Liquidity risk and optimal dividend/investment strategies *

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Abstract

In this paper, we study the problem of determining an optimal control on the dividend and investment policy of a firm operating under uncertain environment and risk constraints. We allow the company to make investment decisions by acquiring or selling producing assets whose value is governed by a stochastic process. The firm may face liquidity costs when it decides to buy or sell assets. We formulate this problem as a multi-dimensional mixed singular and multi-switching control problem and use a viscosity solution approach. We numerically compute our optimal strategies and enrich our studies with numerical results and illustrations.

Keywords: stochastic control, optimal singular / switching problem, viscosity solution, dividend problem, liquidity constraints.

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1 Introduction

In this paper, we consider the problem of determining an optimal control on the dividend and investment policy of a firm operating under uncertain environment and risk constraints. In classical model in corporate finance, it is generally assumed that firm's assets are either infinitely liquid or illiquid. It is particularly the case in the study of optimal dividend and/or investment policy of a firm. In [18], [1], [8], the authors study an optimal dividend problem and consider a stochastic process which represents the cash reserve of the firm. The cash reserve may either grow when the firm makes profits or decrease when the firm is loss-making. The firm goes into bankruptcy when its cash reserve reaches zero. The underlying financial assumption behind the above model is to consider that the firm's assets may be separated into two types of assets, highly liquid assets which may be assimilated as cash reserve, i.e. cash & equivalents, or infinitely illiquid assets, i.e. producing assets that may not be sold. As such, when the cash reserve gets near the bankruptcy point, the firm manager may not be able to inject any cash by selling parts of its non-liquid assets. In [3], the author considers a slightly modified model in which the firm's assets may be liquidated at a positive liquidation value but only once it reaches bankruptcy. The assumptions made in the above models imply that the firm's illiquid assets correspond to producing assets which may be neither increased through investment nor decreased through disinvestment.

Some extensions of the above model are investigated, see for instance [11] where partial and irreversible investment is allowed or [24] which studies the reversible investment case. However, in [11] and [24], the core assumption on the two different types of assets, highly liquid and infinitely illiquid, still remains. In [11], the authors consider a model where the firm manager may be allowed to make a one-off investment. It is the case of a firm which has the opportunity to invest in a new technology that increases its profitability. The firm self-finances the opportunity cost on its cash reserve. In [24], the authors extend the study made in [11] by making the investment reversible. In other words, once installed, the manager can decide to return back to the old technology by receiving some cash compensation. This dividend and investment problem is formulated as a mixed singular/switching control problem. Some other recent studies on optimal dividend problems, such as in [19], consider some more randomness in the model. They consider the problem of optimal dividend distribution for a company in the presence of regime shifts. They assume that the firm cash reserve evolves as a Brownian motion with positive drift that is modulated by a finite state Markov chain, and model the discount rate as a deterministic function of the current state of the chain. Unlike in [24] where the change in profit is due to the manager's investment decision, the regime shift is exogenous in [19]. In all the above studies, it is assumed that a stochastic process X which represents the cash reserve of the firm follows a drifted Brownian motion. The drift represents the average profit that the firm is generating per unit of time. Since the drift is considered to be constant or piecewise constant, the underlying assumption of this model is to consider producing assets as indivisible and may not be sold. As such, producing assets are indeed assumed to be infinitely illiquid. The diffusion part is added to the process in order to model the uncertainty under which the firm is operating and ensures that the cash reserve evolves stochastically.

Another possible financial interpretation of these corporate models studied above is to consider the process X as the net liquidation value of the firm's assets. By doing so, it is implicitly assumed that the firm's assets are infinitely divisible and highly liquid so that the firm manager may sell them at any time and at zero transaction or liquidity cost. In such a setting, the firm does not have to worry about financing issues since any assets may be instantaneously liquidated. It is therefore implicitly assumed that the process X may equally represent the size of the firm. As such, the profit rate generated by the firm should depend on X. A natural model in such a framework is to consider the process X whose drift and volatility coefficients depend on X, for instance, a geometric Brownian motion. In [7], the authors consider the problem of determining an optimal control on the dividend and investment policy of a firm, but under debt constraints. They allow the company to make investment by increasing its outstanding indebtedness, which would impact its capital structure and risk profile, thus resulting in higher interest rate debts. As in the Merton model, they consider that firm value follows a geometric Brownian process. They assume that the firm's assets is highly liquid and may be assimilated to cash equivalents or cash reserve. They formulated their dividend and investment problem as a mixed singular and multi-regime switching control problem.

In our paper, we no longer simplify the optimal dividend and investment problem by assuming that firm's assets are either infinitely illiquid or liquid. For the same reason as highlighted in financial market problems, it is necessary to take into account the liquidity constraints. More precisely, investment (for instance acquiring producing assets) and disinvestment (selling assets) should be possible but not necessarily at their fair value. The firm may have to face some liquidity costs when buying or selling assets. While taking into account liquidity constraints and costs has become the norm in recent financial markets problems, it is still not the case in the corporate finance, to the best of our knowledge, in particular in the studies of optimal dividend and investment strategies. In our paper, we consider the company's assets may be separated in two categories, cash & equivalents, and risky assets which are subjected to liquidity costs. The risky assets are assimilated to producing assets which may be increased when the firm decides to invest or decreased when the firm decides to disinvest. We assume that the price of the risky assets is governed by a stochastic process. The firm manager may buy or sell assets but has to bear liquidity costs. The objective of the firm manager is to find the optimal dividend and investment strategy maximizing its shareholders' value, which is defined as the expected present value of dividends. Mathematically, we formulate this problem as a combined multidimensional singular and multi-regime switching control problem.

In terms of literature, there are many research papers on singular control problems as well as on optimal switching control problems. One of the first corporate finance problems using singular stochastic control theory was the study of the optimal dividend strategy, see for instance [8] and [18]. In the study of optimal switching control problems, a variety of problems are investigated, including problems on management of power station [6], [15], resource extraction [4], firm investment [13], marketing strategy [22], and optimal trading strategies [10], [27]. Other related works on optimal control switching problems include [2] and [23], where the authors employ respectively optimal stopping theory and viscosity techniques to explicitly solve their optimal two-regime switching problem on infinite horizon for one-dimensional diffusions. We may equally refer to [25], for an interesting overview of the area. In the multi-regime switching problems, we may refer to [12], [17], and [26].

However, the studies that are most relevant to our problem are the one investigating combined singular and switching control problems [14], [24], and [7]. By incorporating uncertainty into illiquid assets value, we no longer have to deal with a uni-dimensional control problem but a bi-dimensional singular and multi-regime switching control problem. In such a setting, it is clear that it will be no longer possible to easily get explicit or quasi-explicit optimal strategies. Consequently, to determine the four regions comprising the continuation, dividend and investment/disinvestment regions, numerical resolutions are required.

The plan of the paper is organized as follows. We define the model and formulate our stochastic control problem in the second Section. In Section 3, we characterize our value function as the limit of a sequence of auxiliary functions. The auxiliary functions are defined recursively and each one may be characterized as a unique viscosity solution to its associated HJB equation. This will allow us to get an implementable algorithm approximating our problem. Finally, in Section 4, we numerically compute the value functions and the associated optimal strategies. We further enrich our studies with numerical illustrations.

2 Problem Formulation

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let W and B be two correlated $\mathbb{F}$-Brownian motions, with correlation coefficient c.

We consider a firm which has the ability to make investment or disinvestment by buying or selling producing assets, for instance, factories. We assume that these producing assets are risky assets whose value process S is solution of the following equation:

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right),\; S_0 = s, \tag{2.1}$$

where μ and σ are positive constants.
We denote by $Q_t \in \mathbb{N}$ the number of units of producing assets owned by the company at time t.

We consider a control strategy: $\alpha = ((\tau_i, q_i)_{i\in\mathbb{N}}, Z)$ where τ_i are $\mathbb{F}$-stopping times, corresponding to the investment decision times of the manager, and q_i are $\mathcal{F}_{\tau_i}$-measurable variables valued in $\mathbb{Z}$ and representing the number of producing assets units bought (or sold if $q_i \leq 0$) at time τ_i. When q_i is positive, it means that the firm decides to make investment to increase the assets quantity. Each purchase or sale incurs a fixed cost denoted $\kappa > 0$. The non-decreasing càdlàg process Z represents the total amount of dividends distributed up to time t. Starting from an initial number of assets q and given a control α, the dynamics of the quantity of assets held by the firm is governed by:

$$\begin{cases} dQ_t &= 0 \quad \text{for } \tau_i \leq t < \tau_{i+1}, \\ Q_{\tau_i} &= Q_{\tau_i^-} + q_i, \\ Q_0 &= q, \end{cases} \qquad \text{for } i \in \mathbb{N}. \tag{2.2}$$

Notice that when there is no ambiguity, we use the notation Q_t as above instead of Q_t^α. This remark may apply to X_t and Y_t.
Similarly, starting from an initial cash value x and given a control α, the dynamics of the cash reserve (or more precisely the firm's cash and equivalents) process of the firm is governed by:

$$\begin{cases} dX_t &= rX_t dt + h(Q_t)(bdt + \eta dW_t) - dZ_t, \text{ for } \tau_i \leq t < \tau_{i+1} \\ X_{\tau_i} &= X_{\tau_i^-} - S_{\tau_i} f(q_i) q_i - \kappa, \\ X_0 &= 0, \end{cases} \qquad \text{for } i \in \mathbb{N}. \tag{2.3}$$

where b, r and η are positive constants and h a non-negative, non-decreasing and concave function satisfying $h(q) \leq H$ with $h(1) > 0$ and $H > 0$. The function f represents the liquidity cost function (or impact function with the impact being temporary) and is assumed to be non-negative, non-decreasing, such that $f(0) = 1$.

Remark 2.1. *1.) We assume that the firm profit depends on the number of units of producing assets it owns. With q units of assets, the firm profit per unit of time dt is $h(q)(bdt + \eta dW_t)$. In the case the firm is not allowed to make any investment or disinvestment, i.e. when q is constant, the resulting model is closely related to the Bachelier model which is used in classical problems in corporate finance, see for instance [18].*
2.) The assumption on the function h is quite natural. We assume that h is non-negative, non-decreasing as we consider that the more producing assets units the firm operates, the higher the firm profit. In particular, we assume that $h(0) \geq 0$.

We denote by $Y_t^y = (X_t^x, S_t^s, Q_t^q)$ the solution to (2.1)-(2.3) with initial condition $(X_0^x, S_0^s, Q_0^q) = (x, s, q) := y$. At each time t, the firm's cash value and number of units of producing assets have to remain non-negative i.e. $X_t \geq 0$ and $Q_t \geq 0$, for all $t \geq 0$.

The bankruptcy time is defined as

$$T := T^{y,\alpha} := \inf\{t \geq 0, X_t < 0\}.$$

We define the liquidation value as $L(x, s, q) := x + (sf(-q)q - \kappa)^+$ and notice that $L \geq 0$ on $\mathbb{R}^+ \times (0, +\infty) \times \mathbb{N}$. We introduce the following notation

$$\mathcal{S} := \mathbb{R}^+ \times (0, +\infty) \times \mathbb{N}.$$

The optimal firm value is defined on $\mathcal{S}$, by

$$v(y) = \sup_{\alpha \in \mathcal{A}(y)} J^\alpha(y), \tag{2.4}$$

where $J^\alpha(y) = \mathbb{E}[\int_0^T e^{-\rho u} dZ_u]$, with ρ being a positive discount factor and $\mathcal{A}(y)$ is the set of admissible strategies defined by

$$\begin{aligned}
\mathcal{A}(y) \;\; = \;\; & \{\alpha = ((\tau_i, q_i)_{i\in\mathbb{N}}, Z) : Z \text{ is a predictable and non-decreasing process,} \\
& (\tau_i)_{i\in\mathbb{N}} \text{ is an increasing sequence of stopping times such that } \lim_{i\to+\infty} \tau_i = +\infty \\
& \text{and } q_i \text{ are } \mathbb{F}_{\tau_i} - \text{measurable, and such that } (X_t^{x,\alpha}, Q_t^{q,\alpha}) \in \mathbb{R}^+ \times \mathbb{N}\}.
\end{aligned}$$

We now identify the trivial cases where the value function is infinite.

