An Optimal Dividend and Investment Control Problem under Debt Constraints

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Abstract

This paper concerns with the problem of determining an optimal control on the dividend and investment policy of a firm under debt constraints. We allow the company to make investment by increasing its outstanding indebtedness, which would impact its capital structure and risk profile, thus resulting in higher interest rate debts. Moreover, a high level of debt is also a challenging constraint to any firm as it is no other than the threshold below which the firm value should never go to avoid bankruptcy. It is equally possible for the firm to divest parts of its business in order to decrease its financial debt owed to creditors. In addition, the firm may favor investment by postponing or reducing any dividend distribution to shareholders. We formulate this problem as a combined singular and multi-switching control problem and use a viscosity solution approach to get qualitative descriptions of the solution. We further enrich our studies with a complete resolution of the problem in the two-regime case and provide some numerical illustrations.

Keywords: stochastic control, optimal singular / multi-switching problem, viscosity solution, smooth-fit property, system of variational inequalities, debt constraints.

MSC2000 subject classification: 60G40, 91B70, 93E20.

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1 Introduction

The theory of optimal stochastic control problem, developed in the seventies, has over the recent years once again drawn a significance of interest, especially from the applied mathematics community with the main focus on its applications in a variety of fields including economics and finance. For instance, the use of powerful tools developed in stochastic control theory has provided new approaches and sometime the first mathematical approaches in solving problems arising from corporate finance. It is mainly about finding the best optimal decision strategy for managers whose firms operate under uncertain environment whether it is financial or operational, see [3] and [10]. A number of corporate finance problems have been studied, or at least revisited, with this optimal stochastic control approach.

In this paper, we consider the problem of determining the optimal control on the dividend and investment policy of a firm under debt constraints. There are a number of research on this corporate finance problem. In [7], Décamps and Villeneuve study the interactions between dividend policy and irreversible investment decision in a growth opportunity and under uncertainty. We may equally refer to [20] for an extension of this study, where the authors relax the irreversible feature of the growth opportunity.

As in a large part of the literature in corporate finance, the above papers assume that the firm cash reserve follows a drifted Brownian motion. They also assume that the firm does not have the ability to raise any debt for its investment as it holds no debt in its balance sheet. In our study, as in the Merton model, we consider that firm value follows a geometric Brownian process and more importantly we consider that the firm carries a debt obligation in its balance sheet. However, as in most studies, we still assume that the firm assets is highly liquid and may be assimilated to cash equivalents or cash reserve. We allow the company to make investment and finance it through debt issuance/raising, which would impact its capital structure and risk profile. This debt financing results therefore in higher interest rate on the firm's outstanding debts. More precisely, we model the decisions to raise or redeem some debt obligations as switching decisions controls where each regime corresponds to a specific level debt.

Furthermore, we consider that the manager of the firm works in the interest of the shareholders, but only to a certain extent. Indeed, in the objective function, we introduce a penalty cost P and assume that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy. This penalty cost could represent, for instance, an estimated cost of the negative image upon his/her own reputation due to the bankruptcy under his management leadership. Mathematically, we formulate this problem as a combined singular and multiple-regime switching control problem. Each regime corresponds to a level of debt obligation held by the firm.

In terms of literature, there are many research papers on singular control problems as well as on optimal switching control problems. One of the first corporate finance problems using singular stochastic control theory was the study of the optimal dividend strategy, see for instance [5] and [16]. These two papers focus on the study of a singular stochastic control problem arising from the research on optimal dividend policy for a firm whose cash reserve follows a diffusion model.

In the study of optimal switching control problems, a variety of problems are investigated, including problems on management of power station [4], [14], resource extraction [2], firm investment [11], marketing strategy [18], and optimal trading strategies [8], [24]. Other related works on optimal control switching problems include [1] and [19], where the authors employ respectively optimal stopping theory and viscosity techniques to explicitly solve their optimal two-regime switching problem on infinite horizon for one-dimensional diffusions. We may equally refer to [22], for an interesting overview of the area.

In the above studies, only problems involving the two-regime case are investigated. There are still very few studies on the multi-regime switching problems. The main additional feature in the multiple regime problems consists not only in determining the switching region as opposed to the continuation region, but also in identifying the optimal regime to where to switch. This additional feature sharply increases the complexity of the multi-regime switching problems. Recently, Djehiche, Hamadène and Popier [9], and Hu and Tang [15] have studied optimal multiple switching problems for general adapted processes by means of reflected BSDEs, and they are mainly concerned with the existence and uniqueness of solution to these reflected BSDEs. In [23], the authors investigated an optimal multiple switching problem on infinite horizon for a general one-dimensional diffusion. The switching feature of this problem does not impact its state process but uniquely the profit functions.

The studies that are most relevant to our problem are the one investigating combined singular and switching control problems. Recently an interesting connection between the singular and the switching problems was given by Guo and Tomecek [13]. In [20], the authors studied an optimal dividend problem with reversible technology switching investment and used Bachelier process to model the firm's cash reserve. The firm may decide to switch from an old technology to a new technology in order to increases the drift of the cash without affecting the volatility. They proved that the problem can be decoupled in two pure optimal stopping and singular control problems and provided results which are of quasi-explicit nature.

However, none of the above papers on dividend and investment policies, which provides qualitative solutions, has yet moved away from the basic Bachelier model or the simplistic assumption that firms hold no debt obligations. In our model, unlike [23], switching from one regime, i.e. debt level, to another directly impacts the state process itself. Indeed, the drift of the stochastic differential equation governing the firm value would equally switch as the results of the change in interest rate paid on the outstanding debt. A given level of debt is no other than the threshold below which the firm value should never go to avoid bankruptcy. As such, debt level switching also signifies a change of default constraints on the state process in our optimal control problem. Further original contributions in terms of financial studies of our paper include the feature of the conflicts of interest for firm manager through the presence of the penalty cost in the event of bankruptcy. Studying a mixed singular and multi-switching problem combining with the above financial features including debt constraints and penalty cost turns out to be a major mathematical challenge, especially when our objective is to provide quasi-explicit solutions. In addition, it is always tricky to overcoming the combined difficulties of the singular control with those of the switching control, especially when there are multiple regimes, for instance, building a strict supersolution to our HJB system in order to prove the comparison principle.

The plan of the paper is organized as follows. We define the model and formulate our stochastic control problem in the second section. In section 3, we characterize our problem as the unique viscosity solution to the associated HJB system and obtain some regularity properties. We find that the associated value functions of our problems are at least of class C^1. In section 4, we obtain qualitative description of our problem. In particular, we show that the state space is divided into continuation, dividend and switching regions, with each of them being union of intervals. The bounds of these intervals may be characterized. Finally, in section 5, we further enrich our studies with a complete resolution of the problem in the two-regime case and provide some numerical illustrations.

