(t,y-x)+\bar p_\alpha(t,y'-x)\right)\left(\int_0^t v^{q'\frac{\beta+\zeta}{\alpha}-1}\,\mathrm{d}v\right)^{\frac{1}{q'}}\\
&\lesssim \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[\frac{1}{s^{\frac{\zeta}{\alpha}}}+\frac{1}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\bar p_\alpha(t,y-x)t^{\frac{\beta+\zeta}{\alpha}}.
\end{aligned}
$$

This finally yields

$$
\mathcal{T}_{p',q'}^{-\beta}[\mathfrak{q}_{x,y}^{s,t}]\lesssim \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\bar p_\alpha(t,y-x). \qquad (6.5.8)
$$

Non-thermic part

Noticing that

$$
\|\mathcal{F}(\phi)\star \mathfrak{q}_{x,y}^{s,t}\|_{L^{p'}}\lesssim \|\mathcal{F}(\phi)\|_{L^1}\|\mathfrak{q}_{x,y}^{s,t}\|_{L^{p'}},
$$

we see that (6.5.8) is also a valid bound for the non-thermic part of $\|\mathfrak{q}_{x,y}^{s,t}\|_{\mathbb{B}_{p',q'}^{-\beta}}$.

This concludes the proof of (6.3.7).

Equation (6.3.8) follows from the same proof, using the Hölder regularity in time of the stable kernel instead of its regularity in space (i.e. (6.1.16) instead of (6.1.17)), as well as the forward spatial regularity of Γ^h instead of that of Γ (i.e. (6.1.21) instead of (6.1.24)).

Equations (6.3.9) and (6.3.10) also follow from the same proof, using the standard pointwise estimate $\forall (y,y',z)\in(\mathbb{R}^d)^3, r\in[0,t)$,

$$
|\partial_t\nabla_y p_\alpha(t-r,y-z)-\partial_t\nabla_y p_\alpha(t-r,y'-z)|\lesssim \frac{|y-y'|^\rho}{(t-r)^{\frac{\rho}{\alpha}+1}}\left(\bar p_\alpha(t-r,y-z)+\bar p_\alpha(t-r,y'-z)\right) \qquad (6.5.9)
$$

and the fact that for the considered time variables, $s\asymp r\asymp \tau_s^h$ to deal with the stable kernel. For the estimate (6.3.10), we also rely on the heat kernel estimate (6.1.21) for the forward spatial regularity of Γ^h

Proof of (6.3.11) **and** (6.3.12)

Proof of (6.3.11)**: Hölder regularity involving the one step transition.**
The proof of (6.3.11) is somehow close to the one in the previous subsection. Denote this time

$$\mathfrak{q}_{x,y}^{s,t,h}(\cdot) := \Gamma^h(0,x,\tau_s^h,\cdot)\left[\nabla_y p_\alpha(t-\tau_s^h, y-\cdot) - \nabla_y p_\alpha\left(t-\tau_s^h, y-\cdot-\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u\right)\right].$$

For $s\in[h,\tau_t^h-h]$, we want to estimate $\left\|\mathfrak{q}_{x,y}^{s,t,h}\right\|_{\mathbb{B}^{-\beta}_{p',q'}} = \|\mathcal{F}(\phi)\star\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}} + \mathcal{T}^{-\beta}_{p',q'}\left[\mathfrak{q}_{x,y}^{s,t,h}\right]$. As above let us start with the thermic part of the norm.

Thermic part

Let us split the thermic part into two parts:

$$\begin{aligned}\mathcal{T}^{-\beta}_{p',q'}[\mathfrak{q}_{x,y}^{s,t,h}]^{q'} &= \int_0^t \frac{\mathrm{d}v}{v} v^{(1+\frac{\beta}{\alpha})q'}\|\partial_v p_\alpha(v,\cdot)\star\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}}^{q'} + \int_t^1 \frac{\mathrm{d}v}{v} v^{(1+\frac{\beta}{\alpha})q'}\|\partial_v p_\alpha(v,\cdot)\star\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}}^{q'}\\ &=: \mathcal{T}^{-\beta,(0,t)}_{p',q'}[\mathfrak{q}_{x,y}^{s,t,h}]^{q'} + \mathcal{T}^{-\beta,(t,1)}_{p',q'}[\mathfrak{q}_{x,y}^{s,t,h}]^{q'}.\end{aligned}$$

For the upper part on $(t,1)$, we use the $L^1-L^{p'}$ convolution inequality to write

$$\mathcal{T}^{-\beta,(t,1)}_{p',q'}[\mathfrak{q}_{x,y}^{s,t,h}]^{q'} \lesssim \int_t^1 v^{q'(1+\frac{\beta}{\alpha})-1}\|\partial_v p_\alpha(v,\cdot)\|_{L^1}^{q'}\|\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}}^{q'}\,\mathrm{d}v.$$

Observe now carefully from the pointwise control (6.2.6) on $\mathfrak{b}_h$, , the spatial regularity (6.1.17) of p_α, the heat kernel bound (6.1.20) and the Lebesgue estimate (6.1.18) that

$$\begin{aligned}\|\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}} &= \left\|\Gamma^h(0,x,\tau_s^h,\cdot)\left[\nabla_y p_\alpha(t-\tau_s^h,y-\cdot) - \nabla_y p_\alpha\left(t-\tau_s^h, y-(\cdot+\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u)\right)\right]\right\|_{L^{p'}}\\ &\lesssim \bar{p}_\alpha(t,y-x)\frac{\left((s-\tau_s^h)^{1-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\right)^\delta}{(t-\tau_s^h)^{\frac{1}{\alpha}+\frac{\delta}{\alpha}}}\left(\frac{1}{\tau_s^{h\,\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right),\end{aligned}$$

using as well Lemma 6.4 for the last inequality, i.e. the drift component is negligible w.r.t. the increment of the noise on the corresponding considered time intervals. Hence,

$$\begin{aligned}\mathcal{T}^{-\beta,(t,1)}_{p',q'}[\mathfrak{q}_{x,y}^{s,t,h}] &\lesssim \bar{p}_\alpha(t,y-x)\frac{\left(h^{\frac{\gamma+1-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\right)^\delta}{(t-\tau_s^h)^{\frac{1}{\alpha}+\frac{\delta}{\alpha}}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\left(\int_t^1 v^{\frac{\beta q'}{\alpha}-1}\,\mathrm{d}v\right)^{\frac{1}{q}'}\\ &\lesssim \bar{p}_\alpha(t,y-x)\frac{h^{\frac{\gamma-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]t^{\frac{\beta}{\alpha}},\end{aligned}\tag{6.5.10}$$

taking $\delta=1$ for the last inequality.

Let us now turn to the lower part, for which we use the following cancellation argument:

$$\begin{aligned}\|\partial_v p_\alpha(v,\cdot)\star\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}}^{p'} &= \int\left|\int \partial_v p_\alpha(v,z-w)\mathfrak{q}_{x,y}^{s,t,h}(w)\,\mathrm{d}w\right|^{p'}\mathrm{d}z\\ &= \int\left|\int \partial_v p_\alpha(v,z-w)\left[\mathfrak{q}_{x,y}^{s,t,h}(w)-\mathfrak{q}_{x,y}^{s,t,h}(z)\right]\mathrm{d}w\right|^{p'}\mathrm{d}z.\end{aligned}\tag{6.5.11}$$

We now introduce a diagonal/off-diagonal splitting based on the position of $|w-z|$ w.r.t. $(\tau_s^h)^{\frac{1}{\alpha}}$.

- **Diagonal case:** $|w-z|\le (\tau_s^h)^{\frac{1}{\alpha}}$**.** In this case, the contribution into brackets in (6.5.11) can be bounded as follows: for all $\delta\in[0,1]$, using (6.1.21) and (6.1.20), we have for $\zeta\in(-\beta,-2\beta)$,

$$\begin{aligned}|\mathfrak{q}^{s,t,h}_{x,y}(w)-\mathfrak{q}^{s,t,h}_{x,y}(z)| &= \Bigg|\Bigg[\Gamma^h(0,x,\tau_s^h,w)\Big(\nabla_y p_\alpha(t-\tau_s^h,y-w)-\nabla_y p_\alpha(t-\tau_s^h,y-(w+\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u))\Big)\\
&\quad - \Gamma^h(0,x,\tau_s^h,z)\Big(\nabla_y p_\alpha(t-\tau_s^h,y-z)-\nabla_y p_\alpha(t-\tau_s^h,y-(z+\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u))\Big)\Bigg]\Bigg|\\
&\lesssim |\Gamma^h(0,x,\tau_s^h,w)-\Gamma^h(0,x,\tau_s^h,z)|\frac{\Big(\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u\Big)^\delta}{(t-\tau_s^h)^{\frac{1+\delta}{\alpha}}}\bar p_\alpha(t-\tau_s^h,y-w)\\
&\quad + \Gamma^h(0,x,\tau_s^h,z)\Big(\int_0^1 \mathrm{d}\lambda\Big|\nabla_y^2 p_\alpha(t-\tau_s^h,y-(z+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u))\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u\\
&\quad - \nabla_y^2 p_\alpha(t-\tau_s^h,y-(w+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u))\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u\Big|\Big)\\
&\lesssim \frac{|w-z|^\zeta (h^{\frac{\gamma+1-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}})^\delta}{(t-\tau_s^h)^{\frac{1}{\alpha}+\frac{\delta}{\alpha}}}\Big[\Big(\bar p_\alpha(\tau_s^h,w-x)+\bar p_\alpha(\tau_s^h,z-x)\Big)\frac{1}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}\bar p_\alpha(t-\tau_s^h,y-w)\Big]\\
&\quad + \bar p_\alpha(\tau_s^h,z-x)\frac{|w-z|^\zeta h^{\frac{\gamma+1-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\zeta}{\alpha}}}\Big(\bar p_\alpha(t-\tau_s^h,y-z)+\bar p_\alpha(t-\tau_s^h,y-w)\Big)\\
&\quad + \bar p_\alpha(\tau_s^h,z-x)p_\alpha(t-\tau_s^h,y-w)\frac{|\int_{\tau_s^h}^s\Big(\mathfrak{b}_h(u,z)-\mathfrak{b}_h(u,w)\Big)\,\mathrm{d}u|^\zeta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\zeta}{\alpha}}}h^{\frac{\gamma+1-\beta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}},\end{aligned}$$