Lemma 2.1. *If we have $r > \rho$ or $\mu > \rho$ then $v(y) = +\infty$ on $\mathcal{S}$.*

Proof: Let $y := (x, s, q) \in \mathcal{S}$.
We first assume that $\rho < r$. At time 0, by choosing to liquidate the firm's assets, we may get $L(y) > 0$ in cash. Then by waiting until a given time $t > 0$, we may obtain $v(y) \geq e^{(r-\rho)t} L(y)$. By letting t going to $+\infty$, we have $v(y) = +\infty$.

We now assume that $\rho < \mu$. First, suppose that $q \geq 1$. In this case, by doing nothing up to time t and then liquidate at time t, for any $t > 0$, we may obtain

$$\begin{aligned}
v(y) \;\; & \geq \;\; \mathbb{E}\Big[e^{-\rho t}\big(X_t^x + qS_t^s f(-q) - \kappa\big)\mathbb{1}_{t<T}\Big] \\
& \geq \;\; -\kappa e^{-\rho t} + qf(-q)\mathbb{E}\big[e^{-\rho t} S_t^s \mathbb{1}_{t<T}\big] \\
& \geq \;\; -\kappa e^{-\rho t} + qf(-q)e^{(\mu-\rho)t} s\mathbb{Q}(t < T),
\end{aligned}$$

where $\mathbb{Q}$ is the probability equivalent to $\mathbb{P}$ defined by its Radon-Nykodim density $\mathcal{E}(\sigma B)$. Under this probability $\mathbb{Q}$ the process $B^{\mathbb{Q}}$ defined by $B_t^{\mathbb{Q}} := B_t - \sigma t$ is a Brownian motion. As we have $L(X_t^x, S_t^s, Q_t^q) \geq X_t^x \geq x + h(q)(bt + \eta W_t)$, we know that if we set

$$C_u := x + h(q)(bu + \eta W_u) \text{ for } u \geq 0 \quad \text{and} \quad \hat{T} = \inf\{u \geq 0 : \; C_u \leq 0\},$$

we have $\hat{T} \leq T$ and then $\mathbb{Q}\big(t < T\big) \geq \mathbb{Q}(t < \hat{T})$. Therefore we deduce from Girsanov Theorem that

$$dC_u = h(q)\eta d\Big[\frac{b}{\eta}t + W_t\Big] = h(q)\eta dW_t^*,$$

where W^* is a brownian motion under the probability $\mathbb{Q}^*$ defined by its Radon-Nykodim density $\mathcal{E}(\frac{b}{\eta}W)$ with respect to $\mathbb{Q}$. We recall that $\hat{T}$ admits the following density function

$$f_{\hat{T}}(u) = \frac{x}{h(q)\eta\sqrt{2\pi u^3}} e^{-\frac{x^2}{2h(q)^2\eta^2 u}} \mathbb{1}_{\{u>0\}}.$$

Therefore, we obtain

$$
\begin{aligned}
\mathbb{Q}(t<\hat{T}) &= \mathbb{E}_{\mathbb{Q}^*}\left[e^{\frac{b^2}{2\eta^2}\hat{T}-\frac{b}{\eta}W^*_{\hat{T}}}\mathbb{1}_{\{t<\hat{T}\}}\right] \\
&= e^{\frac{bx}{h(q)\eta^2}}\int_t^{+\infty} e^{\frac{b^2}{2\eta^2}u} f_{\hat{T}}(u)\,du \\
&= e^{\frac{bx}{h(q)\eta^2}}\int_t^{+\infty} e^{\frac{b^2}{2\eta^2}u}\frac{x}{h(q)\eta\sqrt{2\pi u^3}}e^{-\frac{x^2}{2h(q)^2\eta^2 u}}\,du \\
&= \frac{2h(q)\eta}{x\sqrt{2\pi}}e^{\frac{bx}{h(q)\eta^2}}\int_t^{+\infty}\sqrt{u}e^{\frac{b^2}{2\eta^2}u}\frac{x^2}{2h(q)^2\eta^2u^2}e^{-\frac{x^2}{2h(q)^2\eta^2 u}}\,du \\
&\geq \frac{2h(q)\eta\sqrt{t}}{x\sqrt{2\pi}}e^{\frac{bx}{h(q)\eta^2}}\left[1-e^{-\frac{x^2}{2h(q)^2\eta^2 t}}\right]
\end{aligned}
$$

We conclude the proof by asserting that, for t going to $+\infty$, we have

$$
\lim_{t\to+\infty} e^{(\mu-\rho)t}\mathbb{Q}\big(t<T\big) \geq \lim_{t\to+\infty}\frac{x}{h(q)\eta\sqrt{2\pi t}}e^{\frac{bx}{h(q)\eta^2}}e^{(\mu-\rho)t} = +\infty
$$

and then $v(y)=+\infty$.

For the case where the initial value $q=0$, the control policy to apply is to do nothing up to the stopping time $T^{inv} := \inf\{t\geq 0;\ X_t \geq S_t f(1)+\kappa\}$, which is almost surely finite, then to acquire a unit of producing assets. We may then conclude our proof by applying the policy used in the previous case when $q\geq 1$. □

From this point, we shall assume that the parameters satisfy:

$$
\rho > \max(r,\mu) \tag{2.5}
$$

3 Characterization of auxiliary functions

The aim of this section is to provide an implementable algorithm of our problem. To tackle the stochastic control problem as defined in (2.4), one usual way is to first characterize the value function as a unique solution to its associated HJB equation. The second step is to deduce the optimal strategies from smooth-fit properties and more generally from viscosity solution techniques. The optimal strategies may be characterized by different regions of the state-space, i.e. the continuation region, the dividend region as well as the Buy and Sell regions. In such cases, the solutions may be either of explicit or quasi-explicit nature. However, in a non-degenerate multidimensional setting such as in our problem, getting explicit or quasi-explicit solutions is out of reach.

As such, to solve our control problem, we characterize our value function as the limit of a sequence of auxiliary functions. The auxiliary functions are defined recursively and each one may be characterized as a unique viscosity solution to its associated HJB equation. This will allow us to get an implementable algorithm approximating our problem.

3.1 An approximating sequence of functions

We recall the notation $y = (x, s, q) \in \mathcal{S}$. From this point, we may use alternatively y or (x, s, q). We now introduce the following subsets of $\mathcal{A}(y)$:

$$\mathcal{A}_N(y) := \{\alpha = ((\tau_k, \xi_k)_{k\in\mathbb{N}^*}, Z) \in \mathcal{A}(y) : \tau_k = +\infty \text{ a.s. for all } k \geq N+1\}$$

and the corresponding value function v_N, which describes the value function when the investor is allowed to make at most N interventions (investments or disinvestments):

$$v_N(y) = \sup_{\alpha\in\mathcal{A}_N(y)} J^\alpha(y),\ \forall N \in \mathbb{N} \tag{3.6}$$

We shall show in Proposition 3.4 that the sequence $(v_N)_{N\geq 0}$ goes to v when N goes to infinity, but we first have to carefully study some properties of this sequence.

In the next Proposition, we recall explicit formulas for v_0 and the optimal strategy associated to this singular control problem. This problem is indeed very close to the one solved in the pioneering work of Jeanblanc and Shirayev (see [18]). The only difference in our framework is due to the interest $r \neq 0$ and therefore the cash process X does not follow exactly a Bachelier model. However, proofs and results can easily be adapted to obtain Proposition 3.1 and we will skip the proof.

Proposition 3.1. *There exists $x^*(q) \in [0, +\infty)$ such that*

$$v_0(x, s, q) := \begin{cases} V_q(x) & \text{if} \quad 0 \leq x \leq x^*(q) \\ x - x^*(q) + V_q(x^*(q)) & \text{if} \qquad x \geq x^*(q), \end{cases}$$

where V_q is the $\mathcal{C}^2$ function, solution of the following differential equation

$$\frac{\eta^2 h(q)^2}{2} y'' + (rx + bh(q))y' - \rho y = 0; \quad y(0) = 0,\ y'(x^*(q)) = 1 \text{ and } y''(x^*(q)) = 0. \tag{3.7}$$

Notice that $x \to v_0(x, s, q)$ is a concave and $\mathcal{C}^2$ function on $[0, +\infty)$ and that if $h(0) = 0$, it is optimal to immediately distribute dividends up to bankruptcy therefore $v_0(x, s, 0) = x$.

We now are able to characterize our impulse control problem as an optimal stopping time problem, defined through an induction on the number of interventions N.

Proposition 3.2. *(Optimal stopping)*
For all $(x, s, q, N) \in \mathcal{S} \times \mathbb{N}^*$, we have

$$v_N(x, s, q) = \sup_{(\tau,Z)\in\mathcal{T}\times\mathcal{Z}} \mathbb{E}[\int_0^{T\wedge\tau} e^{-\rho u}\, dZ_u + e^{-\rho\tau} G_{N-1}(X^x_{\tau^-}, S^s_\tau, q)\mathbb{1}_{\{\tau<T\}}], \tag{3.8}$$

where $\mathcal{T}$ is the set of stopping times, $\mathcal{Z}$ the set of predictable and non-decreasing càdlàg processes, and

$$G_{N-1}(x, s, q) := \max_{n\in a(x,s,q)} v_{N-1}\left(\Gamma(y, n)\right) \text{ and } G_{-1} = 0, \tag{3.9}$$

$$\text{with} \qquad a(x, s, q) := \left\{n \in \mathbb{Z} : n \geq -q \text{ and } nf(n) \leq \frac{x-\kappa}{s}\right\}, \tag{3.10}$$

$$\text{and} \qquad \Gamma(y, n) := (x - nf(n)s - \kappa,\ s,\ q + n). \tag{3.11}$$

Proof: For $(y,N) := (x,s,q,N) \in \mathcal{S}\times\mathbb{N}^*$, we set

$$\hat{v}_N(y) = \sup_{(\tau,Z)\in\mathcal{T}\times\mathcal{Z}} \mathbb{E}[\int_0^{T\wedge\tau} e^{-\rho s}\,dZ_s + e^{-\rho\tau}G_{N-1}(X^x_{\tau^-}, S^s_\tau, q)\mathbb{1}_{\{\tau<T\}}].$$

For $\alpha \in \mathcal{A}_N(y)$, we have

$$\begin{aligned} J^\alpha(y) &= \mathbb{E}\Big[\int_0^{\tau_1\wedge T} e^{-\rho s}\,dZ_s + \mathbb{E}\Big[\int_{\tau_1}^T e^{-\rho s}\,dZ_s|\mathcal{F}_{\tau_1\wedge T}\Big]\mathbb{1}_{\{\tau_1<T\}}\Big] \\ &\le \mathbb{E}\Big[\int_0^{\tau_1\wedge T} e^{-\rho s}\,dZ_s + e^{-\rho\tau_1}v_{N-1}(X^x_{\tau_1}, S^s_{\tau_1}, Q^q_{\tau_1})\mathbb{1}_{\{\tau_1<T\}}\Big] \\ &\le \mathbb{E}\Big[\int_0^{\tau_1\wedge T} e^{-\rho s}\,dZ_s + e^{-\rho\tau_1}G_{N-1}(X^x_{\tau_1^-}, S^s_{\tau_1}, q)\mathbb{1}_{\{\tau_1<T\}}\Big] \\ &\le \hat{v}_N(y). \end{aligned}$$

It follows that $v_N(y) \le \hat{v}_N(y)$ on $\mathcal{S}$.