2 The model

We consider a firm whose value follows a process X. The firm also has the possibility to raise its debt level in order to satisfy its financial requirement such as investing in growth opportunities. It may equally pay down its debt.

We consider an admissible control strategy $\alpha = (Z_t, (\tau_n)_{n\geq 0}, (k_n)_{n\geq 0})$, where the non-decreasing càd-làg process Z represents the dividend policy, the nondecreasing sequence of stopping times (τ_n) the switching regime time decisions, and (k_n), which are $\mathcal{F}_{\tau_n}$-measurable valued in $\{1, ..., N\}$, the new value of debt regime at time $t = \tau_n$. Let denote the process $X^{x,i,\alpha}$ as the enterprise value of the company with initial value of x and initially operating with a debt level D_i and which follow the control strategy α. However, in order to reflect the fact that a firm also holds a significant amount of debt obligation either to financial institutions, inland revenues, suppliers or through corporate bonds, we assume that the firm debt level may never get to zero. We assume that the firm assets is cash-like, i.e. the manager may dispose of some part of the company assets and obtain its equivalent in cash. In other words, the process X could be seen as a cash-reserve process used in most papers on optimal dividend policy, see for instance [7], [16].

We assume that the cash-reserve process $X^{x,i,\alpha}$, denoted by X when there is no ambiguity and associated to a strategy $\alpha = (Z_t, (\tau_n)_{n\geq 0}, (k_n)_{n\geq 0})$, is governed by the following stochastic differential equation:

$$dX_t = bX_t dt - r_{I_t} D_{I_t} dt + \sigma X_t dW_t - dZ_t + dK_t \tag{2.1}$$

where

$$\begin{aligned} I_t &= \sum_{n\geq 0} k_n 1_{\tau_n \leq t < \tau_{n+1}},\ I_{0^-} = i \\ k_n &\in \mathbb{I}_N := \{1, ..., N\} \\ D_j &< D_l,\ j < l,\ j, l \in \mathbb{I}_N && (2.2) \\ r_j &< r_l,\ j < l,\ j, l \in \mathbb{I}_N && (2.3) \end{aligned}$$

D_i and r_i represent respectively different levels of debt and their associated interest rate paid on those debts. Relations (2.2) and (2.3) assume that the level of risk of a firm uniquely

depends on the level of its debt, i.e. the higher the debt level, the higher the interest rate that the firm has to pay.
The process K_t represents the cash-flow due to the change in the firm's indebtedness. More precisely,

$$K_t = \sum_{n\geq 0} \left(D_{\kappa_{n+1}} - D_{\kappa_n} - g\right) 1_{\tau_{n+1}\leq t} \tag{2.4}$$

where g represents the additional cost associated with the change of firm's level of debt. It could be seen as the fixed commission cost paid for bank services for arranging debt issuance or debt redemption. Mathematically, it prevents continuous switching of the debt level. We assume that g is small with respect to other quantities in the following way

$$\forall (i,j) \in \mathbb{I}_N : \; i \neq j; 0 < g < \min\left(\frac{\mid (b-r_i)D_i - (b-r_j)D_j \mid}{b}; \; \mid D_i - D_j \mid\right). \tag{2.5}$$

For a given control strategy α, the bankruptcy time is represented by the stopping time T^α defined as

$$T^\alpha = \inf\{t \geq 0, X_t^{x,i,\alpha} \leq D_{I_t}\}. \tag{2.6}$$

When there is no ambiguity, we generally refer to T instead of T^α for the bankruptcy time.
We equally introduce a penalty cost or a liquidation cost $P > 0$, in the case of a holding company looking to liquidate one of its own affiliate or activity. In the case of the penalty, it mainly assumes that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy.
We therefore define the value functions which the manager actually optimizes as follows

$$v_i(x) \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{(i,x)}\left[\int_0^{T^-} e^{-\rho t} dZ_t - Pe^{-\rho T}\right], \quad x \in \mathbb{R}, \; i \in \{1,...,N\}, \tag{2.7}$$

where $\mathcal{A}$ represents the set of admissible control strategies, and ρ the discount rate.
The next step would be to compute the real value function u_i of the shareholders. Indeed, once we obtain the optimal strategy, $\alpha^* = (Z_t^*, (\tau_n^*)_{n\geq 0}, (k_n^*)_{n\geq 0})$ of the above problem (2.7), then we may compute the real shareholders' value by following the strategy α^*, as numerically illustrated in Figure 3 and 4:

$$u_i(x) \;=\; \mathbb{E}^{(i,x)}\left[\int_0^{T^{*,-}} e^{-\rho t} dZ_t^*\right], \quad x \in \mathbb{R}, \; i \in \{1,...,N\}, \;\text{ where }\; T^* = T^{\alpha^*} \tag{2.8}$$

Remark 2.1 If $b > \rho$, the value functions is infinite for any initial value of $x > D_i$ and any regime i. The proof is quite straightforward. We simply consider a sequence of strategy controls which consists in doing nothing up to time t_k, where $(t_k)_{k\geq 1}$ is strictly non-decreasing and goes to infinity when k goes to infinity, and then at t_k, distribute $X_{t_k} - D_{I_{t_k}}$ in dividend and allow the company to become bankrupt. We then need to notice that $\lim\limits_{k\to\infty} \mathbb{E}\left[e^{-\rho t_k} X_{t_k}^x\right] = +\infty$.

For the rest of the paper, we now consider that the discount rate ρ is always bigger than the growth rate b.

3 Viscosity Characterization of the value functions

We first introduce some notations. We denote by $R^{x,i}$ the firm value in the absence of dividend distribution and the ability to change the level of debt, fixed at D_i.

$$dR_t^{x,i} = [bR_t^{x,i} - r_i D_i]dt + \sigma R_t^{x,i} dW_t, \ R_0^{x,i} = x \tag{3.1}$$

The associated second order differential operator is denoted $\mathcal{L}_i$:

$$\mathcal{L}_i \varphi = [bx - r_i D_i]\varphi'(x) + \frac{1}{2}\sigma^2 x^2 \varphi''(x) \tag{3.2}$$

Using the dynamic programming principle, we obtain the associated system of variational inequalities satisfied by the value functions:

$$\begin{aligned} \min\left[-\mathcal{A}_i v_i(x)\ ,\ v_i'(x) - 1\ ,\ v_i(x) - \max_{j\neq i} v_j(x + D_j - D_i - g)\right] &= 0, \ x > D_i, \ i \in \mathbb{I}_N \\ v_i(D_i) &= -P, \end{aligned}$$

where the operator $\mathcal{A}_i$ is defined by $\mathcal{A}_i\phi = \mathcal{L}_i\phi - \rho\phi$.