using thoroughly (6.2.11) from Lemma 6.4. From (6.2.7), we eventually derive (recalling that $t-\tau_s^h\ge h$ and $\gamma+1-\beta-\zeta>0$):

$$\begin{aligned}|\mathfrak{q}^{s,t,h}_{x,y}(w)-\mathfrak{q}^{s,t,h}_{x,y}(z)| &\lesssim \frac{|w-z|^\zeta}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\Big[\Big(\bar p_\alpha(\tau_s^h,w-x)+\bar p_\alpha(\tau_s^h,z-x)\Big)\frac{h^{\frac{(\gamma-\beta)\delta}{\alpha}}}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}\bar p_\alpha(t-\tau_s^h,y-w)\\
&\quad + \bar p_\alpha(\tau_s^h,z-x)\frac{h^{\frac{\gamma-\beta}{\alpha}}}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\Big(\bar p_\alpha(t-\tau_s^h,y-z)+\bar p_\alpha(t-\tau_s^h,y-w)\Big)\Big]\\
&\lesssim \frac{|w-z|^\zeta h^{\frac{(\gamma-\beta)}{\alpha}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\Big[\Big(\bar p_\alpha(\tau_s^h,w-x)+\bar p_\alpha(\tau_s^h,z-x)\Big)\frac{1}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}\bar p_\alpha(t-\tau_s^h,y-w)\\
&\quad + \bar p_\alpha(\tau_s^h,z-x)\frac{1}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\Big(\bar p_\alpha(t-\tau_s^h,y-z)+\bar p_\alpha(t-\tau_s^h,y-w)\Big)\Big],\end{aligned}$$

taking $\delta=1$ for the last inequality. The terms that are *a priori* delicate to integrate in (6.5.11) are those emphasizing a *cross dependence on the integration variables*, namely $p_\alpha(\tau_s^h,z-x)\bar p_\alpha(t-\tau_s^h,y-w)$. Anyhow, in the current diagonal regime $|w-z|\le(\tau_s^h)^{\frac{1}{\alpha}}$, it holds that

$$p_\alpha(\tau_s^h,z-x)\bar p_\alpha(t-\tau_s^h,y-w)\lesssim p_\alpha(\tau_s^h,w-x)\bar p_\alpha(t-\tau_s^h,y-w),$$

which eventually gives, in the considered diagonal regime:

$$\begin{aligned}|\mathfrak{q}^{s,t,h}_{x,y}(w)-\mathfrak{q}^{s,t,h}_{x,y}(z)| &\lesssim \frac{|w-z|^\zeta h^{\frac{(\gamma-\beta)}{\alpha}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}}\left[\frac{1}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}+\frac{1}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\right]\\
&\quad + \Big[\bar p_\alpha(\tau_s^h,w-x)\bar p_\alpha(t-\tau_s^h,y-w)+\bar p_\alpha(\tau_s^h,z-x)\bar p_\alpha(t-\tau_s^h,y-z)\Big].\end{aligned}\tag{6.5.12}$$

- **Off-diagonal case:** $|w-z|>(\tau_s^h)^{\frac{1}{\alpha}}$**.** In that case, the contribution into brackets in (6.5.11) can be bounded as follows: for all $\zeta \in (-\beta, 1]$,

$$
\begin{aligned}
|\mathfrak{q}_{x,y}^{s,t,h}(w) - \mathfrak{q}_{x,y}^{s,t,h}(z)| &= \Bigg| \Bigg[\Gamma^h(0,x,\tau_s^h,w)\Big(\nabla_y p_\alpha(t-\tau_s^h, y-w) - \nabla_y p_\alpha(t-\tau_s^h, y-(w+\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u))\Big) \\
&\quad - \Gamma^h(0,x,\tau_s^h,z)\Big(\nabla_y p_\alpha(t-\tau_s^h, y-z) - \nabla_y p_\alpha(t-\tau_s^h, y-(z+\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u))\Big)\Bigg]\Bigg| \\
&\lesssim \frac{|w-z|^\zeta}{(\tau_s^h)^{\frac{\zeta}{\alpha}}} \frac{1}{(t-\tau_s^h)^{\frac{2}{\alpha}}} \Big[\int_0^1 \bar{p}_\alpha(\tau_s^h, w-x)\,\mathrm{d}\lambda \bar{p}_\alpha(t-\tau_s^h, y-(w+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u)) \\
&\quad \times \Big| \int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u \Big| \\
&\quad + \int_0^1 \mathrm{d}\lambda \bar{p}_\alpha(\tau_s^h, z-x) \bar{p}_\alpha(t-\tau_s^h, y-(z+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u)) \Big| \int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u \Big| \Big] \\
&\le \frac{|w-z|^\zeta}{(\tau_s^h)^{\frac{\zeta}{\alpha}}} \frac{h^{\frac{\gamma-\beta}{\alpha}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}} \Big(\bar{p}_\alpha(\tau_s^h, w-x)\bar{p}_\alpha(t-\tau_s^h, y-w) + \bar{p}_\alpha(\tau_s^h, z-x)\bar{p}_\alpha(t-\tau_s^h, y-z)\Big)
\end{aligned}
$$

using (6.2.6) and (6.2.11) for the last inequality, recalling as well that $t - \tau_s^h \ge h$.

Plugging this control and (6.5.12) into (6.5.11) yields, using the $L^1 - L^{p'}$ convolution inequality:

$$
\|\partial_v p_\alpha(v,\cdot) \star \mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}} \lesssim \frac{v^{-1+\frac{\zeta}{\alpha}}}{(t-\tau_s^h)^{\frac{1}{\alpha}}} \Big[\frac{1}{(\tau_s^h)^{\frac{\zeta}{\alpha}}} + \frac{1}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}} \Big] \left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}} \right] \bar{p}_\alpha(t, y-x).
$$

This eventually gives:

$$
\mathcal{T}_{p',q'}^{-\beta,(0,t)}[\mathfrak{q}_{x,y}^{s,t,h}] \lesssim \bar{p}_\alpha(t,y-x) \frac{h^{\frac{\gamma-\beta}{\alpha}} \|b\|_{L^r - \mathbb{B}_{p,q}^\beta}}{(t-\tau_s^h)^{\frac{1}{\alpha}}} \left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}} \right] \left[\frac{1}{(\tau_s^h)^{\frac{\zeta}{\alpha}}} + \frac{1}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}} \right] t^{\frac{\beta+\zeta}{\alpha}}, \quad (6.5.13)
$$

which together with (6.5.10) gives that the bound for the thermic part indeed corresponds to the one of the statement.

Non Thermic part. Write:

$$
\|\mathcal{F}(\phi) \star \mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}} \le \|\mathcal{F}(\phi)\|_{L^1} \|\mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}},
$$

so that from the previous computations (using again (6.2.6) and (6.2.11))

$$
\|\mathcal{F}(\phi) \star \mathfrak{q}_{x,y}^{s,t,h}\|_{L^{p'}} \lesssim \bar{p}_\alpha(t,y-x) \frac{h^{\frac{\gamma-\beta}{\alpha}} \|b\|_{L^r - \mathbb{B}_{p,q}^\beta}}{(t-\tau_s^h)^{\frac{1}{\alpha}}} \left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}} \right].
$$

From this last inequality, (6.5.10) and (6.5.13), the statement (6.3.11) is proved.