Let $\varepsilon > 0$. There exists $(\tau^*, Z^*) \in \mathcal{T}\times\mathcal{Z}$

$$\hat{v}_N(y) \le \varepsilon + \mathbb{E}[\int_0^{T\wedge\tau^*} e^{-\rho s}\,dZ^*_s + e^{-\rho\tau^*}G_{N-1}(X^x_{\tau^*-}, S^s_{\tau^*}, q)\mathbb{1}_{\{\tau^*<T\}}] \tag{3.12}$$

Therefore, there exists ξ^* a random variable $\mathcal{F}_{\tau^*}$-measurable, taking values in $a(X^x_{\tau^*-}, S^s_{\tau^*}, Q^q_{\tau^*-})$, such that

$$\hat{v}_N(y) \le \varepsilon + \mathbb{E}[\int_0^{T\wedge\tau^*} e^{-\rho s}\,dZ^*_s + e^{-\rho\tau^*}v_{N-1}(X^x_{\tau^*} - \xi^* f(\xi^*)S^s_{\tau^*} - \kappa, S^s_{\tau^*}, q+\xi^*)\mathbb{1}_{\{\tau^*<T\}}]$$

Now, let $\alpha = ((\tau_k,\xi_k)_{k\in\mathbb{N}^*}, Z) \in \mathcal{A}_N(y)$ such that

$$\tau_1 = \tau^*, \quad \xi_1 = \xi^* \quad \text{and} \quad Z_u = Z^*_u \quad \text{for all } 0 \le u \le \tau^*.$$

We have

$$v_N(y) \ \ge\ \mathbb{E}\Big[\int_0^{\tau_1\wedge T} e^{-\rho s}\,dZ_s + \mathbb{E}\Big[\int_{\tau_1}^T e^{-\rho s}\,dZ_s|\mathcal{F}_{\tau_1}\Big]\mathbb{1}_{\{\tau_1<T\}}\Big].$$

As the previous inequality is true for all $\hat{\alpha} := ((\tau_k,\xi_k)_{k>1}, (Z_{\tau^*+u})_{u\ge 0}) \in \mathcal{A}_{N-1}(X^x_{\tau^*}, S^s_{\tau^*}, Q^q_{\tau^*})$ such that $\tau_2 > \tau^*$, we finally obtain that

$$\begin{aligned} v_N(y) &\ge \mathbb{E}\Big[\int_0^{\tau_1\wedge T} e^{-\rho s}\,dZ_s + e^{-\rho\tau_1}v_{N-1}(X^x_{\tau_1}, S^s_{\tau_1}, Q^q_{\tau_1})\mathbb{1}_{\{\tau_1<T\}}\Big] \\ &\ge \mathbb{E}\Big[\int_0^{\tau_1\wedge T} e^{-\rho s}\,dZ_s + e^{-\rho\tau_1}G_{N-1}(X^x_{\tau_1^-}, S^s_{\tau_1}, q)\mathbb{1}_{\{\tau_1<T\}}\Big] \\ &\ge \hat{v}_N(y) - \varepsilon, \end{aligned}$$

which ends the proof. □

3.2 Bounds and convergence of $(v_N)_{N\geq 0}$

We begin by stating a standard result which says that any smooth function, which is supersolution to the HJB equation, is a majorant of the value function.

Proposition 3.3. *Let $N \in \mathbb{N}$ and $\phi = (\phi_q)_{q\in\mathbb{N}}$ be a family of non-negative $\mathcal{C}^2$ functions on $\mathbb{R}^+ \times (0,+\infty)$ such that $\forall q \in \mathbb{N}$ (we may use both notations $\phi(x,s,q) := \phi_q(x,s)$), $\phi_q(0,s) \geq 0$ for all $s \in (0,\infty)$ and*

$$\min\Big[\rho\phi(y) - \mathcal{L}^N\phi(y), \phi(y) - G_{N-1}(y), \frac{\partial\phi}{\partial x}(y) - 1\Big] \quad \geq \quad 0 \tag{3.13}$$

for all $y \in (0,+\infty)\times(0,+\infty)\times\mathbb{N}$, where we have set

$$\begin{aligned}\mathcal{L}^N\varphi \quad &= \quad \frac{\eta^2 h(q)^2}{2}\frac{\partial^2\varphi}{\partial x^2} + (rx + bh(q))\frac{\partial\varphi}{\partial x}\\ &\quad +\mathbb{1}_{\{N>0\}}\left[\frac{\sigma^2 s^2}{2}\frac{\partial^2\varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2\varphi}{\partial s\partial x} + \mu s\frac{\partial\varphi}{\partial s}\right].\end{aligned}$$

then we have $v_N \leq \phi$.

Proof: Given an initial state value $y = (x,s,q) \in \mathcal{S}$, take an arbitrary control $\alpha = (\tau, Z) \in \mathcal{T}\times\mathcal{Z}$, and set for $m > 0$, $\theta_m = \inf\{t \geq 0 : \max(X_t^x, S_t^s) \leq \frac{1}{m} \text{ or } \max(X_t^x, S_t^s) \geq m\} \wedge T \nearrow T$ a.s. when m goes to infinity. Apply then Itô's formula to $e^{-\rho t}\phi(Y_t^y)$ between the stopping times 0 and $\tau_m := T \wedge \tau \wedge \theta_m$. Notice for $0 \leq t < \tau_m$ we have that $Q_t^q = q$. Then we have

$$\begin{aligned}e^{-\rho\tau_m}\phi(Y^y_{\tau_m^-}) \quad &= \quad \phi(y) + \int_0^{\tau_m} e^{-\rho t}\left(-\rho\phi + \mathcal{L}^N\phi\right)(Y_t^y)dt\\ &\quad +\mathbb{1}_{\{N>0\}}\int_0^{\tau_m} e^{-\rho t}\sigma S_t^s\frac{\partial\phi}{\partial s}(Y_t^y)dB_t + \int_0^{\tau_m} e^{-\rho t}\eta h(q)\frac{\partial\phi}{\partial x}(Y_t^y)dW_t\\ &\quad -\int_0^{\tau_m} e^{-\rho t}\frac{\partial\phi}{\partial x}(Y_t^y)dZ_t^c + \sum_{0\leq t<\tau_m} e^{-\rho t}[\phi(Y_t^y) - \phi(Y^y_{t^-})],\end{aligned} \tag{3.14}$$

where Z^c is the continuous part of Z.
Since $\frac{\partial\phi}{\partial x} \geq 1$, we have by the mean-value theorem $\phi(Y_t^y) - \phi(Y^y_{t^-}) \geq X_t^x - X^x_{t^-} = -(Z_t - Z_{t^-})$ for $0 \leq t < \tau_m$. By using also the supersolution inequality of ϕ, taking expectation in the above Itô's formula, and noting that the integrands in the stochastic integral terms are bounded by a constant (depending on m), we have

$$\begin{aligned}\mathbb{E}\Big[e^{-\rho\tau_m}\phi(Y^y_{\tau_m^-})\Big] \quad &\leq \quad \phi(y) - \mathbb{E}\Big[\int_0^{\tau_m} e^{-\rho t}\, dZ_t^c\Big]\\ &\quad -\mathbb{E}\Big[\sum_{0\leq t<\tau_m} e^{-\rho t}(Z_t - Z_{t^-})\Big]\end{aligned}$$

and so

$$\phi(y) \geq \mathbb{E}\Big[\int_0^{\tau_m^-} e^{-\rho t}\, dZ_t + e^{-\rho\tau_m}\phi(Y^y_{\tau_m^-})\Big].$$

By sending m to infinity and recalling that $\phi \geq 0$, with Fatou's lemma, we obtain:

$$\phi(y) \geq \mathbb{E}\Big[\int_0^{(T\wedge\tau)^-} e^{-\rho t}\, dZ_t + e^{-\rho(T\wedge\tau)}\phi(Y^y_{(T\wedge\tau)^-})\Big]. \tag{3.15}$$

Now, as $\phi \geq G_{N-1}$ and recalling that, on $\{\tau < T\}$, there exists $n \in \mathbb{N}$ such that $X^x_{T\wedge\tau} = X^x_{(T\wedge\tau)^-} - nf(n)S^s_{T\wedge\tau} - \kappa$, $S^s_{T\wedge\tau} = S^s_{(T\wedge\tau)^-}$ and $Q^q_{T\wedge\tau} = q + n$. , we obtain

$$\begin{aligned} \phi(Y^y_{(T\wedge\tau)^-}) &\geq v_{N-1}\left(X^x_{(T\wedge\tau)^-} - nf(n)S^s_{T\wedge\tau} - \kappa, S^s_{T\wedge\tau}, q+n\right) \\ &= v_{N-1}(Y^y_{T\wedge\tau}) \quad \text{ on } \{\tau < T\}. \end{aligned} \tag{3.16}$$

Moreover, notice that on $\{T \leq \tau\}$, $v_{N-1}(Y^y_{T\wedge\tau}) = v_{N-1}(Y^y_T) = 0 \leq \phi(Y^y_{(T\wedge\tau)^-})$, hence inequality (3.16) also holds on $\{T \leq \tau\}$ and so a.s. Therefore, plugging into (3.15), we have

$$\phi(y) \geq \mathbb{E}\Big[\int_0^{(T\wedge\tau)^-} e^{-\rho t}\, dZ_t + e^{-\rho(T\wedge\tau)}v_{N-1}(Y^y_{(T\wedge\tau)})\Big].$$

We obtain the required result from the arbitrariness of the control α. □

Corollary 3.1. Bounds*:*
For all $N \in \mathbb{N}^*$ and $(x, s, q) \in \mathcal{S}$, we have

$$L(x, s, q) \leq v_N(x, s, q) \leq x + sq + K \quad \textit{where } \rho K = bH.$$

Proof: We obviously have $v_N(x, s, q) \geq L(x, s, q)$ for $N > 0$ and $v_0(x) \geq x$ as the agent may distribute dividend up to bankruptcy.

We set $\phi(x, s, q) = x + sq + K$ with $K \geq 0$. We obviously have $\phi(0, s, q) \geq L(0, s, q) \geq 0$. We also have $\frac{\partial\phi}{\partial x}(x, s, q) \geq 1$. Moreover we have

$$\begin{aligned} \rho\phi(x, s, q) - \mathcal{L}^0\phi(x, s, q) &= \rho(x + sq + K) - (rx + bh(q)) \\ &\geq (\rho - r)x + \rho sq + \rho K - bh(q) \\ &\geq \rho K - bH \\ &\geq 0 \end{aligned}$$

Hence, if $N = 0$, ϕ satisfies the assumptions of Proposition (3.3) and we have $\phi \geq v_0$ on $\mathcal{S}$. Now, assume that $\phi \geq v_{N-1}$. We still have $\frac{\partial\phi}{\partial x}(x, s, q) \geq 1$ and

$$\begin{aligned} \rho\phi(x, s, q) - \mathcal{L}^N\phi(x, s, q) &= \rho(x + sq + K) - (rx + bh(q)) - \mu sq \\ &\geq (\rho - r)x + (\rho - \mu)sq + \rho K - bh(q) \\ &\geq \rho K - bH \\ &\geq 0 \end{aligned}$$

We conclude by noticing that $n(1 - f(n)) \leq 0$ for all $n \in \mathbb{Z}$ and then

$$\begin{aligned} G_N(x, s, q) &= \max_{n\in a(x,s,q)} v_{N-1}\big(\Gamma(y, n)\big) \\ &= \max_{n\in a(x,s,q)} \phi\big(\Gamma(y, n)\big) \\ &= \phi(x, s, q) - \kappa + s \max_{n\in a(x,s,q)} (n(1 - f(n))) \\ &< \phi(x, s, q). \end{aligned}$$

From Proposition 3.3 again, we obtain that $v_N(x,s,q) \leq x + sq + K$.

□

We are able to conclude on the asymptotic behavior of our approximating sequence of functions. The next Proposition shows that this sequence of functions goes to our value function v when N goes to infinity.