We now state a standard first result for this system of PDE.

Proposition 3.1 *Let $(\varphi_i)_{i\in\mathbb{I}_N}$ smooth enough on (D_i,∞) such that $\varphi_i(D_i^+) := \lim_{x\downarrow D_i} \varphi_i(x) \geq -P$, and*

$$\min\left[-\mathcal{A}_i \varphi_i(x)\ ,\ \varphi_i'(x) - 1, \varphi_i(x) - \max_{j\neq i} \varphi_j(x + D_j - D_i - g)\right] \geq 0, \ \text{for}\, x > D_i, i \in \mathbb{I}_N$$

where we set by convention $\varphi_i(x) = -P$ for $x < D_i$, then we have $v_i \leq \varphi_i$, for all $i \in \mathbb{I}_N$.

Proof: Given an initial state-regime value $(x,i) \in (D_i,\infty) \times \mathbb{I}_N$, take an arbitrary control $\alpha = (Z,(\tau_n),(k_n)) \in \mathcal{A}$, and set for $m > 0$, $\theta_{m,n} = \inf\{t \geq T\wedge\tau_n \ : X_t^{x,i,\alpha} \geq m \ \text{ or } \ X_t^{x,i,\alpha} \leq D_{I_t} + 1/m\}$. Notice that $\theta_{m,n}$ is non-decreasing in m and goes up to T a.s. when m goes to ∞. Apply then Itô's formula to $e^{-\rho t}\varphi_{k_n}(X_t^{x,i,\alpha})$ between the stopping times $T\wedge\tau_n$ and $\tau_{m,n+1} := \tau_{n+1}\wedge\theta_{m,n}$. Notice that for $T\wedge\tau_n \leq t < \tau_{n+1}\wedge\theta_{m,n}$, $X_t^{x,i}$ stays in regime k_n. Then, we have

$$\begin{aligned} e^{-\rho\tau_{m,n+1}}\varphi_{k_n}(X^{x,i}_{\tau^-_{m,n+1}}) &= e^{-\rho(T\wedge\tau_n)}\varphi_{k_n}(X^{x,i}_{T\wedge\tau_n}) + \int_{T\wedge\tau_n}^{\tau_{m,n+1}} e^{-\rho t}(-\rho\varphi_{k_n} + \mathcal{L}_{k_n}\varphi_{k_n})(X_t^{x,i})dt \\ &+ \int_{T\wedge\tau_n}^{\tau_{m,n+1}} e^{-\rho t}\sigma\varphi'_{k_n}(X_t^{x,i})dW_t - \int_{T\wedge\tau_n}^{\tau_{m,n+1}} e^{-\rho t}\varphi'_{k_n}(X_t^{x,i})dZ_t^c \\ &+ \sum_{T\wedge\tau_n < t < \tau_{m,n+1}} e^{-\rho t}\left[\varphi_{k_n}(X_t^{x,i}) - \varphi_{k_n}(X_{t^-}^{x,i})\right], \end{aligned} \tag{3.3}$$

where Z^c is the continuous part of Z.

Since $\varphi'_{k_n} \geq 1$, we have by the mean-value theorem $\varphi_{k_n}(X_t^{x,i}) - \varphi_{k_n}(X_{t^-}^{x,i}) \leq X_t^{x,i} - X_{t^-}^{x,i} = -(Z_t - Z_{t^-})$ for $T \wedge \tau_n < t < \tau_{m,n+1}$.

By using also the supersolution inequality of φ_{k_n}, taking expectation in the above Itô's formula, and noting that the integrand in the stochastic integral term is bounded by a constant (depending on m), we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\rho\tau_{m,n+1}}\varphi_{k_n}(X^{x,i}_{\tau^-_{m,n+1}})\right] &\leq \mathbb{E}\left[e^{-\rho(T\wedge\tau_n)}\varphi_{k_n}(X^{x,i}_{T\wedge\tau_n})\right] - \mathbb{E}\left[\int_{T\wedge\tau_n}^{\tau_{m,n+1}} e^{-\rho t} dZ_t^c\right] \\
&\quad - \mathbb{E}\left[\sum_{T\wedge\tau_n < t < \tau_{m,n+1}} e^{-\rho t}(Z_t - Z_{t^-})\right],
\end{aligned}$$

and so

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_n)}\varphi_{k_n}(X^{x,i}_{T\wedge\tau_n})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_n}^{\tau^-_{m,n+1}} e^{-\rho t} dZ_t + e^{-\rho\tau_{m,n+1}}\varphi_{k_n}(X^{x,i}_{\tau^-_{m,n+1}})\right]$$

Noticing that $\int_{T\wedge\tau_n}^{\tau^-_{m,n+1}} e^{-\rho t} dZ_t + e^{-\rho\tau_{m,n+1}}\varphi_{k_n}(X^{x,i}_{\tau^-_{m,n+1}}) \geq -P$, we may apply Fatou's lemma. Thus by sending m to infinity, we obtain :

$$\begin{aligned}
& \mathbb{E}\left[e^{-\rho(T\wedge\tau_n)}\varphi_{k_n}(X^{x,i}_{T\wedge\tau_n})\right] \\
\geq\ & \mathbb{E}\left[\int_{T\wedge\tau_n}^{(T\wedge\tau_{n+1})^-} e^{-\rho t} dZ_t + e^{-\rho(T\wedge\tau_{n+1})}\varphi_{k_n}(X^{x,i}_{(T\wedge\tau_{n+1})^-})\right].
\end{aligned} \tag{3.4}$$

Now, as $\varphi_{k_n}(x) \geq \varphi_{k_{n+1}}(x + D_{k_{n+1}} - D_{k_n} - g)$ and recalling $X^{x,i}_{T\wedge\tau_{n+1}} = X^{x,i}_{(T\wedge\tau_{n+1})^-} + D_{k_{n+1}} - D_{k_n} - g$ on $\{\tau_{n+1} < T\}$, we have

$$\begin{aligned}
\varphi_{k_n}(X^{x,i}_{(T\wedge\tau_{n+1})^-}) &\geq \varphi_{k_{n+1}}(X^{x,i}_{(T\wedge\tau_{n+1})^-} + D_{k_{n+1}} - D_{k_n} - g) \\
&\geq \varphi_{k_{n+1}}(X^{x,i}_{(T\wedge\tau_{n+1})}) \quad \text{on } \{\tau_{n+1} < T\}.
\end{aligned} \tag{3.5}$$