Proof of (6.3.12)**: yet another control involving the one step transition**
Let us now turn to the proof of (6.3.12). Using the product rule for Besov spaces (6.2.2), we can write

$$\begin{aligned}
&\Big\|\bar{p}_\alpha(\tau_s^h,x,\cdot)\Big(\nabla_y^2 p_\alpha(t-\tau_s^h,y-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))-\nabla_y^2 p_\alpha(t-\tau_s^h,y'-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))\Big)\\
&\qquad\times\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u)\Big\|_{\mathbb{B}^{-\beta}_{p',q'}}\\
&\qquad\lesssim\Big\|\bar{p}_\alpha(\tau_s^h,x,\cdot)\Big(\nabla_y^2 p_\alpha(t-\tau_s^h,y-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))-\nabla_y^2 p_\alpha(t-\tau_s^h,y'-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u))\Big)\Big\|_{\mathbb{B}^{-\beta}_{p',q'}}\\
&\qquad\quad\times\Big\|\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u\Big\|_{\mathbb{B}^{-\beta+\varepsilon}_{\infty,\infty}} \qquad (6.5.14)
\end{aligned}$$

for any $\varepsilon>0$. Then, write, using Lemma 6.3 twice,

$$\begin{aligned}
\Big\|\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u\Big\|_{\mathbb{B}^{-\beta+\varepsilon}_{\infty,\infty}} &\asymp \Big\|\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u\Big\|_{L^\infty}+\sup_{z\neq z'\in(\mathbb{R}^d)^2}\frac{|\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u-\int_{\tau_s^h}^s \mathfrak{b}_h(u,z')\,\mathrm{d}u|}{|z-z'|^{-\beta+\varepsilon}}\\
&\lesssim h^{\frac{\gamma+1-\beta}{\alpha}}+h^{\frac{\gamma+1-\varepsilon}{\alpha}}\lesssim h^{\frac{\gamma+1-\varepsilon}{\alpha}}.
\end{aligned}$$

Equation (6.3.12) then follows from controlling the previous $\mathbb{B}^{-\beta}_{p',q'}$ norm in the same way as for (6.3.11). Namely, denoting this time

$$\begin{aligned}
&\mathfrak{q}^{s,t,h}_{x,y}(\cdot)\\
:=&\Gamma^h(0,x,\tau_s^h,\cdot)\left[\nabla_y^2 p_\alpha\left(t-\tau_s^h,y-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u)\right)-\nabla_y^2 p_\alpha\left(t-\tau_s^h,y'-(\cdot+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,\cdot)\,\mathrm{d}u)\right)\right]
\end{aligned}$$

we have, in the diagonal regime $|z-w|\le (\tau_s^h)^{\frac{1}{\alpha}}$, using (6.1.17) and Lemma 6.3 profusely,

$$
\begin{aligned}
&|\mathfrak{q}^{s,t,h}_{x,y}(w)-\mathfrak{q}^{s,t,h}_{x,y}(z)|\\
&=\Gamma^h(0,x,\tau_s^h,w)\left[\nabla_y^2 p_\alpha\left(t-\tau_s^h,y-(w+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u)\right)-\nabla_y^2 p_\alpha\left(t-\tau_s^h,y'-(w+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u)\right)\right]\\
&\quad-\Gamma^h(0,x,\tau_s^h,z)\left[\nabla_y^2 p_\alpha\left(t-\tau_s^h,y-(z+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u)\right)-\nabla_y^2 p_\alpha\left(t-\tau_s^h,y'-(z+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u)\right)\right]\\
&\lesssim\left[\Gamma^h(0,x,\tau_s^h,w)-\Gamma^h(0,x,\tau_s^h,z)\right]\frac{|y-y'|^\delta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\delta}{\alpha}}}\left[p_\alpha(t-\tau_s^h,y-w)+p_\alpha(t-\tau_s^h,y'-w)\right]\\
&\quad+\bar p_\alpha(\tau_s^h,z-x)\Bigg(|y-y'|\int_0^1\Bigg|\nabla_y^3 p_\alpha\left(t-\tau_s^h,y'+\mu(y-y')-(z+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,z)\,\mathrm{d}u)\right)\\
&\qquad\qquad-\nabla_y^3 p_\alpha\left(t-\tau_s^h,y'+\mu(y-y')-(w+\lambda\int_{\tau_s^h}^s \mathfrak{b}_h(u,w)\,\mathrm{d}u)\right)\Bigg|\,\mathrm{d}\mu\mathbb{1}_{|y-y'|\le(t-\tau_s^h)^{\frac{1}{\alpha}}}\\
&\quad+\frac{|y-y'|^\delta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\delta}{\alpha}+\frac{\zeta}{\alpha}}}\left(\bar p_\alpha(t-\tau_s^h,y-z)+\bar p_\alpha(t-\tau_s^h,y-w)+\bar p_\alpha(t-\tau_s^h,y'-z)+\bar p_\alpha(t-\tau_s^h,y'-w)\right)\\
&\quad\times\left|z-w+\lambda\int_{\tau_s^h}^s(\mathfrak{b}_h(u,z)-\mathfrak{b}_h(u,w))\,\mathrm{d}u\right|^\zeta\mathbb{1}_{|y-y'|>(t-\tau_s^h)^{\frac{1}{\alpha}}}\Bigg)\\
&\lesssim\bar p_\alpha(\tau_s^h,w-x)\frac{|w-z|^\zeta}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}\frac{|y-y'|^\delta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\delta}{\alpha}}}\left[p_\alpha(t-\tau_s^h,y-w)+p_\alpha(t-\tau_s^h,y'-w)\right]\\
&\quad+\bar p_\alpha(\tau_s^h,z-x)\frac{|y-y'|^\delta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\delta+\zeta}{\alpha}}}\left|z-w+\lambda\int_{\tau_s^h}^s(\mathfrak{b}_h(u,z)-\mathfrak{b}_h(u,w))\,\mathrm{d}u\right|^\zeta\\
&\qquad\times[\bar p_\alpha(t-\tau_s^h,y-z)+\bar p_\alpha(t-\tau_s^h,y-w)+\bar p_\alpha(t-\tau_s^h,y'-z)+\bar p_\alpha(t-\tau_s^h,y'-w)]\\
&\lesssim\bar p_\alpha(\tau_s^h,w-x)\frac{|w-z|^\zeta}{(\tau_s^h)^{\frac{\zeta}{\alpha}}}\frac{|y-y'|^\delta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\delta}{\alpha}}}\left[p_\alpha(t-\tau_s^h,y-w)+p_\alpha(t-\tau_s^h,y'-w)\right]\\
&\quad+\bar p_\alpha(\tau_s^h,z-x)\frac{|y-y'|^\delta}{(t-\tau_s^h)^{\frac{2}{\alpha}+\frac{\delta+\zeta}{\alpha}}}|z-w|^\zeta\,[\bar p_\alpha(t-\tau_s^h,y-z)+\bar p_\alpha(t-\tau_s^h,y'-z)+\bar p_\alpha(t-\tau_s^h,y-w)\\
&\qquad+\bar p_\alpha(t-\tau_s^h,y'-w)].
\end{aligned}
$$

We can again get rid of the cross terms in the integration variable in the above inequality recalling that in the considered diagonal regime $|w-z|\le(\tau_s^h)^{\frac{1}{\alpha}}$ it holds that $\bar p_\alpha(\tau_s^h,z-x)\lesssim \bar p_\alpha(\tau_s^h,w-x)$. The rest of the proof is similar to the one of (6.3.11).

6.6 Proof of the heat-kernel estimates of Proposition 6.1

6.6.1 Heat-kernel bounds for the Euler scheme: proof of (6.1.22).

Proof of (6.1.22). First, let us state that the duhamel representation (6.3.2)

$$\Gamma^h(0,x,t,\cdot)=p_\alpha(t,y-x)-\int_0^t \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s,X^h_{\tau_s^h})\cdot\nabla_y p_\alpha(t-s,y-X_s^h)\right]\,\mathrm{d}s \tag{6.6.1}$$

was already obtained in [FJM24] because $\mathfrak{b}_h$ can be seen as a Lebesgue drift.

Similarly to the proof of the main theorem, we will control $\left\|\frac{\Gamma^h(0,x,t,\cdot)}{\bar p_\alpha(t,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}$ for $\rho\in(-\beta,\gamma-\beta)$ using a circular argument.