Proposition 3.4. *(Convergence) For all $y \in \mathcal{S}$, we have*

$$\lim_{N\to+\infty} v_N(y) = v(y).$$

Proof: We obviously have $v_N \leq v_{N+1} \leq v$ for all $N \in \mathbb{N}$. By contradiction, assume that there exists $y \in \mathcal{S}$ such that $v(y) = +\infty$ then for any $M > 0$ there would exist a strategy $\alpha = ((\tau_k, \xi_k)_{k\in\mathbb{N}^*}, Z) \in \mathcal{A}(y)$ such that $M \leq J^\alpha(y)$. As $(\tau_i)_{i\in\mathbb{N}^*}$ is such that $\lim_{i\to+\infty} \tau_i = +\infty$, there exists $N \in \mathbb{N}^*$ such that

$$M \leq J^\alpha(y) \leq \mathbb{E}[\int_0^{T\wedge\tau_N} e^{-\rho s}\, dZ_s] \leq v_N(y).$$

As v_N is bounded, it leads to a contradiction.
For $y \in \mathcal{S}$ and $\varepsilon > 0$, we may now consider a strategy $\alpha = ((\tau_k, \xi_k)_{k\in\mathbb{N}^*}, Z) \in \mathcal{A}(y)$ such that

$$v(y) \leq J^\alpha(y) + \varepsilon.$$

Notice that, as $(\tau_i)_{i\in\mathbb{N}^*}$ is such that $\lim_{i\to+\infty} \tau_i = +\infty$, there exists $N \in \mathbb{N}^*$ such that

$$\begin{aligned} J^\alpha(y) &\leq \mathbb{E}[\int_0^{T\wedge\tau_N} e^{-\rho s}\, dZ_s] + \varepsilon \\ &\leq v_N(y) + \varepsilon, \end{aligned}$$

which ends the proof.

□

3.3 Viscosity characterization of v_N

Let $N > 0$. This subsection is devoted to the characterization of the function v_N as the unique function which satisfies the boundary condition

$$v_N(y) = G_{N-1}(y) \text{ on } \{0\} \times (0,+\infty) \times \mathbb{N}. \tag{3.17}$$

and is a viscosity solution of the following HJB equation:

$$\min\{\rho v_{_N}(y) - \mathcal{L}v_{_N}(y); \frac{\partial v_{_N}}{\partial x}(y) - 1; v_{_N}(y) - G_{N-1}(y)\} = 0 \text{ on } (0,+\infty)^2 \times \mathbb{N}, \tag{3.18}$$

where we have set

$$\mathcal{L}\varphi = \frac{\eta^2 h(q)^2}{2}\frac{\partial^2\varphi}{\partial x^2} + \frac{\sigma^2 s^2}{2}\frac{\partial^2\varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2\varphi}{\partial s\partial x} + (rx + bh(q))\frac{\partial\varphi}{\partial x} + \mu s\frac{\partial\varphi}{\partial s}.$$

It relies on the following Dynamic Programming Principle. Let $\theta \in \mathcal{T}$, $y := (x,s,q) \in \mathcal{S}$ and set $\nu = T \wedge \theta$, we have

$$v_N(y) = \sup_{(\tau,Z)\in\mathcal{T}\times\mathcal{Z}} \mathbb{E}[\int_0^{(\nu\wedge\tau)^-} e^{-\rho s}\, dZ_s + e^{-\rho(\nu\wedge\tau)} v_N\left(X^x_{(\nu\wedge\tau)^-}, S^s_{\nu\wedge\tau}, q\right) \mathbb{1}_{\{\tau<\nu\}}] \tag{3.19}$$

We are now able to establish the main results of this section.

Theorem 3.1. *For all $(N, q) \in \mathbb{N}^* \times \mathbb{N}$, the value function $v_N(\cdot,\cdot,q)$ is continuous on $(0,+\infty)^2$. Moreover v_N is the unique viscosity solution on $(0,+\infty)^2 \times \mathbb{N}$ of the HJB equation (3.18) satisfying the boundary condition (3.17) and the following growth condition*

$$|v_N(x,s,q)| \leq C_1 + C_2 x + C_3 sq, \quad \forall (x,s,q) \in \mathcal{S},$$

for some positive constants C_1, C_2 and C_3.

This result relies on the three following lemmas, which proofs are rather standard and are postponed in appendix for the sake of completeness.

Lemma 3.2. ***Supersolution***
Let $N \in \mathbb{N}^*$. Assume that, for all $0 \leq k \leq N-1$ and $q \in \mathbb{N}$, $v_k(\cdot,\cdot,q)$ is continuous on $(0,+\infty)^2$. The lower semi-continuous envelope of v_N, denoted by v_N^l is a supersolution of equation (3.18)*.*

Lemma 3.3. ***Subsolution***
Let $N \in \mathbb{N}^*$. Assume that, for all $0 \leq k \leq N-1$ and $q \in \mathbb{N}$, $v_k(\cdot,\cdot,q)$ is continuous on $(0,+\infty)^2$. The upper semi-continuous envelope of v_N, denoted by v_N^u is a subsolution of equation (3.18)*.*

Lemma 3.4. ***Comparison Principle***
Assume that u is a upper semi-continuous viscosity subsolution on $(0,+\infty)^2 \times \mathbb{N}$ of the HJB equation (3.18), and that w is a lower semi-continuous viscosity supersolution on $\mathcal{S}$ of the HJB equation (3.18), satisfying the boundary condition $\limsup_{y \to \bar{y}} u(y) \leq \liminf_{y \to \bar{y}} w(y)$, for all $\bar{y} \in \{0\} \times (0,+\infty) \times \mathbb{N}$, and the linear growth condition :

$$|u(x,s,q)| + |w(x,s,q)| \leq C_1 + C_2 x + C_3 sq, \quad \forall (x,s,q) \in \mathcal{S},$$

for some positive constants C_1, C_2 and C_3. Then,

$$u(y) \leq w(y) \quad \forall y \in (0,+\infty)^2 \times \mathbb{N}.$$

4 Numerical Results

In this paragraph, we present some numerical results by approximating the solution of the HJB equation (3.18). To solve the HJB equation (3.18) arising from the stochastic control problem (3.6), we choose to use a finite difference scheme which leads to the construction of an approximating Markov chain. The convergence of the scheme can be shown using standard arguments as in [21]. We may equally refer to [5], [16], and [20] for numerical schemes involving singular control problems.
Numerical tests are performed with the following set of parameters values:
$\rightarrow$ $r = 0.05, \quad \mu = 0.08, \quad b = 0.1, \quad \rho = 0.1.$
$\rightarrow$ $\sigma = 0.2, \quad \eta = 0.2, \quad c = 0.01.$
$\rightarrow$ Liquidation cost and function: $\kappa = 0.1, \quad f(q) = \exp(\lambda q) \quad s.t. \quad \lambda = 10^{-7}.$
$\rightarrow$ Firm profit rate: $h(q) = 5\sqrt{1+q}$.

Figures 1: Description of different regions and optimal investment/dividend policy in (x, s) for a fixed numbers of producing assets units q_0.
We plot the shape of the optimal regions in function of (x, s) for a fixed number of producing assets $q_2 > q_1 > q_0$. We may distinguish four regions: buy, sell, dividend and continuation regions. We may clearly make the following observations

- As the assets price gets higher, the dividend region shrinks in favor of the buy region. Indeed, the firm has to hold sufficient amount of cash in order to be able to invest in more expensive assets.

- However, for very high assets price, the buy region does not exist any more. Financially, it means that for very high assets price, it is no longer optimal to invest in the assets and it is preferable to distribute dividend as if investment opportunities no longer exist.

- The sell region appears as the firm's cash reserve gets close to zero. Indeed, the firm has to make a disinvestment decision in order to inject cash into its balance, therefore avoiding bankruptcy.

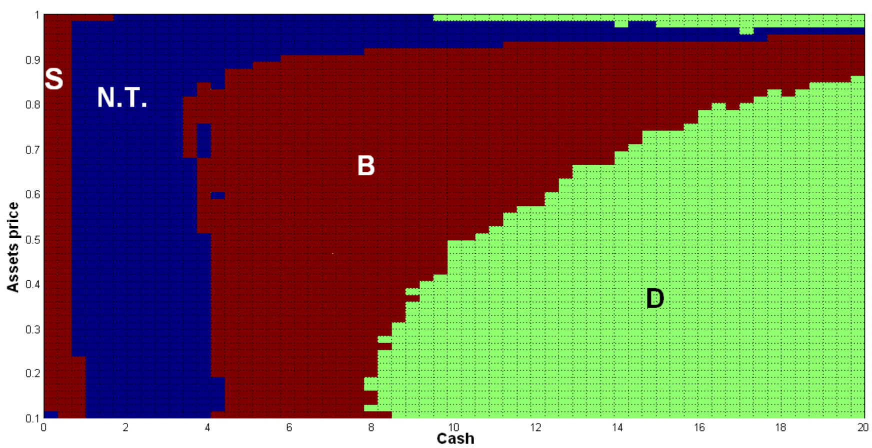

Figure 1: *Description of different regions, in (x, s) for a fixed q_0.*

Figures 2 and 3 : Description of regions in (x, s) for respectively q_1 and q_2 with $q_2 > q_1 > q_0$. We observe that the buy region significantly gets smaller for higher number of assets $q_1 > q_0$ and completely disappear for $q_2 > q_1$. These observations are explained by the concavity of the function h.

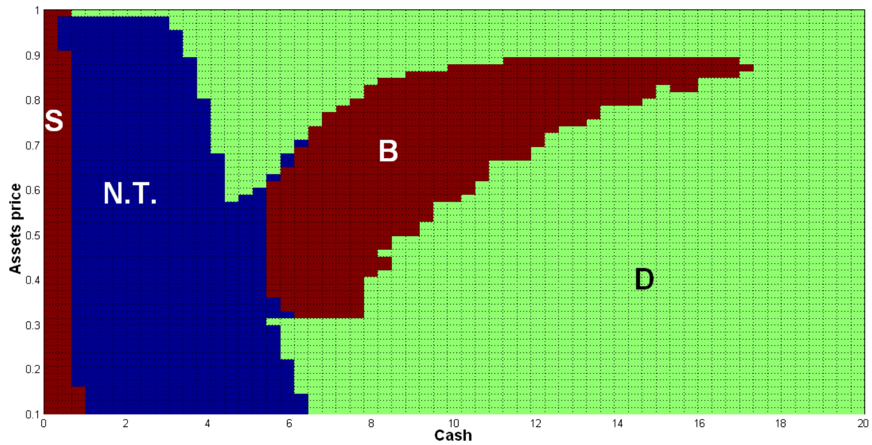

Figure 2: *Description of different regions, in* (x, s) *for* $q_1 > q_0$.

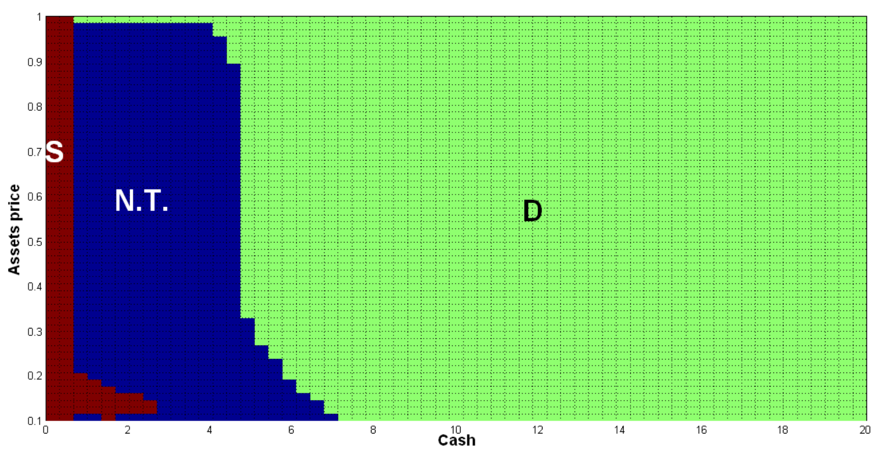

Figure 3: *Description of different regions, in* (x, s) *for* $q_2 > q_1$.