Moreover, notice that on $\{T \leq \tau_{n+1}\}$, $X_T^{x,i} \leq D_n$, hence $\varphi_{k_n}(X^{x,i}_{(T\wedge\tau_{n+1})}) = \varphi_{k_n}(X_T^{x,i}) = -P$ and $\varphi_{k_{n+1}}(X^{x,i}_{(T\wedge\tau_{n+1})}) = \varphi_{k_{n+1}}(X_T^{x,i}) = -P$, we see that inequality (3.5) also holds on $\{T \leq \tau_{n+1}\}$ and so a.s. Therefore, plugging into (3.4), we have

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_n)}\varphi_{k_n}(X^{x,i}_{T\wedge\tau_n})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_n}^{(T\wedge\tau_{n+1})^-} e^{-\rho t} dZ_t + e^{-\rho(T\wedge\tau_{n+1})}\varphi_{k_{n+1}}(X^{x,i}_{T\wedge\tau_{n+1}})\right].$$

By iterating the previous inequality for all n, we then obtain

$$\begin{aligned}
\varphi_i(x) &\geq \mathbb{E}\left[\int_0^{(T\wedge\tau_n)^-} e^{-\rho t} dZ_t + e^{-\rho(T\wedge\tau_n)}\varphi_{k_n}(X^{x,i}_{T\wedge\tau_n})\right], \\
&\geq \mathbb{E}\left[\int_0^{(T\wedge\tau_n)^-} e^{-\rho t} dZ_t - e^{-\rho(T\wedge\tau_n)}P\right], \quad \forall n \geq 0,
\end{aligned}$$

since $\varphi_{k_n} \geq -P$. By sending n to infinity, we obtain the required result from the arbitrariness of the control α. □

Corollary 3.1 *If* $\max_{i\in\mathbb{I}_N}(b-r_i)D_i \leq -\rho P$*, an optimal policy is the immediate consumption.*

Proof: It is easy to see that, in this case, the set of functions $v_i(x) = x - D_i - P$, for $i \in \{1,...,N\}$, satisfy the previous system of variational inequalities. It is a direct application of Proposition 3.1. □

Throughout the paper, we now assume that the following assumption holds

$$P > \min_{j\in\mathbb{I}_N} \frac{r_j - b}{\rho} D_j. \tag{A-1}$$

In the following Corollary, we show a linear growth condition on the value functions.

Corollary 3.2 *For all* $i \in \mathbb{I}_N$*, and for all* $x \in (D_i, \infty)$*, we have*

$$v_i(x) \leq x - D_i + \max_{j\in\mathbb{I}_N} \frac{b - r_j}{\rho} D_j$$

Proof: We set $\forall i \in \mathbb{I}_N$,

$$\varphi_i(x) = \begin{cases} x - D_i + \max_{i\in\mathbb{I}_N} \frac{b-r_i}{\rho} D_i \ , & x > D_i \\ -P, & x \leq D_i. \end{cases}$$

A straightforward computation shows that φ_i, $i \in \mathbb{I}_N$, satisfy the supersolution properties and the associated assumptions. Indeed it is clear that $\varphi_i(D_i^+) := \lim_{x\downarrow D_i} \varphi_i(x) \geq -P$ and $\varphi_i(x) = -P$ for $x < D_i$ and we equally have the following inequality

$$\min\left[-\mathcal{A}_i\varphi_i(x)\ ,\ \varphi_i'(x) - 1\ ,\ \varphi_i(x) - \max_{j\neq i}\varphi_j(x + D_j - D_i - g)\right] \geq 0,\ x \geq D_i$$

□

We shall assume that the following dynamic programming principle holds: for any $(x, i) \in [D_i, \infty) \times \mathbb{I}_N$, we have

$$\begin{aligned} (\mathbf{DP}) \ \ v_i(x) \ \ = \ \ & \sup_{\alpha=((Z),(\tau_n),(k_n))\in\mathcal{A}} \mathbb{E}\left[\int_0^{(T\wedge\theta\wedge\tau_1)^-} e^{-\rho t} dZ_t \right. \\ & \left. + \ e^{-\rho(T\wedge\theta\wedge\tau_1)} \left(v_i(X^{x,i}_{T\wedge\theta}) 1_{T\wedge\theta<\tau_1} + v_{k_1}(X^{x,i}_{\tau_1}) 1_{\tau_1\leq T\wedge\theta}\right)\right], \end{aligned} \tag{3.6}$$

where θ is any stopping time, possibly depending on $\alpha \in \mathcal{A}$ in (3.6).

The next result states the initial-boundary data for the value functions.

Proposition 3.2 *The value functions* v_i *are continuous on* (D_i, ∞) *and satisfy*

$$v_i(D_i^+) \ := \lim_{x\downarrow D_i} v_i(x) \ \ = \ \ -P. \tag{3.7}$$

Proof: a) We first prove (3.7). For $x > D_i$, let us consider the process $R^{x,i}$, defined in (3.1), and denote $\theta_i = \inf\{t \geq 0 \ : R_t^{x,i} = D_i\}$.

Let consider the geometric Brownian process $R^{x,0}$ defined as the solution to

$$dR_t = bR_t dt + \sigma R_t dW_t,\ R_0 = x,$$

and denote $\theta_{i,0} = \inf\{t \geq 0\ :R_t^{x,0} = D_i\}$. Notice that

$$\begin{aligned} R_t^{x,i} &\leq R_t^{x,0}\ ,\ \forall i \in \mathbb{I}_N,\ t \geq 0 \\ \theta_i &\leq \theta_{i,0},\ \forall i \in \mathbb{I}_N. \end{aligned} \tag{3.8}$$

Fix some $r \in (0, g)$ such that $D_i < x < D_i + r$, and denote $\theta_{r,i} = \inf\{t \geq 0\ :R_t^{x,i} = r + D_i\}$ and $\theta_{r,i,0} = \inf\{t \geq 0\ :R_t^{x,0} = r + D_i\}$. A straight calculation gives us

$$\mathbb{P}\left[\theta_{i,0} > \theta_{r,i,0}\right] = \frac{s(x) - s(D_i)}{s(D_i + r) - s(D_i)},$$

where s is a scale function of the process $R^{x,0}$. From the continuity of s at D_i, we deduce that

$$\mathbb{P}\left[\theta_{i,0} > \theta_{r,i,0}\right] \quad \rightarrow \quad 0, \quad \text{ as } \ x \downarrow D_i.$$

Notice that $\theta_{r,i,0} < \theta_{r,i}$ and combined with (3.8), we obtain $\mathbb{P}\left[\theta_i > \theta_{r,i}\right] \leq \mathbb{P}\left[\theta_{i,0} > \theta_{r,i,0}\right]$. As such,

$$\mathbb{P}\left[\theta_i > \theta_{r,i}\right] \quad \rightarrow \quad 0, \quad \text{ as } \ x \downarrow D_i. \tag{3.9}$$

Let $\alpha = (Z, (\tau_n)_{n\geq 1}, k_{n\geq 1})$ be an arbitrary policy in $\mathcal{A}$, and denote $\eta = T \wedge \theta_{r,i} = T^{x,i,\alpha} \wedge \theta_{r,i}$. For $t \leq \eta$, from the definition of an admissible control, there is no regime shift. As such, for $t \leq \eta$, we have $X_t^{x,i} \leq R_t^{x,i} \leq R_t^{x,0}$. We also have $T^{x,i} \leq \theta_i$.