Control of the supremum norm

We write the Duhamel formula as follows:

$$\begin{aligned}\Gamma^h(0,x,t,y) &= p_\alpha(t,y-x) - \int_0^h \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s,X^h_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X^h_s)\right]\mathrm{d}s\\ &\quad - \int_h^t \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s,X^h_{\tau^h_s})\cdot\nabla_y p_\alpha(t-s,y-X^h_s)\right]\mathrm{d}s\\ &=: \Delta_1(y)+\Delta_2(y)+\Delta_3(y).\end{aligned}$$

For Δ_2, using (6.2.5), (6.2.11) and (6.1.15),

$$\begin{aligned}|\Delta_2| &= \left|\int_0^h\int \mathfrak{b}_h(s,x)\cdot p_\alpha\left(s,z-x-\int_0^s \mathfrak{b}_h(r,x)\,\mathrm{d}r\right)\nabla_y p_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\right|\\ &\lesssim \int_0^h\int s^{-\frac{d}{\alpha p}+\frac{\beta}{\alpha}}\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\bar{p}_\alpha(s,z-x)(t-s)^{-\frac{1}{\alpha}}\bar{p}_\alpha(t-s,y-z)\,\mathrm{d}z\,\mathrm{d}s\\ &\lesssim \bar{p}_\alpha(t,y-x)\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}\left(\int_0^h s^{-\frac{dr'}{\alpha p}+\frac{\beta r'}{\alpha}}(t-s)^{-\frac{r'}{\alpha}}\,\mathrm{d}s\right)^{\frac{1}{r'}}\\ &\lesssim \bar{p}_\alpha(t,y-x)h^{1-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}}t^{-\frac{1}{\alpha}} \lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}h^{\frac{1}{\alpha}-\frac{\beta}{\alpha}}t^{-\frac{1}{\alpha}} \lesssim \bar{p}_\alpha(t,y-x)h^{\frac{\gamma}{\alpha}}t^{-\frac{\beta}{\alpha}}.\end{aligned} \tag{6.6.2}$$

For Δ_3, using the harmonicity of the stable kernel, for $\rho>-\beta$,

$$\begin{aligned}|\Delta_3| &= \left|\int_h^t\int \Gamma^h(0,x,\tau^h_s,z)\mathfrak{b}_h(s,z)\cdot\nabla_y p_\alpha\left(t-\tau^h_s,y-z-\int_{\tau^h_s}^s \mathfrak{b}_h(r,z)\,\mathrm{d}r\right)\mathrm{d}z\,\mathrm{d}s\right|\\ &\lesssim \int_h^t \left\|\frac{\Gamma^h(0,x,\tau^h_s,\cdot)}{\bar{p}_\alpha(\tau^h_s,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}\|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\\ &\qquad\times\left\|\bar{p}_\alpha(\tau^h_s,\cdot-x)\nabla_y p_\alpha\left(t-\tau^h_s,y-\cdot-\int_{\tau^h_s}^s \mathfrak{b}_h(r,\cdot)\,\mathrm{d}r\right)\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\mathrm{d}s.\end{aligned}$$

Using a triangular inequality, (6.3.11) (in which we can trivially replace Γ^h with p_α) and (6.3.4), we have, for $c=c_0$ and for any $\zeta\in(0,1]$,

$$\begin{aligned}&\left\|\bar{p}_\alpha(\tau^h_s,\cdot-x)\nabla_y p_\alpha\left(t-\tau^h_s,y-\cdot-\int_{\tau^h_s}^s \mathfrak{b}_h(r,\cdot)\,\mathrm{d}r\right)\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ &\lesssim \left\|\bar{p}_\alpha(\tau^h_s,\cdot-x)\left[\nabla_y p_\alpha\left(t-\tau^h_s,y-\cdot-\int_{\tau^h_s}^s \mathfrak{b}_h(r,\cdot)\,\mathrm{d}r\right)-\nabla_y p_\alpha\left(t-\tau^h_s,y-\cdot\right)\right]\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ &\quad+\left\|\bar{p}_\alpha(\tau^h_s,\cdot-x)\nabla_y p_\alpha\left(t-\tau^h_s,y-\cdot\right)\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\\ &\lesssim \bar{p}_\alpha(t,y-x)\left(1+h^{\frac{\gamma-\beta}{\alpha}}\right)\frac{t^{\frac{\beta}{\alpha}}}{s^{\frac{\rho}{\alpha}}(t-\tau^h_s)^{\frac{1}{\alpha}}}\left[\frac{1}{(\tau^h_s)^{\frac{d}{\alpha p}}}+\frac{1}{(t-\tau^h_s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{(\tau^h_s)^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau^h_s)^{\frac{\zeta}{\alpha}}}\right].\end{aligned} \tag{6.6.3}$$

Set now, for $s\in(0,T]$,

$$\tilde{g}_{h,\rho}(s) := \left\|\frac{\Gamma^h(0,x,s,\cdot)}{\bar{p}_\alpha(s,\cdot-x)}\right\|_{L^\infty} + s^{\frac{\rho}{\alpha}}\sup_{z\neq z'\in(\mathbb{R}^d)^2}\left|\frac{\frac{\Gamma^h(0,x,s,z)}{\bar{p}_\alpha(s,z-x)}-\frac{\Gamma^h(0,x,s,z')}{\bar{p}_\alpha(s,z'-x)}}{|z-z'|^\rho}\right| \gtrsim s^{\frac{\rho}{\alpha}}\left\|\frac{\Gamma^h(0,x,s,\cdot)}{\bar{p}_\alpha(s,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}. \tag{6.6.4}$$

Plugging this and (6.6.3) into Δ_3 along with (6.2.8) yields

$$|\Delta_3| \lesssim \bar{p}_\alpha\,(t, y-x) \int_h^t \tilde{g}_{h,\rho}(\tau_s^h)\|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \times \frac{t^{\frac{\beta}{\alpha}}}{s^{\frac{\rho}{\alpha}}(t-\tau_s^h)^{\frac{1}{\alpha}}}\left[\frac{1}{(\tau_s^h)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau_s^h)^{\frac{d}{\alpha p}}}\right]\left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{(\tau_s^h)^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau_s^h)^{\frac{\zeta}{\alpha}}}\right] \mathrm{d}s.$$

Note that, on the considered time interval, $s \asymp \tau_s^h$, yielding

$$\begin{aligned}|\Delta_3| &\lesssim \bar{p}_\alpha\,(t, y-x) \sup_{s\in(h,T]} \tilde{g}_{h,\rho}(s) \int_0^t \frac{t^{\frac{\beta}{\alpha}}}{s^{\frac{\rho}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}} + \frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right] \mathrm{d}s \\ &\lesssim \bar{p}_\alpha(t, y-x) t^{\frac{\gamma-\beta-\rho}{\alpha}} \sup_{s\in(h,T]} \tilde{g}_{h,\rho}(s). \end{aligned} \tag{6.6.5}$$

Gathering (6.6.2) and (6.6.5), we get recalling that ρ can be chosen so that $\gamma - \beta - \rho > 0$,

$$\left\|\frac{\Gamma^h(0,x,t,\cdot)}{\bar{p}_\alpha(t,\cdot - x)}\right\|_{L^\infty} \lesssim 1 + T^{\frac{\gamma-\beta-\rho}{\alpha}} \sup_{s\in(h,T]} \tilde{g}_{h,\rho}(s). \tag{6.6.6}$$

Control of the Hölder modulus

We will now control the ρ-Hölder modulus of $\Gamma^h(0,x,t,\cdot)/\bar{p}_\alpha(t,\cdot - x)$ in the diagonal regime, i.e. for $(y,y) \in (\mathbb{R}^d)^2$ such that $|y-y'| \le t^{\frac{1}{\alpha}}$ (since otherwise the required control is trivial). Similarly to the proof of the main theorem, let us write, using (6.1.17) (which readily extends to $\bar{p}_\alpha$ instead of p_α),

$$\begin{aligned}\left|\frac{\Gamma^h(0,x,t,y)}{\bar{p}_\alpha(t,y-x)} - \frac{\Gamma^h(0,x,t,y')}{\bar{p}_\alpha(t,y'-x)}\right| &\le \left|\frac{\Gamma^h(0,x,t,y) - \Gamma^h(0,x,t,y')}{\bar{p}_\alpha(t,y-x)}\right| + \Gamma^h(0,x,t,y')\left|\frac{1}{\bar{p}_\alpha(t,y-x)} - \frac{1}{\bar{p}_\alpha(t,y'-x)}\right| \\ &\lesssim \left|\frac{\Gamma^h(0,x,t,y) - \Gamma^h(0,x,t,y')}{\bar{p}_\alpha(t,y-x)}\right| + \frac{|y-y'|^\rho}{t^{\frac{\rho}{\alpha}}}\left\|\frac{\Gamma^h(0,x,t,\cdot)}{\bar{p}_\alpha(t,\cdot - x)}\right\|_{L^\infty}. \end{aligned} \tag{6.6.7}$$

The error expansion for $|\Gamma^h(0,x,t,y) - \Gamma^h(0,x,t,y')|$ writes

$$\begin{aligned}\Gamma^h(0,x,t,y) - \Gamma^h(0,x,t,y') &= p_\alpha(t,y-x) - p_\alpha(t,y'-x) \\ &\quad - \int_0^h \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s, X^h_{\tau_s^h}) \cdot (\nabla_y p_\alpha(t-s, y - X_s^h) - \nabla_{y'} p_\alpha(t-s, y'-X_s^h))\right] \mathrm{d}s \\ &\quad - \int_h^t \mathbb{E}_{0,x}\left[\mathfrak{b}_h(s, X^h_{\tau_s^h}) \cdot (\nabla_y p_\alpha(t-s, y - X_s^h) - \nabla_{y'} p_\alpha(t-s, y'-X_s^h))\right] \mathrm{d}s \\ &=: \Delta_1(y,y') + \Delta_2(y,y') + \Delta_3(y,y'),\end{aligned}$$

where with a slight abuse of notation we do not emphasize the dependence of these quantities on x, t and *a priori* on h (we will actually prove that the estimates are uniform w.r.t. this last parameter). For Δ_1, using (6.1.17), we have

$$|\Delta_1(y,y')| = |p_\alpha(t,y-x) - p_\alpha(t,y'-x)| \lesssim \frac{|y-y'|^\rho}{t^{\frac{\rho}{\alpha}}}\bar{p}_\alpha(t,y-x). \tag{6.6.8}$$