Figures 4 and 5: The value function for different values of the transaction cost κ and the liquidation factor λ. We plot the value function for fixed s and q. We can see that the higher are the transaction cost κ and the liquidation factor λ, the lower is the firm value. Higher costs make the firm more careful in distributing dividends.

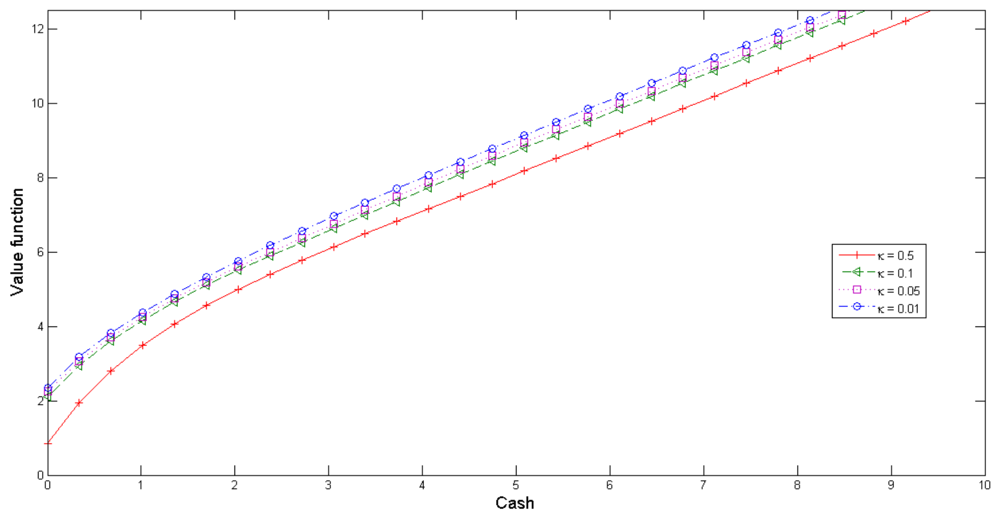

Figure 4: *The value function sliced in x for different values of κ.*

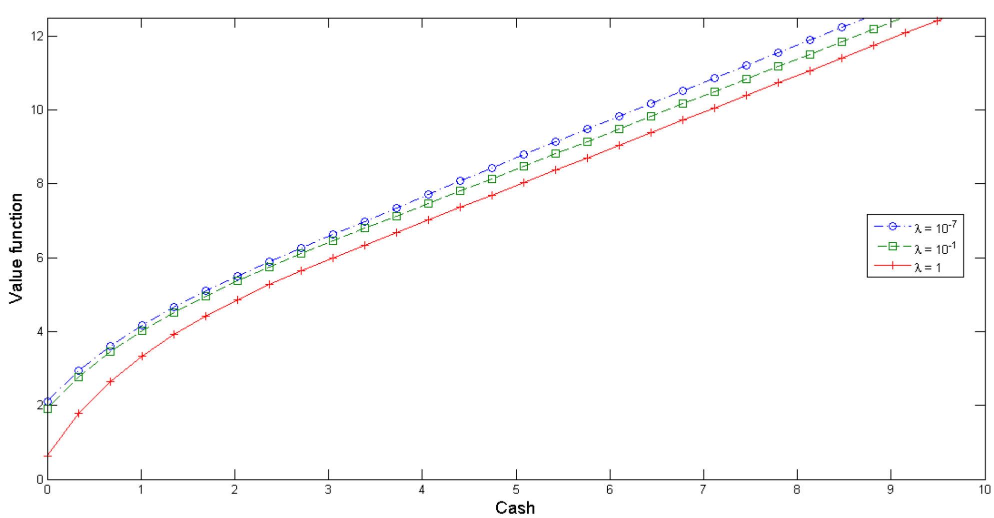

Figure 5: *The value function sliced in x for different values of λ.*

Appendix

Proof of Lemma 3.2:

Consider any $\bar{y} := (\bar{x}, \bar{s}, \bar{q}) \in (0, +\infty)^2 \times \mathbb{N}$ and let $\varphi(.,.,\bar{q})$ a $\mathcal{C}^2$ function on $(0, +\infty)^2$ such that $v_N^l(\bar{y}) = \varphi(\bar{y})$ and $v_N^l - \varphi \geq 0$ in a neighborhood of $\bar{y}$ denoted by $\bar{B}_\varepsilon(\bar{y}) := (\bar{x} - \varepsilon, \bar{x} + \varepsilon) \times (\bar{s} - \varepsilon, \bar{s} + \varepsilon) \times \{\bar{q}\}$ where $0 < \varepsilon < \min(\bar{x}, \bar{s})$.
On the one hand, we obviously have $v_N \geq G_{N-1}$ and as G_{N-1} is continuous, it implies that $v_N^l \geq G_{N-1}$. Therefore, we have

$$\varphi(\bar{y}) = v_N^l(\bar{y}) \geq G_{N-1}(\bar{y}). \tag{4.20}$$

On the other hand, let us consider the admissible control $\hat{\alpha} = \big((\hat{\tau}_i, \hat{q}_i)_{i\in\mathbb{N}}, \hat{Z}\big)$ where we decide to never make an impulse, i.e. $\hat{\tau}_1 = +\infty$, while the dividend policy is defined by $\hat{Z} = \delta$ for $t \geq 0$, with $0 \leq \delta \leq \varepsilon$. We know that there exists a sequence $(\bar{x}_m, \bar{s}_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty} (\bar{x}_m, \bar{s}_m) = (\bar{x}, \bar{s}) \quad \text{and} \quad \lim_{m\to+\infty} v_N(\bar{x}_m, \bar{s}_m, \bar{q}) = v_N^l(\bar{x}, \bar{s}, \bar{q}).$$

With the same notation, we set $\bar{y}_m = (\bar{x}_m, \bar{s}_m, \bar{q})$.
We define the exit time $\tau_\varepsilon^m := \inf\{t \geq 0, Y_t^{\bar{y}_m} \notin \bar{B}_\varepsilon(\bar{y})\}$. We notice that $\tau_\varepsilon^m < T$.
From the dynamic programming principle (see (3.19)), if we set $\gamma_m := v_N(\bar{y}_m) - \varphi(\bar{y}_m) \geq 0$ and $\nu_m = \tau_\varepsilon^m \wedge h_m$ where $(h_m)_{m\geq 0}$ is a positive sequence such that $\lim_{m\to+\infty} h_m = 0$ and $\lim_{m\to+\infty} \gamma_m / h_m = 0$, then we have

$$\begin{aligned}
\varphi(\bar{y}_m) &= v_N(\bar{y}_m) - \gamma_m \\
&\geq \mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\, d\hat{Z}_t + e^{-\rho\nu_m} v_N\big(Y_{\nu_m}^{\bar{y}_m}\big)\Big] - \gamma_m \\
&\geq \mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\, d\hat{Z}_t + e^{-\rho\nu_m} v_N^l\big(Y_{\nu_m}^{\bar{y}_m}\big)\Big] - \gamma_m \\
&\geq \mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\, d\hat{Z}_t + e^{-\rho\nu_m} \varphi\big(Y_{\nu_m}^{\bar{y}_m}\big)\Big] - \gamma_m.
\end{aligned} \tag{4.21}$$

Applying Itô's formula to the process $e^{-\rho t}\varphi(Y_t^{\bar{y}_m})$ between 0 and ν_m and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}\Big[e^{-\rho\nu_m}\varphi(Y_{\nu_m}^{\bar{y}_m})\Big] &= \varphi(\bar{y}_m) + \mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\,(-\rho\varphi + \mathcal{L}\varphi)(Y_t^{\bar{y}_m})dt\Big] \\
&\quad + \mathbb{E}\Bigg[\sum_{0\leq t<\nu_m} e^{-\rho t}[\varphi(Y_t^{\bar{y}_m}) - \varphi(Y_{t^-}^{\bar{y}_m})]\Bigg].
\end{aligned} \tag{4.22}$$

Combining relations (4.21) and (4.22), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\,(\rho\varphi - \mathcal{L}\varphi)(Y_t^{\bar{y}_m})dt\Big] - \mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\, d\hat{Z}_t\Big] \\
&\quad - \mathbb{E}\Bigg[\sum_{0\leq t<\nu_m} e^{-\rho t}[\varphi(Y_t^{\bar{y}_m}) - \varphi(Y_{t^-}^{\bar{y}_m})]\Bigg] \;\geq\; -\gamma_m.
\end{aligned} \tag{4.23}$$

• If we take $\delta = 0$, we notice that Y is continuous on $[0, \nu_m]$ and only the first term of relation (4.23) is non zero. By dividing the above inequality by h_m and letting m going to infinity, it follows from the smoothness of φ and the continuity of the coefficients that

$$(\rho\varphi - \mathcal{L}\varphi)(\bar{y}) \geq 0. \tag{4.24}$$

• If we take now $\delta > 0$ in (4.23), we notice that $\hat{Z}$ jumps only at $t = 0$ with size δ, hence

$$\mathbb{E}\Big[\int_0^{\nu_m^-} e^{-\rho t}\,(\rho\varphi - \mathcal{L}\varphi)(Y_t^{\bar{y}_m})dt\Big] - \delta - (\varphi(\bar{x}_m - \delta, \bar{s}_m, \bar{q}) - \varphi(\bar{x}_m, \bar{s}_m, \bar{q})) \geq -\gamma_m. \tag{4.25}$$

By sending m to infinity, and then dividing by δ and letting $\delta \to 0$, we obtain

$$\frac{\partial\varphi}{\partial x}(\bar{x}, \bar{s}, \bar{q}) - 1 \geq 0. \tag{4.26}$$

We conclude by combining (4.20), (4.24) and (4.26) to obtain the required supersolution property

$$\min\{\rho\varphi(\bar{x}, \bar{s}, \bar{q}) - \mathcal{L}\varphi(\bar{x}, \bar{s}, \bar{q}); \frac{\partial\varphi}{\partial x}(\bar{x}, \bar{s}, \bar{q}) - 1; v_N^l(\bar{x}, \bar{s}, \bar{q}) - G_{N-1}(\bar{x}, \bar{s}, \bar{q})\} \geq 0. \tag{4.27}$$

Proof of Lemma 3.3: Consider any $\bar{y} := (\bar{x}, \bar{s}, \bar{q}) \in (0, +\infty)^2 \times \mathbb{N}$ and let $\varphi(., ., \bar{q})$ a $\mathcal{C}^2$ function on $(0, +\infty)^2$ such that $v_N^u(\bar{y}) = \varphi(\bar{y})$ and $v_N^u - \varphi \leq 0$ in a neighborhood of $\bar{y}$, denoted by $\bar{B}_\varepsilon(\bar{y}) := (\bar{x} - \varepsilon, \bar{x} + \varepsilon) \times (\bar{s} - \varepsilon, \bar{s} + \varepsilon) \times \{\bar{q}\}$ where $0 < \varepsilon < \min(\bar{x}, \bar{s})$.