We then write :

$$\begin{aligned} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] &= \mathbb{E}\left[\int_0^{\eta^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[1_{T>\eta} \int_\eta^{T^-} e^{-\rho t} dZ_t\right] \\ &\leq \mathbb{E}\left[Z_{\eta^-}\right] + \mathbb{E}\left[\mathbb{E}\left[1_{T>\eta} \int_\eta^{T^-} e^{-\rho t} dZ_t \middle| \mathcal{F}_{\theta_{r,i}^-}\right]\right] \\ &\leq \mathbb{E}\left[R_\eta^{x,0} - D_i\right] + \mathbb{E}\left[1_{T>\theta_{r,i}} \mathbb{E}\left[\int_{\theta_{r,i}}^{T^-} e^{-\rho t} dZ_t \middle| \mathcal{F}_{\theta_{r,i}^-}\right]\right] \\ &\leq \mathbb{E}\left[R_\eta^{x,0} - D_i\right] + \mathbb{E}\left[1_{T>\theta_{r,i}} e^{-\rho\theta_{r,i}} \left(v_i\left(X_{\theta_{r,i}^-}^{x,i}\right) + P\right)\right], \end{aligned} \tag{3.10}$$

where we also used in the second inequality the fact that on $\{T > \eta\}$, $\eta = \theta_{r,i}$, and $\theta_{r,i}$ is a predictable stopping time. Now, since v_i is nondecreasing, we have $v_i(X_{\theta_{r,i}^-}^{x,i}) \leq v_i(D_i + r)$. Moreover, recalling that $T \leq \theta_i$, inequalities (3.10) and (3.9) yield

$$0 \quad \leq \quad \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] \tag{3.11}$$

$$\leq \quad \mathbb{E}\left[\sup_{0\leq t\leq\theta_i} R_t^{x,0} - D_i\right] + \left(v_i(D_i + r) + P\right)\mathbb{P}[\theta_i > \theta_{r,i}] \quad \longrightarrow \quad 0, \quad \text{ as } \ x \downarrow D_i. \tag{3.12}$$

Furthermore, using the fact that $T^{x,i} \leq \theta_i \leq \theta_{i,0}$, we have

$$\mathbb{E}\left[-Pe^{-\rho T}\right] \leq -P\mathbb{E}\left[e^{-\rho\theta_{i,0}}\right]$$

Noticing that $\theta_{i,0}$ is the hitting time of a drifted Brownian, it is straightforward that $\mathbb{E}\left[e^{-\rho\theta_{i,0}}\right] \longrightarrow 1, \quad \text{as } x \downarrow D_i$ and recalling (3.11), we obtain

$$\begin{aligned} -P \ \leq \ v_i(x) \quad &\leq \quad \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t - Pe^{-\rho T}\right] \\ &\leq \quad \mathbb{E}\left[\sup_{0\leq t\leq\theta_i} R_t^{x,0} - D_i\right] + v_i(D_i+r)\mathbb{P}[\theta_i > \theta_{r,i}] - P\mathbb{E}\left[e^{-\rho\theta_{i,0}}\right] \quad \longrightarrow \quad -P, \quad \text{ as } x \downarrow D_i. \end{aligned}$$

We may therefore conclude that $v_i(D_i^+) = -P$.

b) We now turn to the continuity of the value functions v_i. Let $\gamma > 0$ and $x \in (D_i, \infty)$. We set

$$T^\gamma \ = \ \inf\{t \geq 0; \ R_t^{x,i} \geq x + \gamma\}.$$

We now consider a control strategy $\alpha = (Z, (\tau_n), k_n)$, where $\tau_1 > T^\gamma$ and $Z_t = 0, \ \forall\, t < \tau_1$. Notice that $\forall \ t < \tau_1, \ X_t^{x,i} = R_t^{x,i}$. Applying the programming dynamic principle (DP), we obtain

$$v_i(x) \quad \geq \quad \mathbb{E}\left[e^{-\rho X_{T^\gamma \wedge T^{x,i}}} v_i(X_{T^\gamma\wedge T^{x,i}})\right],$$

therefore

$$\begin{aligned} v_i(x+\gamma) - v_i(x) \quad &\leq \quad \mathbb{E}\left[(1-e^{-\rho T^\gamma}) v_i(x+\gamma) 1_{T^\gamma < T^{x,i}} + v_i(x+\gamma) 1_{T^\gamma \geq T^{x,i}}\right] \\ &\quad\ + \mathbb{E}\left[Pe^{-\rho T^{x,i}} 1_{T^\gamma \geq T^{x,i}}\right] \\ &\leq \quad v_i(x+\gamma)\left(1 - \mathbb{E}[e^{-\rho T^\gamma}]\right) + (v_i(x+\gamma) + P)\mathbb{P}(T^\gamma \geq T^{x,i}). \end{aligned}$$

Using the non-decreasing property of the value functions, for $\gamma \leq \gamma_0$, we have

$$v_i(x+\gamma) - v_i(x) \leq \quad v_i(x+\gamma_0)\left(1 - \mathbb{E}[e^{-\rho T^\gamma}]\right) + (v_i(x+\gamma_0) + P)\mathbb{P}(T^\gamma \geq T^{x,i}). \quad (3.13)$$

Using the same arguments as in the above proof of the right continuity of v_i at D_i^+, we may obtain

$$\begin{aligned} \mathbb{P}(T^\gamma \geq T^{x,i}) \quad &\longrightarrow \quad 0, \quad \text{ as } \ \gamma \downarrow 0, \ \text{ and} \\ \mathbb{E}[e^{-\rho T^\gamma}] \quad &\longrightarrow \quad 1, \quad \text{ as } \ \gamma \downarrow 0. \end{aligned}$$

Given the finiteness of v_i as show in Corollary 3.2, we obtain that the right-hand side of the (3.13) goes to zero as $\gamma \downarrow 0$. We may therefore conclude the right-continuity of v_i. An analog argument gives us the left-continuity. □

We then have the PDE characterization of the value functions v_i.

Theorem 3.1 *The value functions v_i, $i \in \mathbb{I}_N$, are continuous on (D_i, ∞), and are the unique viscosity solutions on (D_i, ∞) with linear growth condition and boundary data $v_i(D_i)$ $= -P$, to the system of variational inequalities :*

$$\min\left[-\mathcal{A}_i v_i(x)\ ,\ v_i'(x) - 1\ ,\ v_i(x) - \max_{j \neq i} v_j(x + D_j - D_i - g)\right] = 0,\ x > D_i. \tag{3.14}$$

Actually, we obtain some more regularity results on the value functions.