For Δ_2, using (6.2.5), (6.2.11) and (6.1.17),

$$
\begin{aligned}
|\Delta_2(y,y')| &= \left| \int_0^h \int \mathfrak{b}_h(s,x)\cdot p_\alpha\left(s, z-x-\int_0^s \mathfrak{b}_h(r,x)\,\mathrm{d}r\right)[\nabla_y p_\alpha(t-s,y-z) - \nabla_{y'} p_\alpha(t-s,y'-z)]\,\mathrm{d}z\,\mathrm{d}s \right| \\
&\lesssim \int_0^h \int s^{-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \bar{p}_\alpha(s,z-x) \frac{|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} [\bar{p}_\alpha(t-s,y-z) + \bar{p}_\alpha(t-s,y'-z)]\,\mathrm{d}z\,\mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)|y-y|^\rho \|b\|_{L^r-\mathbb{B}^\beta_{p,q}} \left(\int_0^h s^{-\frac{dr'}{\alpha p}+\frac{\beta r'}{\alpha}} (t-s)^{-\frac{r'(\rho+1)}{\alpha}}\,\mathrm{d}s \right)^{\frac{1}{r'}} \\
&\lesssim \bar{p}_\alpha(t,y-x)|y-y|^\rho h^{1-\frac{1}{r}-\frac{d}{\alpha p}+\frac{\beta}{\alpha}} t^{-\frac{1+\rho}{\alpha}} \\
&\lesssim \bar{p}_\alpha(t,y-x)|y-y|^\rho h^{\frac{\gamma}{\alpha}} h^{\frac{1}{\alpha}-\frac{\beta}{\alpha}} t^{-\frac{1+\rho}{\alpha}} \lesssim \bar{p}_\alpha(t,y-x)\frac{|y-y|^\rho}{t^{\frac{\rho}{\alpha}}} h^{\frac{\gamma}{\alpha}} t^{-\frac{\beta}{\alpha}}.
\end{aligned}
\tag{6.6.9}
$$

For Δ_3, using the harmonicity of the stable kernel, for $\rho > -\beta$,

$$
\begin{aligned}
&|\Delta_3(y,y')| \\
=& \Bigg| \int_h^t \int \Gamma^h(0,x,\tau^h_s,z)\mathfrak{b}_h(s,z) \\
&\quad \cdot \left[\nabla_y p_\alpha\left(t-\tau^h_s, y-z-\int_{\tau^h_s}^s \mathfrak{b}_h(r,z)\,\mathrm{d}r\right) - \nabla_{y'} p_\alpha\left(t-\tau^h_s, y'-z-\int_{\tau^h_s}^s \mathfrak{b}_h(r,z)\,\mathrm{d}r\right)\right] \mathrm{d}z\,\mathrm{d}s \Bigg| \\
\lesssim& \int_h^t \left\| \frac{\Gamma^h(0,x,\tau^h_s,\cdot)}{\bar{p}_\alpha(\tau^h_s,\cdot-x)} \right\|_{\mathbb{B}^\rho_{\infty,\infty}} \|\mathfrak{b}_h(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \\
&\times \left\| p_\alpha(\tau^h_s,\cdot-x) \left[\nabla_y p_\alpha\left(t-\tau^h_s, y-\cdot-\int_{\tau^h_s}^s \mathfrak{b}_h(r,\cdot)\,\mathrm{d}r\right) - \nabla_{y'} p_\alpha\left(t-\tau^h_s, y'-\cdot-\int_{\tau^h_s}^s \mathfrak{b}_h(r,\cdot)\,\mathrm{d}r\right)\right] \right\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}s.
\end{aligned}
$$

Similarly to the computations performed to prove (6.3.12) (in which we again take p_α instead of Γ^h) and the definition of $\tilde{g}_{h,\rho}(s)$ along with (6.2.8) yields, for $\zeta \in (-\beta, 1]$,

$$
\begin{aligned}
|\Delta_3(y,y')| &\lesssim \bar{p}_\alpha(t,y-x) \int_h^t \tilde{g}_{h,\rho}(\tau^h_s) \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \\
&\qquad \times \frac{t^{\frac{\beta}{\alpha}}|y-y'|^\rho(1+h^{\frac{\gamma-\beta}{\alpha}})}{s^{\frac{\rho}{\alpha}}(t-\tau^h_s)^{\frac{1+\rho}{\alpha}}} \left[\frac{1}{(\tau^h_s)^{\frac{d}{\alpha p}}} + \frac{1}{(t-\tau^h_s)^{\frac{d}{\alpha p}}} \right] \left[1 + \frac{t^{\frac{\zeta}{\alpha}}}{(\tau^h_s)^{\frac{\zeta}{\alpha}}} + \frac{t^{\frac{\zeta}{\alpha}}}{(t-\tau^h_s)^{\frac{\zeta}{\alpha}}} \right] \mathrm{d}s \\
&\lesssim \bar{p}_\alpha(t,y-x)|y-y'|^\rho t^{\frac{\gamma-\beta-2\rho}{\alpha}} \sup_{s\in(h,T]} \tilde{g}_{h,\rho}(s),
\end{aligned}
\tag{6.6.10}
$$

using the fact that $s \asymp \tau^h_s$ on the considered time interval. We then have, plugging (6.6.8), (6.6.9) and (6.6.10) into (6.6.7),

$$
\left| \frac{\Gamma^h(0,x,t,y)}{\bar{p}_\alpha(t,y-x)} - \frac{\Gamma^h(0,x,t,y')}{\bar{p}_\alpha(t,y'-x)} \right| \lesssim \frac{|y-y'|^\rho}{t^{\frac{\rho}{\alpha}}} T^{\frac{\gamma-\beta-\rho}{\alpha}} \sup_{s\in(h,T]} \tilde{g}_{h,\rho}(s),
\tag{6.6.11}
$$

which, together with (6.6.6) and recalling that $\gamma-\beta-\rho$ can be chosen to be positive, concludes the proof of (6.1.22). □

6.6.2 Time sensitivity of the heat-kernel: proof of (6.1.26).

Let us assume $t' \geq t$ and $0 \leq t'-t \leq \frac{t}{2}$ and write from the Duhamel representation (6.3.1) of the density that:

$$\begin{aligned}\Gamma(0,x,t,y)-\Gamma(0,x,t',y) =& p_\alpha(t,y-x)-p_\alpha(t',y-x)\\ &-\int_0^t \mathbb{E}_{t_k,x}\Big[b(s,X_s)\cdot\Big(\nabla_y p_\alpha(t-s,y-X_s)-\nabla_y p_\alpha(t'-s,y-X_s)\Big)\Big]\,\mathrm{d}s\\ &+\int_t^{t'} \mathbb{E}_{t_k,x}\Big[b(s,X_s)\cdot\Big(\nabla_y p_\alpha(t'-s,y-X_s)\Big)\Big]\,\mathrm{d}s. \end{aligned}\tag{6.6.12}$$

Thus, from (6.1.16) we get:

$$\begin{aligned}|\Gamma(0,x,t,y)-\Gamma(0,x,t',y)| \lesssim& \Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}}\bar p_\alpha(t,y-x)\\ &+\int_0^t\Big|\int \Gamma(0,x,s,z)b(s,z)\Big(\nabla_y p_\alpha(t'-s,y-z)-\nabla_y p_\alpha(t-s,y-z)\Big)\,\mathrm{d}z\Big|\,\mathrm{d}s\\ &+\int_t^{t'}\Big|\int \Gamma(0,x,s,z)b(s,z)\nabla_y p_\alpha(t'-s,y-z)\,\mathrm{d}z\Big|\,\mathrm{d}s.\end{aligned}$$

Using now the product rule (6.2.2) and (6.1.25), (6.3.4) (taking therein $\zeta\in(-\beta,1)$), we get for $\rho>-\beta$,