Let us argue by contradiction by assuming on the contrary that $\exists\, \delta > 0$ s.t. $\forall y \in \bar{B}_\varepsilon(\bar{y})$ we have

$$\begin{aligned} \rho\varphi(y) - \mathcal{L}\varphi(y) &> \delta, \qquad (4.28)\\ \frac{\partial\varphi}{\partial x}(y) - 1 &> \delta, \qquad (4.29)\\ v_N^u(y) - G_{N-1}(y) &> \delta. \qquad (4.30) \end{aligned}$$

We know that there exists a sequence $(\bar{x}_m, \bar{s}_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty}(\bar{x}_m, \bar{s}_m) = (\bar{x}, \bar{s}) \quad \text{and} \quad \lim_{m\to+\infty} v_N(\bar{x}_m, \bar{s}_m, \bar{q}) = v_N^u(\bar{x}, \bar{s}, \bar{q}).$$

Let $\bar{y}_m := (\bar{x}_m, \bar{s}_m, \bar{q}) \in B_\varepsilon(\bar{y})$. For any admissible control $\alpha = \big((\tau_i, q_i)_{i\in\mathbb{N}^*}, Z\big)$, consider the exit time $\tau_\varepsilon^m = \inf\{t \geq 0, Y_t^{\bar{y}_m} \notin \bar{B}_\varepsilon(\bar{y})\}$. We notice that $\tau_\varepsilon^m < T$. Applying Itô's formula to the process $e^{-\rho t}\varphi(Y_t^{\bar{y}_m})$ between 0 and $(\tau_\varepsilon^m \wedge \tau_1)^-$ and by noting that before $(\tau_\varepsilon^m \wedge \tau_1)^-$, $Y_t^{\bar{y}_m}$ stays in the ball $\bar{B}_\varepsilon(\bar{y})$, we obtain

$$\begin{aligned} \mathbb{E}\Big[e^{-\rho(\tau_\varepsilon^m\wedge\tau_1)^-}\varphi(Y^{\bar{y}_m}_{(\tau_\varepsilon^m\wedge\tau_1)^-})\Big] &= \varphi(\bar{y}_m) + \mathbb{E}\Big[\int_0^{(\tau_\varepsilon^m\wedge\tau_1)^-} e^{-\rho t}\,(-\rho\varphi + \mathcal{L}\varphi)(Y_t^{\bar{y}_m})dt\Big]\\ &\quad - \mathbb{E}\Big[\int_0^{(\tau_\varepsilon^m\wedge\tau_1)^-} e^{-\rho t}\,\frac{\partial\varphi}{\partial x}(Y_t^{\bar{y}_m})dZ_t^c\Big]\\ &\quad + \mathbb{E}\Bigg[\sum_{0\leq t<\tau_\varepsilon^m\wedge h} e^{-\rho t}[\varphi(Y_t^{\bar{y}_m}) - \varphi(Y_{t^-}^{\bar{y}_m})]\Bigg]. \qquad (4.31) \end{aligned}$$

From Taylor's formula and (4.29), and noting that $\Delta X^{\bar x}_t = -\Delta Z_t$ for all $0 \le t < \tau^m_\varepsilon \wedge \tau_1$, we have

$$\begin{aligned}\varphi(Y^{\bar y_m}_t) - \varphi(Y^{\bar y_m}_{t^-}) &= \Delta X^{\bar x}_t \frac{\partial \varphi}{\partial x}(Y^{\bar y_m}_{t^-}) \\ &\le -(1+\delta)\Delta Z_t. \end{aligned} \tag{4.32}$$

Plugging the relations (4.28), (4.29) and (4.32) into (4.31), we obtain

$$\begin{aligned}\varphi(\bar y_m) &\ge \mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dZ_t + e^{-\rho(\tau^m_\varepsilon\wedge\tau_1)^-}\varphi(Y^{\bar y_m}_{(\tau^m_\varepsilon\wedge\tau_1)^-})\Big] \\ &\quad +\delta\Big(\mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dt\Big] + \mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dZ_t\Big]\Big) \\ &\ge \mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dZ_t + e^{-\rho\tau^{m-}_\varepsilon}\varphi(Y^{\bar y}_{\tau^{m-}_\varepsilon})1_{\tau^m_\varepsilon<\tau_1} + e^{-\rho\tau_1^-}\varphi(Y^{\bar y_m}_{\tau_1^-})1_{\tau_1\le\tau^m_\varepsilon}\Big] \\ &\quad +\delta\Big(\mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dt\Big] + \mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dZ_t\Big]\Big). \end{aligned} \tag{4.33}$$

<u>First step:</u> On $\{\tau^m_\varepsilon < \tau_1\}$, we notice that while $Y^{\bar y_m}_{\tau^{m-}_\varepsilon} \in \bar B_\varepsilon(\bar y)$, $Y^{\bar y_m}_{\tau^m_\varepsilon}$ is either on the boundary $\partial \bar B_\varepsilon(\bar y)$ or out of $\bar B_\varepsilon(\bar y)$. However, there is some random variable γ valued in $[0,1]$ s.t.

$$\begin{aligned} X^{(\gamma)} &:= X^{\bar x_m}_{\tau^{m-}_\varepsilon} + \gamma \Delta X^{\bar x_m}_{\tau^m_\varepsilon}, \\ &= X^{\bar x_m}_{\tau^{m-}_\varepsilon} - \gamma\Delta Z_{\tau^m_\varepsilon} \quad \in \{\bar x - \varepsilon, \bar x + \varepsilon\}, \end{aligned}$$

hence, $Y^{(\gamma)} := (X^{(\gamma)}, S^{\bar s_m}_{\tau^m_\varepsilon}, Q^{\bar q}_{\tau^m_\varepsilon})$ is on the boundary $\partial\bar B_\varepsilon(\bar y)$.
Following the same arguments as in (4.32), we have

$$\varphi(Y^{(\gamma)}) - \varphi(Y^{\bar y_m}_{\tau^{m-}_\varepsilon}) \le -\gamma(1+\delta)\Delta Z_{\tau^m_\varepsilon}. \tag{4.34}$$

Noting that $X^{(\gamma)} = X^{\bar x_m}_{\tau^m_\varepsilon} + (1-\gamma)\Delta Z_{\tau_\varepsilon}$, we have

$$v^u_N(Y^{(\gamma)}) \ge v^u_N(Y^{\bar y_m}_{\tau^m_\varepsilon}) + (1-\gamma)\Delta Z_{\tau^m_\varepsilon}. \tag{4.35}$$

Recalling that $\varphi(Y^{(\gamma)}) \ge v^u_N(Y^{(\gamma)})$, inequalities (4.34) and (4.35) imply

$$\varphi(Y^{\bar y_m}_{\tau^{m-}_\varepsilon}) \ge v^u_N(Y^{\bar y_m}_{\tau^m_\varepsilon}) + (1+\gamma\delta)\Delta Z_{\tau^m_\varepsilon}. \tag{4.36}$$

<u>Second step:</u> On $\{\tau_1 \le \tau^m_\varepsilon\}$, we notice that $Y^{\bar y_m}_{\tau_1^-} \in \bar B_\varepsilon(\bar y)$, thus $v^u_N(Y^{\bar y_m}_{\tau_1^-}) \le \varphi(Y^{\bar y_m}_{\tau_1^-})$. From the assumption (4.30) we obtain

$$\varphi(Y^{\bar y_m}_{\tau_1^-}) \ \ge\ G_{N-1}(Y^{\bar y_m}_{\tau_1^-}) + \delta. \tag{4.37}$$

Plugging (4.36) and (4.37) into (4.33) we have

$$\begin{aligned}\varphi(\bar y_m) &\ge \mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dZ_t + e^{-\rho\tau^m_\varepsilon} v^u_N(Y^{\bar y_m}_{\tau^m_\varepsilon})1_{\tau_\varepsilon<\tau_1} + e^{-\rho\tau_1}G_{N-1}(Y^{\bar y_m}_{\tau_1^-})1_{\tau_1\le\tau^m_\varepsilon}\Big] \\ &\quad +\delta\,\mathbb{E}\Big[\int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dt + \int_0^{(\tau^m_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\, dZ_t + \gamma e^{-\rho\tau^m_\varepsilon}\Delta Z_{\tau^m_\varepsilon}1_{\tau_\varepsilon<\tau_1} + e^{-\rho\tau_1}1_{\tau_1\le\tau^m_\varepsilon}\Big] \\ &\quad +\mathbb{E}\Big[e^{-\rho\tau^m_\varepsilon}\Delta Z_{\tau^m_\varepsilon}1_{\tau^m_\varepsilon<\tau_1}\Big]. \end{aligned} \tag{4.38}$$

We now claim that there exists a constant $c_0 > 0$ such that for any admissible control

$$
\begin{aligned}
c_0 \quad \leq \quad & \mathbb{E}\Big[\int_0^{(\tau_\varepsilon^m \wedge \tau_1)^-} e^{-\rho t}\, dt + \int_0^{(\tau_\varepsilon^m \wedge \tau_1)^-} e^{-\rho t}\, dZ_t\Big] \\
& + \mathbb{E}\Big[\gamma e^{-\rho \tau_\varepsilon^m} \Delta Z_{\tau_\varepsilon^m} 1_{\tau_\varepsilon^m < \tau_1} + e^{-\rho \tau_1} 1_{\tau_1 \leq \tau_\varepsilon^m}\Big].
\end{aligned} \tag{4.39}
$$

The C^2 function $\psi(x,s,q) = c_0[1 - \frac{(x-\bar{x}_m)^2}{\varepsilon^2}]$, with

$$
0 < c_0 \leq \min\left\{\Big(\rho + \frac{2}{\varepsilon}(r(\bar{x}+\varepsilon) + bH) + \frac{\eta^2}{\varepsilon^2}H\Big)^{-1}, \frac{\varepsilon}{2}\right\}
$$

satisfies

$$
\begin{cases}
\min\{-\rho\psi(x,s,q) + \mathcal{L}\psi(x,s,q) + 1; -\frac{\partial \psi}{\partial x}(x,s,q) + 1; -\psi(x,s,q)+1\} \geq 0 \text{ on } \bar{B}_\varepsilon(\bar{y}), \\
\psi(x,s,q) = 0 \text{ on } \partial \bar{B}_\varepsilon(\bar{y}).
\end{cases} \tag{4.40}
$$

Applying Itô's formula, we then obtain

$$
\begin{aligned}
0 < c_0 = \psi(\bar{y}_m) \quad \leq \quad & \mathbb{E}\Big[e^{-\rho(\tau_\varepsilon^m \wedge \tau_1)^-} \psi(Y^{\bar{y}_m}_{(\tau_\varepsilon^m \wedge \tau_1)^-})\Big] \\
& + \mathbb{E}\Big[\int_0^{(\tau_\varepsilon^m \wedge \tau_1)^-} e^{-\rho t}\, dt + \int_0^{(\tau_\varepsilon^m \wedge \tau_1)^-} e^{-\rho t}\, dZ_t\Big].
\end{aligned} \tag{4.41}
$$

Noting that $\frac{\partial \psi}{\partial x}(x,s,q) \leq 1$, we have

$$
\psi(Y^{\bar{y}}_{\tau_\varepsilon^{m-}}) - \psi(Y^{(\gamma)}) \leq X^{\bar{x}_m}_{\tau_\varepsilon^{m-}} - X^{(\gamma)} = \gamma \Delta Z_{\tau_\varepsilon^m}.
$$

Plugging into (4.41), we obtain

$$
\begin{aligned}
0 < c_0 \quad \leq \quad & \mathbb{E}\Big[e^{-\rho \tau_1} \psi(Y^{\bar{y}_m}_{\tau_1^-}) 1_{\tau_1 \leq \tau_\varepsilon^m}\Big] + \mathbb{E}\Big[\int_0^{(\tau_\varepsilon^m \wedge \tau_1)^-} e^{-\rho t}\, dt\Big] \\
& + \mathbb{E}\Big[\int_0^{(\tau_\varepsilon^m \wedge \tau_1)^-} e^{-\rho t}\, dZ_t\Big] + \mathbb{E}\Big[e^{-\rho \tau_\varepsilon^m} \gamma \Delta Z_{\tau_\varepsilon^m} 1_{\tau_\varepsilon^m < \tau_1}\Big].
\end{aligned} \tag{4.42}
$$

Since $\psi \leq 1$ for all $y \in \bar{B}_\varepsilon(\bar{y})$, this proves the claim (4.39).