Proposition 3.3 *The value functions v_i, $i \in \mathbb{I}_N$, are C^1 on (D_i, ∞). Moreover, if we set for $i \in \mathbb{I}_N$:*

$$\mathcal{S}_i = \left\{x \geq D_i\ , v_i(x) = \max_{j \neq i} v_j(x + D_j - D_i - g),\right\} \tag{3.15}$$

$$\mathcal{D}_i = \overline{\mathrm{int}\left(\{x \geq D_i\ , v_i'(x) = 1\}\right)}, \tag{3.16}$$

$$\mathcal{C}_i = (D_i, \infty) \setminus (\mathcal{S}_i \cup \mathcal{D}_i), \tag{3.17}$$

then v_i is C^2 on the open set $\mathcal{C}_i \cup \mathrm{int}(\mathcal{D}_i) \cup \mathrm{int}(\mathcal{S}_i)$ of (D_i, ∞), and we have in the classical sense

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad x \in \mathcal{C}_i.$$

$\mathcal{S}_i$, $\mathcal{D}_i$, and $\mathcal{C}_i$ respectively represent the switching, dividend, and continuation regions when the outstanding debt is at regime i.

The proofs of Theorem 3.1 and Proposition 3.3 are becoming quite standard in particular for the viscosity property and regularity of the value functions. However, for the sake of completeness we will provide the proof of uniqueness property, which is always of interest, especially in our case of a combined singular and multi-switching problem, in the Appendix.

4 Qualitative results on the switching regions

For $i,\ j \in \mathbb{I}_N$ and $x \in [D_i, +\infty)$, we introduce some notations:

$$\delta_{i,j} = D_j - D_i, \quad \Delta_{i,j} = (b - r_j)D_j - (b - r_i)D_i \quad \text{and} \quad x_{i,j} = x + \delta_{i,j} - g.$$

We set $x_i^* = \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\}$ for all $i \in \mathbb{I}_N$

We equally define $\mathcal{S}_{i,j}$ as the switching region from debt level i to j.

$$\mathcal{S}_{i,j} = \{x \in (D_i,\ +\infty), v_i(x) = v_j(x_{i,j})\}.$$

From the definition (3.15) of the switching regions, we have the following elementary decomposition property :

$$\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{i,j},\ i \in \mathbb{I}_N.$$

We now begin with two obvious results. Since there is a fixed switching cost $g > 0$, it is not optimal to continuously change your debt structure. Moreover, if it is optimal to distribute dividends and to switch to another regime, it is still optimal to distribute dividend after the regime switch.

Lemma 4.1 *Let $i,\ j \in \mathbb{I}_N$ such that $i \neq j$. Assume that there exists $x \in \mathcal{S}_{i,j}$ then we have*

i) $x_{i,j} := x + D_j - D_i - g \notin \mathcal{S}_j$.

ii) $v_i'(x) = v_j'(x_{i,j})$.

Especially, if $x \in \mathcal{S}_{i,j} \cap \mathcal{D}_i$ then $x_{i,j} \in \mathcal{D}_j \setminus \mathcal{S}_j$.

Proof: For $k \in \mathbb{I}_N \setminus \{j\}$, we have

$$v_j(x_{i,j}) = v_i(x) \geq v_k(x_{i,k}).$$

As v_k is strictly non-decreasing, we get $v_j(x_{i,j}) > v_k(x_{i,k} - g)$.
Let $h \in \mathbb{R}$. For h going to 0, we have

$$\begin{aligned} v_j(x_{i,j} + h) &= v_j(x_{i,j}) + hv_j'(x_{i,j}) + \mathrm{o}(h) \\ &= v_i(x) + hv_j'(x_{i,j}) + \mathrm{o}(h) \\ &= v_i(x + h) + h(v_j'(x_{i,j}) - v_i'(x)) + \mathrm{o}(h). \end{aligned}$$

As $v_i(x+h) \geq v_j(x_{i,j} + h)$, we obtain

$$h(v_j'(x_{i,j}) - v_i'(x)) \leq \mathrm{o}(h).$$

Hence, we have $v_j'(x_{i,j}) = v_i'(x)$. □

In the following Lemma, we state that there exists a finite level of cash such that it is optimal to distribute dividends up to this level.

Lemma 4.2 *For all $i \in \mathbb{I}_N$, we have $x_i^* := \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\} < +\infty$.*

Proof: Assume that there exists $k \in \{0, ..., N-1\}$ such that $x_i^* < +\infty$ for all $i \in \mathbb{I}_k := \{i_1, ..., i_k\} \subset \mathbb{I}_N$. Notice that $\mathbb{I}_k = \emptyset$ if $k = 0$. We will show that there exists $j \in \mathbb{I}_N \setminus \mathbb{I}_k$ such that $x_j^* < +\infty$.
From Corollary 3.2 and Proposition 3.3, we deduce that, for all $i \in \mathbb{I}_N$, the function $x \to v_i(x) - x$ is continuous, non decreasing and bounded. We set $a_i := \lim\limits_{x \to +\infty} (v_i(x) - x)$. Moreover, for all $(i,\ j) \in \mathbb{I}_N$ such that $i \neq j$, we have

$$a_j - (a_i + \delta_{j,i} - g) = \lim_{x \to +\infty} (v_j(x) - v_i(x + \delta_{j,i} - g)) \geq 0.$$

Let $j_0 \in \mathbb{I}_N \setminus \mathbb{I}_k$ such that $a_{j_0} + D_{j_0} = \max_{j \in \mathbb{I}_N \setminus \mathbb{I}_k} (a_j + D_j)$. For all $j \in \mathbb{I}_N \setminus \mathbb{I}_k$, we have $a_j + \delta_{j_0,j} - g < a_{j_0}$.
It is easy to see that there exists $\bar{x} \in [D_j, +\infty)$ satisfying the following conditions:

$$\begin{aligned} v_{j_0}(\bar{x}) &> \bar{x} + \max_{j \in \mathbb{I}_N \setminus \mathbb{I}_k; j_0 \neq j} (a_j + \delta_{j_0,j} - g), \\ \rho v_{j_0}(\bar{x}) &> b\bar{x} - r_{j_0} D_{j_0}, \\ \bar{x} &> x_i^* - (\delta_{j_0,i} - g), \quad \forall i \in \mathbb{I}_k. \end{aligned}$$