$$\begin{aligned}&|\Gamma(0,x,t,y)-\Gamma(0,x,t',y)| \lesssim \Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}}\bar p_\alpha(t,y-x)\\ &+\int_0^t \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\Big\|\frac{\Gamma(0,x,s,\cdot)}{\bar p_\alpha(s,\cdot-x)}\Big\|_{\mathbb{B}^\rho_{\infty,\infty}}\|\bar p_\alpha(s,\cdot-x)\Big(\nabla_y p_\alpha(t-s,y-\cdot)-\nabla_y p_\alpha(t'-s,y-\cdot)\Big)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\ &+\int_t^{t'} \|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\Big\|\frac{\Gamma(0,x,s,\cdot)}{\bar p_\alpha(s,\cdot-x)}\Big\|_{\mathbb{B}^\rho_{\infty,\infty}}\|\bar p_\alpha(s,\cdot-x)\nabla_y p_\alpha(t'-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\ &\lesssim \bar p_\alpha(t,x-y)\Bigg(\Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}}+\int_0^t\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}s^{-\frac{\rho}{\alpha}}\frac{(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}}{(t-s)^{\frac{\gamma-\varepsilon+1}{\alpha}}}(t')^{\frac{\beta}{\alpha}}\Bigg[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\Bigg]\Bigg[\frac{(t')^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{(t')^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\Bigg]\,\mathrm{d}s\\ &+\int_t^{t'}\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}s^{-\frac{\rho}{\alpha}}\frac{1}{(t'-s)^{\frac{1}{\alpha}}}(t')^{\frac{\beta}{\alpha}}\Bigg[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t'-s)^{\frac{d}{\alpha p}}}\Bigg]\Bigg[\frac{(t')^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{(t')^{\frac{\zeta}{\alpha}}}{(t'-s)^{\frac{\zeta}{\alpha}}}\Bigg]\,\mathrm{d}s\Bigg)\\ &\lesssim \bar p_\alpha(t,x-y)\Bigg(\Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}}\\ &+\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}(t')^{\frac{\beta}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}\Big(\int_0^t\frac{s^{-\frac{\rho}{\alpha}r'}}{(t-s)^{\frac{\gamma-\varepsilon+1}{\alpha}r'}}\Bigg[\frac{1}{s^{\frac{dr'}{\alpha p}}}+\frac{1}{(t-s)^{\frac{dr'}{\alpha p}}}\Bigg]\Bigg[\frac{t^{\frac{\zeta r'}{\alpha}}}{s^{\frac{\zeta r'}{\alpha}}}+\frac{t^{\frac{\zeta r'}{\alpha}}}{(t-s)^{\frac{\zeta r'}{\alpha}}}\Bigg]\,\mathrm{d}s\Big)^{\frac{1}{r'}}\\ &+\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}(t')^{\frac{\beta-\rho}{\alpha}}\Big(\int_t^{t'}\frac{1}{(t'-s)^{\frac{r'}{\alpha}}}\Bigg[\frac{1}{s^{\frac{dr'}{\alpha p}}}+\frac{1}{(t'-s)^{\frac{dr'}{\alpha p}}}\Bigg]\frac{(t')^{\frac{r'\zeta}{\alpha}}}{(t'-s)^{\frac{r'\zeta}{\alpha}}}\,\mathrm{d}s\Big)^{\frac{1}{r'}}\Bigg)\end{aligned}$$

recalling that, since $t'-t\leq t/2$, $t'/s\lesssim 1$, $s\in[t,t']$. Hence,

$$\begin{aligned}&|\Gamma(0,x,t,y)-\Gamma(0,x,t',y)|\\ &\lesssim \bar p_\alpha(t,x-y)\Bigg(\Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}}+\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}t^{\frac{\beta}{\alpha}}\Big[(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}t^{1-\frac{1}{r}-\frac{\rho}{\alpha}-\frac{\gamma-\varepsilon+1}{\alpha}-\frac{d}{p\alpha}}+t^{\frac{\zeta}{\alpha}-\frac{\rho}{\alpha}}(t'-t)^{1-\frac{1}{r}-\frac{1}{\alpha}-\frac{d}{p\alpha}-\frac{\zeta}{\alpha}}\Big]\Bigg)\\ &\lesssim \bar p_\alpha(t,x-y)\Bigg(\Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}}+\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}t^{-\frac{\beta+\rho}{\alpha}+\frac{\varepsilon}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}\Bigg).\end{aligned}$$

We thus get:

$$\Big\|\frac{\Gamma(0,x,t,\cdot)-\Gamma(0,x,t',\cdot)}{\bar p_\alpha(t',x-\cdot)}\Big\|_{L^\infty}\lesssim\Big(\frac{t'-t}{t}\Big)^{\frac{\gamma-\varepsilon}{\alpha}},\tag{6.6.13}$$

provided $\beta+\rho<\gamma$, which can always be achieved taking $\rho=-\beta+\eta$ for $\eta<\gamma$. It now remains to control the thermic part of the Besov norm, i.e. with the notation of (6.2.1) the quantity

$$\mathcal{T}^{\rho}_{\infty,\infty}\Big(\frac{\Gamma(0,x,t,\cdot)-\Gamma(0,x,t',\cdot)}{\bar p_\alpha(t',\cdot-x)}\Big):=\sup_{v\in[0,1]}v^{1-\frac{\rho}{\alpha}}\|\partial_v\tilde p_\alpha\star\frac{(\Gamma(0,x,t,\cdot)-\Gamma(0,x,t',\cdot))}{\bar p_\alpha(t',\cdot-x)}\|_{L^\infty}.$$

Write now from the expansion (6.6.12):

$$\begin{aligned}&\mathcal{T}^{\rho}_{\infty,\infty}\Big(\frac{\Gamma(0,x,t,\cdot)-\Gamma(0,x,t',\cdot)}{\bar p_\alpha(t',\cdot-x)}\Big)\\
\le&\mathcal{T}^{\rho}_{\infty,\infty}\Big(\frac{p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x)}{\bar p_\alpha(t',\cdot-x)}\Big)\\
&+\mathcal{T}^{\rho}_{\infty,\infty}\Big(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_0^t\int\Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\Big)\\
&+\mathcal{T}^{\rho}_{\infty,\infty}\Big(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_t^{t'}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\,\mathrm{d}s\Big)=:\sum_{i=1}^{3}\mathcal{T}^{\rho}_{\infty,\infty,i}(x,t,t').\end{aligned}\tag{6.6.14}$$

Write:

$$T^{\rho}_{\infty,\infty,1}(x,t,t')=\sup_{v\in[0,1]}v^{1-\frac{\rho}{\alpha}}\Big\|\partial_v\tilde p_\alpha(v,\cdot)\star\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}\Big\|_{L^\infty}.$$

Recall that t,t' are assumed to be small and that, in that setting, t appears as a natural cutting level in the study of the thermic part of the norm. Namely, for $v\ge t$ we readily get from (6.1.16) and (6.1.16)

$$\Big\|\partial_v\tilde p_\alpha(v,\cdot)\star\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}\Big\|_{L^\infty}\lesssim\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}\frac{1}{v}.$$

In particular:

$$\sup_{v\in[t,1]}v^{1-\frac{\rho}{\alpha}}\Big\|\partial_v\tilde p_\alpha(v,\cdot)\star\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}\Big\|_{L^\infty}\lesssim\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}\frac{1}{t^{\frac{\rho}{\alpha}}}.\tag{6.6.15}$$

Write now for $v\in[0,t]$, for all $y\in\mathbb{R}^d$:

$$\begin{aligned}&\Big|\partial_v\tilde p_\alpha(v,\cdot)\star\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}(y)\Big|\\
=&\Big|\int\partial_v\tilde p_\alpha(v,y-z)\Big(\frac{(p_\alpha(t,z-x)-p_\alpha(t',z-x))}{\bar p_\alpha(t',z-x)}-\frac{(p_\alpha(t,y-x)-p_\alpha(t',y-x))}{\bar p_\alpha(t',y-x)}\Big)\,\mathrm{d}z\Big|\\
\lesssim&\frac{1}{v}\Big(\int\bar p_\alpha(v,y-z)\mathbb{1}_{|y-z|>t^{\frac{1}{\alpha}}}2\Big\|\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}\Big\|_{L^\infty}\,\mathrm{d}z\\
&+\int\bar p_\alpha(v,y-z)\mathbb{1}_{|y-z|\le t^{\frac{1}{\alpha}}}\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}\frac{|y-z|^{\rho}}{t^{\frac{\rho}{\alpha}}}\,\mathrm{d}z\Big)\end{aligned}$$

using a spatial Taylor expansion and (6.1.16) (recalling as well that the diagonal regime holds for the *non thermic* densities) for the last inequality. Hence, from (6.6.13),

$$\begin{aligned}&\Big|\partial_v\tilde p_\alpha(v,\cdot)\star\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}(y)\Big|\\
\lesssim&\frac{1}{v}\frac{1}{t^{\frac{\rho}{\alpha}}}\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}\int\bar p_\alpha(v,y-z)|y-z|^{\rho}dz\lesssim\frac{1}{v^{1-\frac{\rho}{\alpha}}}\frac{1}{t^{\frac{\rho}{\alpha}}}\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}.\end{aligned}$$

Hence,

$$\sup_{v\in[0,t]} v^{1-\frac{\rho}{\alpha}}\|\partial_v\tilde p_\alpha(v,\cdot)\star\frac{(p_\alpha(t,\cdot-x)-p_\alpha(t',\cdot-x))}{\bar p_\alpha(t',\cdot-x)}\|_{L^\infty}\lesssim\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}\frac{1}{t^{\frac{\rho}{\alpha}}},$$

which together with (6.6.15) yields

$$T^{\rho}_{\infty,\infty,1}(x,t,t')\lesssim\frac{|t-t'|^{\frac{\gamma-\varepsilon}{\alpha}}}{t^{\frac{\gamma-\varepsilon}{\alpha}}}\frac{1}{t^{\frac{\rho}{\alpha}}}. \tag{6.6.16}$$

which is precisely the expected expected bound.