Finally, by taking the supremum over all admissible control α, and using the dynamic programming principle (3.19), Equation (4.38) implies that $\varphi(\bar{y}_m) \geq v_N^u(\bar{y}_m) + \delta c_0$, which leads to a contradiction when m goes to infinity. Thus we obtain the required viscosity subsolution property:

$$
\min\{\rho\varphi(\bar{x},\bar{s},\bar{q}) - \mathcal{L}\varphi(\bar{x},\bar{s},\bar{q}); \frac{\partial \varphi}{\partial x}(\bar{x},\bar{s},\bar{q}) - 1; v_N^u(\bar{x},\bar{s},\bar{q}) - G_{N-1}(\bar{x},\bar{s},\bar{q})\} \leq 0. \tag{4.43}
$$

□

Proof of Lemma 3.4:

Step 1. We first construct a strict supersolution to the system with suitable perturbation of w. We set

$$
g(y) = A + B(x+sq+1) + D(x+sq+1)^p, \quad p \in (1,2), \quad y \in \mathcal{S},
$$

where

$$A = \frac{BHb + bH + 1}{\rho} + C_1 + C_3\kappa, \quad B = 2 \quad \text{and} \quad D = B\big(\frac{\rho - \max(r,\mu)}{2Hb}\big).$$

We then define for all $\gamma \in (0,1)$, the lower semi-continuous function on $(0,+\infty)^2 \times \mathbb{N}$ by:

$$w^\gamma = (1-\gamma)w + \gamma g.$$

Let $y \in (0,+\infty)^2 \times \mathbb{N}$. For all $\gamma \in (0,1)$, we then see that :

$$\begin{aligned} w^\gamma(y) - G_{N-1}(y) &\geq (1-\gamma)\big(w(y) - G_{N-1}(y)\big) + \gamma\big(g(y) - G_{N-1}(y)\big) \\ &\geq \gamma\kappa, \end{aligned} \tag{4.44}$$

where the last inequality comes from the facts that as a supersolution $w(y) \geq G_{N-1}(y)$ and that for all $n \in \mathcal{A}(y)$, Corollary 3.1 implies that:

$$\begin{aligned} v_{N-1}\big(\Gamma(y,n)\big) &\leq x - \kappa - nsf(n) + s(q+n) + \frac{bH}{\rho} \\ &\leq \frac{bH}{\rho} + (x + sq + 1) + sn(1 - f(n)) - \kappa \\ &\leq g(y) - \kappa. \end{aligned} \tag{4.45}$$

Furthermore, we also easily obtain

$$\frac{\partial g}{\partial x}(y) - 1 = B + p(x + sq + 1)^{p-1} - 1 \geq 1. \tag{4.46}$$

Recalling that $h(q) \leq H$ for all $q \in \mathbb{N}$, a straight calculation gives

$$\begin{aligned} \rho g(y) - \mathcal{L}g(y) &= \rho\big(A + B(x+sq+1) + D(x+sq+1)^p\big) - Dc\sigma h(q)\eta q s p(p-1)(x+sq+1)^{p-2} \\ &\quad -\frac{\eta^2 h^2(q)}{2} Dp(p-1)(x+sq+1)^{p-2} - \frac{s^2q^2\sigma^2}{2} Dp(p-1)(x+sq+1)^{p-2} \\ &\quad -\big(rx + bh(q)\big)\big(B + Dp(x+sq+1)^{p-1}\big) - \mu s\big(Bq + Dqp(x+sq+1)^{p-1}\big) \\ &\geq \rho A - BbH + B\big(\rho - \max(r,\mu)\big)(x+sq+1) - Dc\sigma H\eta p(p-1)(x+sq+1)^p \\ &\quad -D\frac{\eta^2H^2}{2} p(p-1)(x+sq+1)^p - D\frac{\sigma^2}{2}p(p-1)(x+sq+1)^p \\ &\quad -DbHp(x+sq+1)^{p-1} - D\max(r,\mu)p(x+sq+1)^p + D\rho(x+sq+1)^p \\ &\geq \rho A - BbH + (x+sq+1)\Big(B\big(\rho - \max(r,\mu)\big) - DbHp(x+sq+1)^{p-2}\Big) \\ &\quad +D\Big(\rho - \max(r,\mu)p - \big(\frac{\eta^2H^2}{2} + \frac{\sigma^2}{2} + c\sigma H\eta\big)p(p-1)\Big)(x+sq+1)^p. \end{aligned}$$

Using that $p(x+sq+1)^{p-2} \leq 2$ for all $p \in (1,2)$ and replacing A by its value, we obtain

$$\begin{aligned} \rho g(y) - \mathcal{L}g(y) &\geq 1 + (x+sq+1)\big(B(\rho - \max(r,\mu)) - 2DbH\big) \\ &\quad +D\big(\rho - \max(r,\mu)p - \big(\frac{\eta^2H^2}{2} + \frac{\sigma^2}{2} + c\sigma H\eta\big)p(p-1)\big)(x+sq+1)^p. \end{aligned}$$

Recalling that $\rho > \max(r,\mu)$, we can choose $p \in (1,2)$ s.t.

$$\xi := \rho - \max(r,\mu)p - (\frac{\eta^2 H^2}{2} + \frac{\sigma^2}{2} + c\sigma H\eta)p(p-1) > 0,$$

we then have

$$\rho g(y) - \mathcal{L}g(y) \quad \geq \quad 1 + D\xi(x + sq + 1)^p, \quad \forall y \in (0,+\infty)^2 \times \mathbb{N}. \tag{4.47}$$

Combining (4.45), (4.46) and (4.47), we obtain that w^γ is a strict supersolution of equation (3.18): $\forall y \in (0,+\infty)^2 \times \mathbb{N}$, we have

$$\min\{\rho w^\gamma(y) - \mathcal{L}w^\gamma(y); \frac{\partial w^\gamma}{\partial x}(y) \ - \ 1; w^\gamma(y) - G_{N-1}(y)\} \geq \gamma \min(1,\kappa) := \delta. \tag{4.48}$$

Step 2. In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0,1)$:

$$\sup_{y \in (0,+\infty)^2 \times \mathbb{N}} (u - w^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0. We argue by contradiction and suppose that there exist some $\gamma \in (0,1)$ s.t.

$$\theta := \sup_{y \in (0,+\infty)^2 \times \mathbb{N}} (u - w^\gamma) > 0. \tag{4.49}$$

Notice that $u(y) - w^\gamma(y)$ goes to $-\infty$ as x, s and q go to infinity. For any $\bar{y} \in \{0\} \times (0,+\infty) \times \mathbb{N}$, we also have

$$\limsup_{y \to \bar{y}} u(y) - w^\gamma(y) \leq \gamma(\liminf_{y \to \bar{y}} w(y) - A) \leq 0.$$

Hence, by semi-continuity of the functions u and w^γ, there exists $y_0 = (x_0, s_0, q_0) \in (0,+\infty)^2 \times \mathbb{N}$ s.t.

$$\theta = u(y_0) - w^\gamma(y_0).$$

For any $\varepsilon > 0$, we consider the the functions

$$\begin{aligned} \Phi_\varepsilon(y,y') &= u(y) - w^\gamma(y') - \varphi_\varepsilon(y,y') \\ \varphi_\varepsilon(y,y') &= \frac{1}{4}|y - y_0|^4 + \frac{1}{2\varepsilon}|y - y'|^2, \end{aligned}$$

for all y, $y' \in \mathcal{S}$. By standard arguments in comparison principle, the function Φ_ε attains its maximum in $(y_\varepsilon, y'_\varepsilon) \in ((0,+\infty)^2 \times \mathbb{N})^2$, which converges (up to a subsequence) to (y_0, y_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon \to 0} \frac{|y_\varepsilon - y'_\varepsilon|^2}{\varepsilon} = 0. \tag{4.50}$$

Applying Theorem 3.2 in [9], we get the existence of two 2×2 symmetric matrices M^ε and N^ε s.t.:

$$\begin{aligned} (p^\varepsilon, M^\varepsilon) &\in J^{2,+}u(y_\varepsilon), \\ (d^\varepsilon, N^\varepsilon) &\in J^{2,-}w^\gamma(y'_\varepsilon), \end{aligned}$$

and

$$\begin{pmatrix} M^\varepsilon & 0 \\ 0 & -N^\varepsilon \end{pmatrix} \le D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) + \varepsilon\big(D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon)\big)^2, \tag{4.51}$$

where

$$\begin{aligned}
p^\varepsilon &= D_y\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) = \begin{pmatrix} (x_\varepsilon - x_0)|y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon) \\ (s_\varepsilon - s_0)|y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon}(s_\varepsilon - s'_\varepsilon) \end{pmatrix} \\
d^\varepsilon &= -D_{y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) = \begin{pmatrix} \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon) \\ \frac{1}{\varepsilon}(s_\varepsilon - s'_\varepsilon) \end{pmatrix}, \\
D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) &= \begin{pmatrix} 2(x_\varepsilon - x_0)^2 + |y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon} & 2(x_\varepsilon - x_0)(s_\varepsilon - s_0) & \frac{-1}{\varepsilon} & 0 \\ 2(x_\varepsilon - x_0)(s_\varepsilon - s_0) & 2(s_\varepsilon - s_0)^2 + |y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon} & 0 & \frac{-1}{\varepsilon} \\ \frac{-1}{\varepsilon} & 0 & \frac{1}{\varepsilon} & 0 \\ 0 & \frac{-1}{\varepsilon} & 0 & \frac{1}{\varepsilon} \end{pmatrix}.
\end{aligned}$$

By writing the viscosity subsolution property of u and the viscosity supersolution property (4.48) of w^γ, we have the following inequalities:

$$\min\Big\{\rho u(y_\varepsilon) - \big(rx_\varepsilon + bh(q_\varepsilon)\big)p_1^\varepsilon - \mu s_\varepsilon p_2^\varepsilon \quad -c\sigma\eta s_\varepsilon h(q_\varepsilon)M_{12}^\varepsilon - \tfrac{\eta^2 h^2(q_\varepsilon)}{2}M_{11}^\varepsilon - \tfrac{\sigma^2 s_\varepsilon^2}{2}M_{22}^\varepsilon; \\ p_1^\varepsilon - 1; u(y_\varepsilon) - G_{N-1}(y_\varepsilon)\Big\} \le 0 \tag{4.52}$$

$$\min\Big\{\rho w^\gamma(y'_\varepsilon) - \big(rx'_\varepsilon + bh(q'_\varepsilon)\big)d_1^\varepsilon - \mu s'_\varepsilon d_2^\varepsilon \quad -c\sigma\eta s'_\varepsilon h(q'_\varepsilon)N_{12}^\varepsilon - \tfrac{\eta^2 h^2(q'_\varepsilon)}{2}N_{11}^\varepsilon - \tfrac{\sigma^2 (s'_\varepsilon)^2}{2}N_{22}^\varepsilon; \\ d_1^\varepsilon - 1; w^\gamma(y'_\varepsilon) - G_{N-1}(y'_\varepsilon)\Big\} \ge \delta \tag{4.53}$$

We then distinguish three cases:

• *Case 1*: $u(y_\varepsilon) - G_{N-1}(y_\varepsilon) \le 0$ in (4.52).
From the definition of $(y_\varepsilon, y'_\varepsilon)$, we have

$$\begin{aligned}
\theta &= u(y_0) - w^\gamma(y_0) \\
&\le u(y_\varepsilon) - w^\gamma(y'_\varepsilon) - \varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) \\
&\le G_{N-1}(y_\varepsilon) - G_{N-1}(y'_\varepsilon) - \delta - \varphi_\varepsilon(y_\varepsilon, y'_\varepsilon).
\end{aligned}$$

Now, letting ε going to 0, we deduce from equation (4.50) and the continuity of G_{N-1} that $0 < \theta \le -\delta < 0$, which is obviously a contradiction.