At this point, we introduce a continuous function defined on $[D_i,\ +\infty)$:

$$\hat{V}(x) = \left\{ \begin{array}{cl} v_{j_0}(x) & \text{if} \quad x < \bar{x} \\ x - \bar{x} + v_{j_0}(\bar{x}) & \text{if} \quad x \geq \bar{x} \end{array} \right.$$

Let $x \geq \bar{x}$. We have

$$-\mathcal{A}_{j_0}\hat{V}(x) = (\rho - b)(x - \bar{x}) + [\rho v_{j_0}(\bar{x}) - (b\bar{x} - r_{j_0}D_{j_0})] > 0.$$

Moreover, for $j \in \mathbb{I}_N \setminus \mathbb{I}_k$ such that $j \neq j_0$, we have

$$\hat{V}(x) \geq x + a_j + \delta_{j_0,j} - g \geq v_j(x + \delta_{j_0,j} - g).$$

For $i \in \mathbb{I}_k$, we have

$$\begin{array}{rcl} v_i(x + \delta_{j_0,i} - g) - \hat{V}(x) & = & x + \delta_{j_0,i} - g - x_i^* + v_i(x_i^*) - (x - \bar{x} + v_{j_0}(\bar{x})) \\ & = & v_i(x_i^*) - x_i^* + (\bar{x} + \delta_{j_0,i} - g) - v_{j_0}(\bar{x}) \\ & \leq & v_i(\bar{x} + \delta_{j_0,i} - g) - v_{j_0}(\bar{x}) \\ & \leq & 0. \end{array}$$

Finally, for all $j \in \mathbb{I}_N \setminus \{j_0\}$, $\hat{V}(x) \leq v_{j_0}(x) \leq v_j(x - \delta_{j,j_0} + g)$.
As $\hat{V}'(x) = 1$, $\hat{V}$ is a continuous solution of equation (3.14). From Theorem 3.1, we deduce that $v_{j_0} = \hat{V}$ and $x_{j_0}^* \leq \bar{x}$. □

Now, we shall study properties of x_i^* and more generally, properties of left-boundaries of $\mathcal{D}_i$ in the sense as detailed in the following definition.

Definition 4.1 *Let $i \in \mathbb{I}_N$ and $x \in (D_i, +\infty)$. x is a left-boundary of $\mathcal{D}_i$ if there exists $\varepsilon > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$ with values in $(D_i, x) \setminus \mathcal{D}_i$ such that*

$$[x, x + \varepsilon] \in \mathcal{D}_i \quad \textit{and} \quad \lim_{n\to+\infty} y_n = x.$$

Remark 4.1 *Notice that, if $x_i^* > D_i$ then x_i^* is a left-boundary of $\mathcal{D}_i$.*

In order to compute the dividend regions, we establish the following lemma.

Lemma 4.3 *Let $i,\ j \in \mathbb{I}_N$ such that $j \neq i$. We assume that there exists $\hat{x}_i$ a left-boundary of $\mathcal{D}_i$.*

i) Assume that $\hat{x}_i \notin \mathcal{S}_i$, then we have $(b - r_i)D_i > -\rho P$ and $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_iD_i$. As $x \to \rho v_i(x) - bx + r_iD_i$ is increasing, it implies that

$$\rho v_i(x) < bx - r_iD_i \textit{ on } (D_i, \hat{x}_i) \quad \textit{and} \quad \rho v_i(x) > bx - r_iD_i \textit{ on } (\hat{x}_i, +\infty).$$

ii) Assume that $\hat{x}_i \in \mathcal{S}_{i,j}$ then we have

ii.a) $[\hat{x}_i,\ \hat{x}_i + \varepsilon] \subset \mathcal{S}_{i,j}$ and $\hat{x}_i + \delta_{i,j} - g$ is a left-boundary of $\mathcal{D}_j$.

ii.b) $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_iD_i + \Delta_{i,j} - bg$ and $\Delta_{i,j} > 0$.

ii.c) $\forall k \in \mathbb{I}_N - \{i, j\}, \hat{x}_i \notin \mathcal{S}_{i,k}$.

Notice that the last equality implies that $-\rho P + bg < (b - r_j)D_j$.

Remark 4.2 *We have* $\rho v_i(\hat{x}_i) \geq b\hat{x}_i - r_i D_i$, $\forall i \in \mathbb{I}_N$.

Proof:
i). We assume that $\hat{x}_i \notin \mathcal{S}_i$. As $\mathcal{S}_i$ is closed and $\hat{x}_i > D_i$, we can choose $\varepsilon > 0$ such that $(\hat{x}_i - \varepsilon,\ \hat{x}_i + \varepsilon) \cap \mathcal{S}_i = \emptyset$. Moreover, $v_i' \geq 1$ and $v_i'(\hat{x}_i) = 1$ so there exists a sequence $(y_n)_{n\in\mathbb{N}} \in (\hat{x}_i - \varepsilon,\ \hat{x}_i) \cap \mathcal{C}_i$ such that $\lim_{n\to+\infty} y_n = \hat{x}_i$ and $v_i''(y_n) \leq 0$. We have

$$0 \geq v_i''(y_n) = \frac{2}{\sigma^2 y_n^2}\left(\rho v_i(y_n) - (by_n - r_i D_i)v_i'(y_n)\right)$$

and letting n going to infinity, we get

$$0 \geq \rho v_i(\hat{x}_i) - (b\hat{x}_i - r_i D_i) = \lim_{y\to\hat{x}_i; y>\hat{x}_i} -\mathcal{A}_i v_i(y) \geq 0,$$

leading to the desired equality.

Now let us show that $(b - r_i)D_i > -\rho P$.
Assume that $(b - r_i)D_i \leq -\rho P$, we then obtain the following inequality:

$$v_i(\hat{x}_i) \leq \frac{b}{\rho}(\hat{x}_i - D_i) - P,$$

leading to a contradiction as $\rho > b$ and $v_i(\hat{x}_i) \geq \hat{x}_i - D_i - P$.

ii). We assume that $\hat{x}_i \in \mathcal{S}_{i,j}$.
ii.a). *We first prove that* $[\hat{x}_i,\ \hat{x}_i + \varepsilon] \subset \mathcal{S}_{i,j}$ *and that* $\hat{x}_j := \hat{x}_i + \delta_{i,j} - g$ *is a left-boundary of* $\mathcal{D}_j$.
Let $y \in [\hat{x}_i,\ \hat{x}_i + \varepsilon]$. We have

$$v_j(y + \delta_{i,j} - g) \leq v_i(y) = y - \hat{x}_i + v_i(\hat{x}_i) = y - \hat{x}_i + v_j(\hat{x}_j).$$