Let us now turn to $\mathcal{T}^{\rho}_{\infty,\infty,3}(x,t,t')$

$$\mathcal{T}^{\rho}_{\infty,\infty,3}(x,t,t')=\mathcal{T}^{\rho}_{\infty,\infty}\Big(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_t^{t'}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\,\mathrm{d}s\Big).$$

We proceed with the same previous dichotomy for the time variable:

- For $v\in[t,1]$ write:

$$\begin{aligned}
&\left\|\partial_v\tilde p_\alpha(v,\cdot)\star\left(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_t^{t'}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\,\mathrm{d}s\right)\right\|_{L^\infty}\\
\lesssim&\frac{C}{v}\int_t^{t'}\left\|\frac{1}{\bar p_\alpha(t',\cdot-x)}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\right\|_{L^\infty}\mathrm{d}s\\
\lesssim&\frac{1}{v}\int_t^{t'}\|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}\sup_{y\in\mathbb{R}^d}\frac{1}{\bar p_\alpha(t',\cdot-y)}\left\|\frac{\Gamma(0,x,s,\cdot)}{\bar p_\alpha(s,x,\cdot)}\right\|_{\mathbb{B}^{\rho}_{\infty,\infty}}\|\bar p_\alpha(s,x,\cdot)\nabla p_\alpha(t'-s,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}\mathrm{d}s\\
\lesssim&\frac{1}{v}\int_t^{t'}\|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}}s^{-\frac{\rho}{\alpha}}\frac{1}{(t'-s)^{\frac{1}{\alpha}}}(t')^{\frac{\beta}{\alpha}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t'-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{(t')^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t'^{\frac{\zeta}{\alpha}}}{(t'-s)^{\frac{\zeta}{\alpha}}}\right]\mathrm{d}s,
\end{aligned}$$

where we used (6.1.25) and (6.3.4) for the last inequality, where $\rho,\zeta>-\beta$. We get:

$$\begin{aligned}
&\left\|\partial_v\tilde p_\alpha(v,\cdot)\star\left(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_t^{t'}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\,\mathrm{d}s\right)\right\|_{L^\infty}\\
\lesssim&\frac{1}{v}\|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}}t^{-\frac{\rho}{\alpha}+\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}}\left(\int_t^{t'}\frac{1}{(t'-s)^{\frac{r'}{\alpha}}}\left[\frac{1}{(t'-s)^{\frac{dr'}{\alpha p}}}\right]\left[\frac{1}{(t'-s)^{\frac{\zeta r'}{\alpha}}}\right]\mathrm{d}s\right)^{\frac{1}{r'}}\\
\lesssim&\frac{1}{v}\|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}}t^{-\frac{\rho}{\alpha}}(t'-t)^{1-\frac{1}{r}-(\frac{1}{\alpha}+\frac{d}{p\alpha}+\frac{\zeta}{\alpha})}=\frac{1}{v}\|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}}t^{-\frac{\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}+\frac{-2\beta-\zeta}{\alpha}+\frac{\varepsilon}{\alpha}}.
\end{aligned}$$

Eventually, choosing $-2\beta-\zeta+\varepsilon=\rho$.

$$\begin{aligned}
&\sup_{v\in[t,1]}v^{1-\frac{\rho}{\alpha}}\left\|\partial_v\tilde p_\alpha(v,\cdot)\star\left(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_t^{t'}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\,\mathrm{d}s\right)\right\|_{L^\infty}\\
\lesssim&(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}t^{-\frac{\rho}{\alpha}}.
\end{aligned} \tag{6.6.17}$$

- For $v\in[0,t]$, write for all $y\in\mathbb{R}^d$:

$$
\begin{aligned}
&|\partial_v\tilde p_\alpha(v,\cdot)\star\left(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_t^{t'}\int\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,\cdot-z)\,\mathrm{d}z\,\mathrm{d}s\right)(y)|\\
=&\left|\int_t^{t'}\int\int\partial_v\tilde p_\alpha(v,y-w)\Big(\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,w-z)}{\bar p_\alpha(t',w-x)}-\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,y-z)}{\bar p_\alpha(t',y-x)}\Big)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\right|\\
\lesssim&\int_t^{t'}\int|\partial_v\tilde p_\alpha(v,y-w)|\left|\int\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,w-z)}{\bar p_\alpha(t',w-x)}\right.\\
&\left.-\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,y-z)}{\bar p_\alpha(t',y-x)}\,\mathrm{d}z\right|\,\mathrm{d}w\,\mathrm{d}s\\
\lesssim&\int_t^{t'}\|b(s,\cdot)\|_{B^\beta_{p,q}}\int|\partial_v\tilde p_\alpha(v,y-w)|\left\|\frac{\Gamma(0,x,s,\cdot)}{\bar p_\alpha(s,\cdot-x)}\right\|_{B^\rho_{\infty,\infty}}\\
&\times\left\|\frac{\bar p_\alpha(s,\cdot-x)\nabla p_\alpha(t'-s,w-\cdot)}{\bar p_\alpha(t',w-x)}-\frac{\bar p_\alpha(s,\cdot-x)\nabla p_\alpha(t'-s,y-\cdot)}{\bar p_\alpha(t',y-x)}\right\|_{B^{-\beta}_{p',q'}}\,\mathrm{d}w\,\mathrm{d}s\\
\underset{(6.3.6)}{\lesssim}&\int_t^{t'}\|b(s,\cdot)\|_{B^\beta_{p,q}}\left(\int\frac{\bar p_\alpha(v,y-w)}{v}|w-y|^\zeta\,\mathrm{d}w\right)\frac{s^{-\frac{\rho}{\alpha}}(t')^{\frac{\beta}{\alpha}}}{(t'-s)^{\frac{\zeta+1}{\alpha}}}\left[\frac{1}{(t'-s)^{\frac{d}{\alpha p}}}+\frac{1}{s^{\frac{d}{\alpha p}}}\right]\left[\frac{(t')^{\frac{\zeta}{\alpha}}}{(t'-s)^{\frac{\zeta}{\alpha}}}+\frac{(t')^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}\right]\\
\lesssim&v^{-1+\frac{\zeta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}(t')^{-\frac{\rho}{\alpha}+\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}}\left(\int_t^{t'}\frac{\mathrm{d}s}{(t'-s)^{r'(\frac{1+\zeta}{\alpha}+\frac{d}{\alpha p}+\frac{\zeta}{\alpha})}}\right)^{\frac{1}{r'}}\\
\lesssim&v^{-1+\frac{\zeta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}(t')^{-\frac{\rho}{\alpha}+\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}}(t'-t)^{1-\frac{1}{r}-(\frac{1}{\alpha}+\frac{d}{p\alpha}+2\frac{\zeta}{\alpha})}\\
\lesssim&v^{-1+\frac{\zeta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}(t')^{-\frac{\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}+\frac{-2\zeta-2\beta+\varepsilon}{\alpha}}\lesssim v^{-1+\frac{\zeta}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}(t')^{-\frac{\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}
\end{aligned}
$$

provided $-2\zeta-2\beta+\varepsilon\geq0\iff\zeta\leq-\beta+\frac{\varepsilon}{2}$. Together with (6.6.17) we eventually get:

$$
\mathcal{T}^\rho_{\infty,\infty,3}(x,t,t')\lesssim(t')^{-\frac{\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}}. \tag{6.6.18}
$$

Let us now turn to $\mathcal{T}^\rho_{\infty,\infty,2}(x,t,t')$

$$
\mathcal{T}^\rho_{\infty,\infty,2}(x,t,t')=\mathcal{T}^\rho_{\infty,\infty}\Big(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_0^t\int\Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\Big).
$$

We proceed with the same previous dichotomy for the time variable:

- For $v\in[t,1]$ write:

$$
\begin{aligned}
&\left\|\partial_v\tilde p_\alpha(v,\cdot)\star\left(\frac{1}{\bar p_\alpha(t',\cdot-x)}\int_0^t\int\Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\right)\right\|_{L^\infty}\\
\lesssim&\frac{C}{v}\int_0^t\left\|\frac{1}{\bar p_\alpha(t',\cdot-x)}\int\Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\right\|_{L^\infty}\,\mathrm{d}s\\
\lesssim&\frac{1}{v}\int_0^t\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}\sup_{y\in\mathbb{R}^d}\frac{1}{\bar p_\alpha(t',y-x)}\left\|\frac{\Gamma(0,x,s,\cdot)}{\bar p_\alpha(s,\cdot-x)}\right\|_{\mathbb{B}^\rho_{\infty,\infty}}\left\|\bar p_\alpha(s,\cdot-x)\Big(\nabla p_\alpha(t'-s,y-\cdot)-\nabla p_\alpha(t-s,y-\cdot)\Big)\right\|_{\mathbb{B}^{-\beta}_{p',q'}}\,\mathrm{d}s\\
\lesssim&\frac{1}{v}\int_0^t\|b(s,\cdot)\|_{\mathbb{B}^\beta_{p,q}}s^{-\frac{\rho}{\alpha}}(t-t')^{\frac{\gamma-\varepsilon}{\alpha}}\frac{t^{\frac{\beta}{\alpha}}}{(t-s)^{\frac{1}{\alpha}+\frac{\gamma-\varepsilon}{\alpha}}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]\left[1+\frac{t^{\frac{\zeta}{\alpha}}}{s^{\frac{\zeta}{\alpha}}}+\frac{t^{\frac{\zeta}{\alpha}}}{(t-s)^{\frac{\zeta}{\alpha}}}\right]\,\mathrm{d}s,
\end{aligned}
$$