• *Case 2*: $(x_\varepsilon - x_0)|y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon) - 1 = p_1^\varepsilon - 1 \le 0$ in (4.52).
Notice by (4.53), we have

$$\frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon) - 1 = d_1^\varepsilon - 1 \ge \delta,$$

which implies in this case

$$(x_\varepsilon - x_0)|y_\varepsilon - y_0|^2 \le -\delta.$$

By sending ε to zero, we obtain again a contradiction.
• *Case 3:* $\rho u(y_\varepsilon) - \big(rx_\varepsilon + bh(q_\varepsilon)\big)p_1^\varepsilon - \mu s_\varepsilon p_2^\varepsilon - c\sigma\eta s_\varepsilon h(q_\varepsilon)M_{12}^\varepsilon - \frac{\eta^2 h^2(q_\varepsilon)}{2}M_{11}^\varepsilon - \frac{\sigma^2 s_\varepsilon^2}{2}M_{22}^\varepsilon \leq 0$ in (4.52).
From (4.53), we have

$$\rho w^\gamma(y'_\varepsilon) - \big(rx'_\varepsilon + bh(q'_\varepsilon)\big)d_1^\varepsilon - \mu s'_\varepsilon d_2^\varepsilon - c\sigma\eta s'_\varepsilon h(q'_\varepsilon)N_{12}^\varepsilon - \frac{\eta^2 h^2(q'_\varepsilon)}{2}N_{11}^\varepsilon - \frac{\sigma^2 (s'_\varepsilon)^2}{2}N_{22}^\varepsilon \geq \delta,$$

which implies in this case

$$\begin{aligned} \rho\big(u(y_\varepsilon) - w^\gamma(y'_\varepsilon)\big) - r(x_\varepsilon p_1^\varepsilon - x'_\varepsilon d_1^\varepsilon) - b(h(q_\varepsilon)p_1^\varepsilon - h(q'_\varepsilon)d_1^\varepsilon) - \mu(s_\varepsilon p_1^\varepsilon - s'_\varepsilon d_1^\varepsilon) \\ -c\sigma\eta(s_\varepsilon h(q_\varepsilon)M_{12}^\varepsilon - s'_\varepsilon h(q'_\varepsilon)N_{12}^\varepsilon) - \frac{\eta^2}{2}(h^2(q_\varepsilon)M_{11}^\varepsilon - h^2(q'_\varepsilon)N_{11}^\varepsilon) \\ -\frac{\sigma^2}{2}\big((s_\varepsilon)^2 M_{22}^\varepsilon - (s'_\varepsilon)^2 N_{22}^\varepsilon\big) \leq -\delta \end{aligned} \tag{4.54}$$

We have that

$$x_\varepsilon p_1^\varepsilon - x'_\varepsilon d_1^\varepsilon = x_\varepsilon(x_\varepsilon - x_0)|y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon)^2.$$

From (4.50) we have that this last quantity goes to zero when ε goes to zero. Using the same argument, we also have that the quantity $s_\varepsilon p_1^\varepsilon - s'_\varepsilon d_1^\varepsilon$ goes to zero when ε goes to zero. Using that $h(q) \leq H$ for all $q \in \mathbb{N}$ we have

$$\begin{aligned} h(q_\varepsilon)p_1^\varepsilon - h(q'_\varepsilon)d_1^\varepsilon &\leq h(q_\varepsilon)(x_\varepsilon - x_0)|y_\varepsilon - y_0|^2 + \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon)\big(h(q_\varepsilon) - h(q'_\varepsilon)\big) \\ &\leq h(q_\varepsilon)(x_\varepsilon - x_0)|y_\varepsilon - y_0|^2 + H\frac{(x_\varepsilon - x'_\varepsilon)^2 + (q_\varepsilon - q'_\varepsilon)^2}{\varepsilon}. \end{aligned}$$

Again, by (4.50), this last quantity goes to zero when ε goes to zero.
Moreover, assuming that $c \neq 0$, from (4.51), we have

$$\begin{aligned} c\sigma\eta(s_\varepsilon h(q_\varepsilon)M_{12}^\varepsilon - s'_\varepsilon h(q'_\varepsilon)N_{12}^\varepsilon) + \frac{\eta^2}{2}(h^2(q_\varepsilon)M_{11}^\varepsilon - h^2(q'_\varepsilon)N_{11}^\varepsilon) \\ +\frac{c^2\sigma^2}{2}\big((s_\varepsilon)^2 M_{22}^\varepsilon - (s'_\varepsilon)^2 N_{22}^\varepsilon\big) \leq \Upsilon \end{aligned} \tag{4.55}$$

where

$$\Upsilon = \Lambda\Big(D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) + \varepsilon\big(D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon)\big)^2\Big)\Lambda^\intercal$$

with

$$\Lambda = \frac{1}{\sqrt{2}}\big(\eta h(q_\varepsilon), c\sigma s_\varepsilon, \eta h(q'_\varepsilon), c\sigma s'_\varepsilon\big)^\intercal$$

Here $\intercal$ denotes the transpose operator.
From (4.51), we have also

$$\frac{c^2\sigma^2}{2}\big((s_\varepsilon)^2 M_{22}^\varepsilon - (s'_\varepsilon)^2 N_{22}^\varepsilon\big) \leq \bar{\Upsilon} \tag{4.56}$$

where

$$\bar{\Upsilon} = \bar{\Lambda}\Big(D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon) + \varepsilon\big(D^2_{y,y'}\varphi_\varepsilon(y_\varepsilon, y'_\varepsilon)\big)^2\Big)\bar{\Lambda}^\intercal$$

with

$$\bar{\Lambda} = \frac{1}{\sqrt{2}}\big(0, c\sigma s_\varepsilon, 0, c\sigma s'_\varepsilon\big)^\intercal$$

Combining (4.55) and (4.56) and the fact that $-1 \leq c \leq 1$, we obtain

$$c\sigma\eta(s_\varepsilon h(q_\varepsilon)M^\varepsilon_{12} - s'_\varepsilon h(q'_\varepsilon)N^\varepsilon_{12}) + \frac{\eta^2}{2}(h^2(q_\varepsilon)M^\varepsilon_{11} - h^2(q'_\varepsilon)N^\varepsilon_{11}) + \frac{\sigma^2}{2}\big((s_\varepsilon)^2M^\varepsilon_{22} - (s'_\varepsilon)^2N^\varepsilon_{22}\big) \leq \Upsilon + \frac{1-c^2}{c^2}\bar{\Upsilon}.$$

After some straightforward calculations and using (4.50) and that $h(q) \leq H$ for all $q \in \mathbb{N}$, we have that Υ and $\bar{\Upsilon}$ go to zero when ε goes to zero.
The case of $c = 0$ is treated in the same way by choosing $\Lambda = \frac{1}{\sqrt{2}}\big(\eta h(q_\varepsilon), 0, \eta h(q'_\varepsilon), 0\big)^\intercal$ and $\bar{\Lambda} = \frac{1}{\sqrt{2}}\big(0, \sigma s_\varepsilon, 0, \sigma s'_\varepsilon\big)^\intercal$.
Finally, using all the above arguments and the continuity of u and w^γ we can see that when ε goes to zero in the inequality (4.54), we obtain the required contradiction: $\rho\theta \leq -\delta < 0$. This ends the proof.

□

References

[1] Asmussen S., Taksar M.: Controlled diffusion models for optimal dividend payout. Insurance Math. Econom., 20, pp. 1-15 (1997).

[2] Bayraktar E., EgamiE. : On the One-Dimensional Optimal Switching Problem. Mathematics of Operations Research, 2010, 35 (1), 140-159 (2010).

[3] Boguslavskaya E.: On Optimization of dividend flow for a company in a presence of liquidation value. Working paper. http://www.boguslavsky.net/fin/index.html.

[4] Brekke K., Oksendal B.: Optimal switching in an economic activity under uncertainty. SIAM J. Cont. Optim., 32, 1021-1036 (1994).

[5] Budhiraja A., Ross K.: Convergent Numerical Scheme for Singular Stochastic Control with State Constraints in a Portfolio Selection Problem. SIAM J. Control Optim., 45(6), 2169-2206 (2007).

[6] Carmona R., Ludkovski M. : Valuation of Energy Storage: An Optimal Switching Approach. Quantitative Finance, 10(4), 359-374, (2010).

[7] Chevalier E., Ly Vath V., Scotti S. : An Optimal Dividend and Investment Control Problem under Debt Constraints. Siam Journal on Financial Mathematics, SIAM J. Finan. Math., 4(1), 297 - 326 (2013).

[8] Choulli T. , Taksar M., Zhou X.Y. : A diffusion model for optimal dividend distribution for a company with constraints on risk control. SIAM J. Cont. Optim., 41, 1946-1979 (2003).

[9] Crandall M., Ishii H., Lions P.L. : User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc., 27, 1-67 (1992).

[10] Dai M., Zhang Q., Zhu Q. : Trend Following Trading under a Regime Switching Model. SIAM Journal on Financial Mathematics 1, 780-810 (2010).

[11] Décamps J.P., Villeneuve S. : Optimal dividend policy and growth option. Finance and Stochastics, 11, 3-27 (2005).

[12] Djehiche B., Hamadène S., Popier A. : A finite horizon optimal switching problem. SIAM J. Control and Optimization, Volume 48, Issue 4, pp. 2751-2770 (2009).

[13] Duckworth K. and M. Zervos : A model for investment decisions with switching costs. Annals of Applied Probability, 11, 239-250 (2001).

[14] Guo, X., Tomecek, P. : Connections between singular control and optimal switching. SIAM J. Control Optim., 47(1) (2008): 421-443.

[15] Hamadène S., Jeanblanc M. : On the Starting and Stopping Problem: Application in reversible investments. Mathematics of Operation Research, vol. 32 no. 1 182-192 (2007).

[16] Hindy A., Huang C., Zhu H. : Numerical Analysis of a Free-Boundary Singular Control Problem in Financial Economics. J. Econom. Dynam. Control, 21, 297-327 (1993).

[17] Hu Y., Tang S. : Multi-dimensional BSDE with oblique reflection and optimal switching. Prob. Theory and Related Fields, 147(1-2) 89-121 (2010).

[18] Jeanblanc M., Shiryaev A. : Optimization of the flow of dividends. Russian Math. Survey, 50, 257-277 (1995).

[19] Jiang Z., Pistorius M. : Optimal dividend distribution under Markov regime switching. Finance and Stochastics, 16, 449-476 (2012).

[20] Jin Z., G. Yin, Zhu C. : Numerical solutions of optimal risk control and dividend optimization policies under a generalized singular control formulation. Automatica, 48(8), 1489-1501 (2012).

[21] Kushner, H., Dupuis, P. : Numerical Methods for Stochstic Control Problems in Continuous Time. Stochastic Modelling and Applied Probability, Volume 24, Springer, New York second edition (2001).

[22] Lon P. C., Zervos M. : A model for optimally advertising and launching a product. Mathematics of Operations Research, 36, 363-376 (2011).

[23] Ly Vath V., Pham H. : Explicit solution to an optimal switching problem in the two-regime case. SIAM J. Cont. Optim., 46, 395-426 (2007).

[24] Ly Vath V., Pham H., Villeneuve S.: A mixed singular/switching control problem for a dividend policy with reversible technology investment. Annals of Applied Probability, 2008, 18, pp. 1164-1200 (2008).

[25] Pham H. : Continuous-time Stochastic Control and Optimization with Financial Applications. Stochastic Modelling and Applied Probability, Vol. 61, 232p (2009).

[26] Pham H., Ly Vath V., Zhou X.Y. : Optimal switching over multiple regimes. SIAM Journal on Control and Optimization, 48, 2217-2253 (2009).

[27] Zervos M., Johnson T., Alazemi F. : Buy-low and sell-high investment strategies. Mathematical Finance, 23, pp.560-578 (2013).