using (6.3.8) (with p_α in place of Γ^h) for the last inequality. Thus,

$$\begin{aligned}
&\left\|\partial_v \tilde{p}_\alpha(v,\cdot)\star\left(\frac{1}{\bar{p}_\alpha(t',\cdot-x)}\int_0^t\int \Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\right)\right\|_{L^\infty}\\
\lesssim&\frac{1}{v}(t-t')^{\frac{\gamma-\varepsilon}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}t^{\frac{\beta+\zeta}{\alpha}}\left(\int_0^t s^{-r'\frac{\rho}{\alpha}}\frac{1}{(t-s)^{r'(\frac{1}{\alpha}+\frac{\gamma-\varepsilon}{\alpha})}}\left[\frac{1}{s^{\frac{d}{\alpha p}}}+\frac{1}{(t-s)^{\frac{d}{\alpha p}}}\right]^{r'}\left[\frac{1}{s^{\frac{\zeta}{\alpha}}}+\frac{1}{(t-s)^{\frac{\zeta}{\alpha}}}\right]^{r'}\right)^{\frac{1}{r'}}\\
\lesssim&\frac{1}{v}(t-t')^{\frac{\gamma-\varepsilon}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}t^{\frac{\beta+\zeta}{\alpha}}t^{1-\frac{1}{r'}-\left(\frac{\rho}{\alpha}+\frac{1}{\alpha}+\frac{\gamma-\varepsilon}{\alpha}+\frac{d}{\alpha p}+\frac{\zeta}{\alpha}\right)}\le\frac{1}{v}(t-t')^{\frac{\gamma-\varepsilon}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}t^{-\frac{\rho+\beta-\varepsilon}{\alpha}},
\end{aligned}$$

which yields, taking $\rho+\beta-\varepsilon\le 0$,

$$\begin{aligned}
&\sup_{v\in[t,1]}v^{1-\frac{\rho}{\alpha}}\left\|\partial_v \tilde{p}_\alpha(v,\cdot)\star\left(\frac{1}{\bar{p}_\alpha(t',\cdot-x)}\int_0^t\int \Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\right)\right\|_{L^\infty}\\
\le&(t-t')^{\frac{\gamma-\varepsilon}{\alpha}}\|b\|_{L^r-\mathbb{B}^\beta_{p,q}}t^{-\frac{\rho}{\alpha}}.
\end{aligned}\tag{6.6.19}$$

- For $v\in[0,t]$: write for all $y\in\mathbb{R}^d$:

$$\begin{aligned}
&|\partial_v \tilde{p}_\alpha(v,\cdot)\star\left(\frac{1}{\bar{p}_\alpha(t',\cdot-x)}\int_0^t\int \Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,\cdot-z)-\nabla p_\alpha(t-s,\cdot-z)\Big)\,\mathrm{d}z\,\mathrm{d}s\right)(y)|\\
=&\Bigg|\int_0^{t-2|t'-t|}\int\int\partial_v\tilde{p}_\alpha(v,y-w)\Big(\frac{\Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,w-z)-\nabla p_\alpha(t-s,w-z)\Big)}{\bar{p}_\alpha(t',w-x)}\\
&-\frac{\Gamma(0,x,s,z)b(s,z)\Big(\nabla p_\alpha(t'-s,y-z)-\nabla p_\alpha(t-s,y-z)\Big)}{\bar{p}_\alpha(t',y-x)}\Big)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\Bigg|\\
&+\left|\int_{t-2|t'-t|}^t\int\int\partial_v\tilde{p}_\alpha(v,y-w)\Big(\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,w-z)}{\bar{p}_\alpha(t',w-x)}-\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t'-s,y-z)}{\bar{p}_\alpha(t',y-x)}\Big)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\right|\\
&+\left|\int_{t-2|t'-t|}^t\int\int\partial_v\tilde{p}_\alpha(v,y-w)\Big(\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t-s,w-z)}{\bar{p}_\alpha(t',w-x)}-\frac{\Gamma(0,x,s,z)b(s,z)\nabla p_\alpha(t-s,y-z)}{\bar{p}_\alpha(t',y-x)}\Big)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\right|\\
=:&(T_1+T_2+T_3)(v,t,t',x,y).
\end{aligned}$$

Note that the terms $(T_2+T_3)(v,t,t',x,y)$ can be handled just as we did before for the lower cut in the thermic variable for $\mathcal{T}^{\rho}_{\infty,\infty,3}(x,t,t')$. On the other hand:

$$\begin{aligned}
&T_1(v,t,t',x,y)\\
&\lesssim \left|\int_0^1 \mathrm{d}\lambda \int_0^{t-2|t'-t|} \int\int \partial_v \tilde{p}_\alpha(v,y-w)\Big(\frac{\Gamma(0,x,s,z)b(s,z)\partial_u \nabla p_\alpha(u,w-z)|_{u=t+\lambda(t'-t)-s}}{\bar{p}_\alpha(t',w-x)}\right.\\
&\left. - \frac{\Gamma(0,x,s,z)b(s,z)\partial_u \nabla p_\alpha(u,y-z)|_{u=t+\lambda(t'-t)-s}}{\bar{p}_\alpha(t',y-x)}\Big)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\right|(t'-t)\\
&\lesssim \int_0^1 \mathrm{d}\lambda \int_0^{t-2|t'-t|} \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \int |\partial_v \tilde{p}_\alpha(v,y-w)| \left\|\frac{\Gamma(0,x,s,\cdot)}{\bar{p}_\alpha(s,\cdot-x)}\right\|_{\mathbb{B}^{\rho}_{\infty,\infty}}\\
&\times \left\|\frac{\bar{p}_\alpha(s,\cdot-x)\partial_u \nabla p_\alpha(u,w-\cdot)|_{u=t+\lambda(t'-t)-s}}{\bar{p}_\alpha(t',w-x)} - \frac{\bar{p}_\alpha(s,\cdot-x)\partial_u \nabla p_\alpha(u,y-\cdot)|_{u=t+\lambda(t'-t)-s}}{\bar{p}_\alpha(t',y-x)}\right\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}w\,\mathrm{d}s(t'-t)\\
\underset{(6.3.6)}{\lesssim}& \int_t^{t'} \|b(s,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} s^{-\frac{\rho}{\alpha}} \left(\int \frac{\bar{p}_\alpha(v,y-w)}{v}|w-y|^{\zeta}\,\mathrm{d}w\right) \frac{(t')^{\frac{\beta}{\alpha}}}{(t'-s)^{\frac{1+\zeta}{\alpha}+1}} \left[\frac{1}{(t'-s)^{\frac{d}{\alpha p}}} + \frac{1}{s^{\frac{d}{\alpha p}}}\right]\\
&\times \left[(t')^{\frac{\zeta}{\alpha}}\left(\frac{1}{(t'-s)^{\frac{\zeta}{\alpha}}} + \frac{1}{s^{\frac{\zeta}{\alpha}}}\right) + 1\right](t'-t)\,\mathrm{d}s\\
&\lesssim v^{-1+\frac{\zeta}{\alpha}} \|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} (t')^{\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}} (t'-t)^{\frac{\gamma-\varepsilon}{\alpha}} \left(\int_0^{t-2|t'-t|} \frac{\mathrm{d}s}{s^{r'\frac{\rho}{\alpha}}(t'-s)^{r'(\frac{1+\zeta}{\alpha}+\frac{d}{\alpha p}+\frac{\zeta}{\alpha}+\frac{\gamma-\varepsilon}{\alpha})}}\right)^{\frac{1}{r'}}\\
&\lesssim v^{-1+\frac{\zeta}{\alpha}} \|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} (t')^{\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}} (t'-t)^{\frac{\gamma-\varepsilon}{\alpha}} t^{1-\frac{1}{r}-(\frac{1}{\alpha}+\frac{d}{p\alpha}+2\frac{\zeta}{\alpha}+\frac{\gamma-\varepsilon}{\alpha})-\frac{\rho}{\alpha}}\\
&\lesssim v^{-1+\frac{\zeta}{\alpha}} \|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} (t')^{\frac{\beta}{\alpha}+\frac{\zeta}{\alpha}} (t'-t)^{\frac{\gamma-\varepsilon}{\alpha}} t^{\frac{-2\zeta-2\beta+\varepsilon}{\alpha}-\frac{\rho}{\alpha}} \lesssim v^{-1+\frac{\zeta}{\alpha}} \|b\|_{L^r-\mathbb{B}^{\beta}_{p,q}} (t')^{-\frac{\rho}{\alpha}} (t'-t)^{\frac{\gamma-\varepsilon}{\alpha}},
\end{aligned}$$

provided $-\frac{\beta}{\alpha} - \frac{\zeta}{\alpha} + \frac{\varepsilon}{2\alpha} > 0$ for the above integral to converge. These computations, together with (6.6.19) eventually yields:

$$\mathcal{T}^{\rho}_{\infty,\infty,2}(x,t,t') \lesssim (t')^{-\frac{\rho}{\alpha}}(t'-t)^{\frac{\gamma-\varepsilon}{\alpha}},$$

which together with (6.6.18), (6.6.16) and (6.6.14) gives the claim.

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