On the other hand, $v_j' \geq 1$ so $y - \hat{x}_i + v_j(\hat{x}_j) \leq v_j(y + \delta_{i,j} - g)$. It follows that $v_j(y + \delta_{i,j} - g) = v_i(y)$ and $[\hat{x}_i,\ \hat{x}_i + \varepsilon] \subset \mathcal{S}_{i,j}$.
Moreover, we have proved that $[\hat{x}_j,\ \hat{x}_j + \varepsilon] \subset \mathcal{D}_j$.
We assume that there exists $\eta > 0$ such that $(\hat{x}_j - \eta, \hat{x}_j) \subset \mathcal{D}_j$ and show that it leads to a contradiction. Let $x \in (\hat{x}_j - \eta, \hat{x}_j)$. We have

$$\begin{aligned} v_j(x) &= x - \hat{x}_j + v_j(\hat{x}_j) \\ &= (x - \delta_{i,j} + g) - \hat{x}_i + v_i(\hat{x}_i) \\ &> v_i(x - \delta_{i,j} + g). \end{aligned}$$

The last inequality follows from the fact that $\hat{x}_i$ is a left-boundary of $\mathcal{D}_i$ and contradicts the fact that v_i is solution of equation (3.14). Hence, to show that $\hat{x}_j$ is a left-boundary of $\mathcal{D}_j$ it remains to prove that $\hat{x}_j > D_j$. However, if it was not the case, we would have,

$$\begin{aligned} v_i(\hat{x}_i) &= v_j(\hat{x}_i + \delta_{i,j} - g), \text{ since } \hat{x}_i \in \mathcal{S}_{i,j}, \\ &= v_j(D_j) = -P. \end{aligned}$$

But $\hat{x}_i = \hat{x}_j - \delta_{i,j} + g = D_i + g$, leading to the contradiction $-P = v_i(D_i) < v_i(D_i + g) = -P$.

ii.b). *We now prove that* $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_i D_i + \Delta_{i,j} - bg$ *and* $\Delta_{i,j} := (b - r_j)D_j - (b - r_i)D_i > 0$.

From Lemma 4.1, we know that $\hat{x}_j \notin \mathcal{S}_j$. Therefore it follows from step i) that $(b - r_j)D_j > -\rho P$ and $\rho v_j(\hat{x}_j) = b\hat{x}_j - r_j D_j$. We obtain

$$\begin{aligned} \rho v_i(\hat{x}_i) &= \rho v_j(\hat{x}_i + \delta_{i,j} - g) \\ &= \rho v_j(\hat{x}_j) \\ &= b\hat{x}_j - r_j D_j \\ &= b\hat{x}_i + b(\delta_{i,j} - g) - r_j D_j \\ &= b\hat{x}_i - r_i D_i + \Delta_{i,j} - bg. \end{aligned} \tag{4.1}$$

As $\rho v_i(\hat{x}_i) - (b\hat{x}_i - r_i D_i) = \lim\limits_{y \to \hat{x}_i; y > \hat{x}_i} \rho v_i(y) - \mathcal{L}_i v_i(y) \geq 0$, we have $\rho v_i(\hat{x}_i) \geq b\hat{x}_i - r_i D_i$ and then $\Delta_{i,j} \geq bg > 0$.

ii.c). *It remains to show that* $\forall k \in \mathbb{I}_N - \{i, j\}, \hat{x}_i \notin \mathcal{S}_{i,k}$.
This fact is an elementary result as highlighted earlier because if there exists $k \in \mathbb{I}_N - \{i, j\}$ such that $\hat{x}_i \in \mathcal{S}_{i,k} \cap \mathcal{S}_{i,j}$, it would implies that $\Delta_{i,k} = \Delta_{i,j}$.
Relation (4.1) gives us the last equality. □

Corollary 4.1 *Let* $i \in \mathbb{I}_N$. *We have the following results:*

i) If $x_i^* \notin \mathcal{S}_i$, *then either* $x_i^* = D_i$ *or*
$(b - r_i)D_i > -\rho P$, *and* $\rho v_i(x_i^*) = bx_i^* - r_i D_i$.

ii) If there exists $j \in \mathbb{I}_N - \{i\}$, *such that* $x_i^* \in \mathcal{S}_{i,j}$, *then*
we have $\Delta_{i,j} > 0$ *and* $(b - r_j)D_j > -\rho P + bg$ *and* $\rho v_i(x_i^*) = bx_i^* - r_i D_i + \Delta_{i,j} - bg$.

Proof: These results are straightforward from Lemma 4.3 –(i) and –(iib) and Remark 4.1. □

We now turn to the following result which basically states that when it is optimal to distribute dividend and/or to switch regime, then it is still optimal when the firm is richer.

Lemma 4.4 *Let* $(i,\ j) \in \mathbb{I}_N^2$ *such that* $i \neq j$. *If* $(x_i^*,\ +\infty) \cap \mathcal{S}_{i,j} \neq \emptyset$ *then there exists* $y_{i,j}^* \in [x_i^*,\ +\infty)$ *such that*

$$[x_i^*,\ +\infty) \cap \mathcal{S}_{i,j} = [y_{i,j}^*,\ +\infty) \quad \textit{and} \quad \rho v_i(y_{i,j}^*) = by_{i,j}^* - r_i D_i + \Delta_{i,j} - bg.$$

Proof: We set $y^*_{i,j} = \inf\Big([x^*_i,\ +\infty) \cap \mathring{\mathcal{S}}_{i,j}\Big)$. If $x^*_i \in \mathcal{S}_{i,j}$, the result has been proved in Corollary 4.1. We now assume that $x^*_i < y^*_{i,j}$. Let $y > y^*_{i,j}$. Using the same argument as in ii) of Corollary 4.1, we may get $v_j(y + \delta_{i,j} - g) = v_i(y)$ and $[y^*_{i,j},\ +\infty) \subset \mathcal{S}_{i,j}$.
Moreover, we know that $y^*_{i,j} + \delta_{i,j} - g \notin \mathcal{S}_j$ and $\mathcal{S}_j$ is a closed set, so there exists $\varepsilon > 0$ such that $[\bar{y}^*_{i,j} - \varepsilon,\ \bar{y}^*_{i,j}] \cap \mathcal{S}_j = \emptyset$ where we set $\bar{y}^*_{i,j} = y^*_{i,j} + \delta_{i,j} - g$. As $y^*_{i,j} \in \mathring{\mathcal{D}}_i$, we can find a sequence $(y_k)_{k\in\mathbb{N}}$ with values in $[\bar{y}^*_{i,j} - \varepsilon,\ \bar{y}^*_{i,j}]$, such that $y_k \notin \mathcal{D}_j$ (if not we may obtain a contradiction by straightforwardly showing that $y^*_{i,j} > \inf\Big([x^*_i,\ +\infty) \cap \mathring{\mathcal{S}}_{i,j}\Big)$.), i.e.

$$\forall k \in \mathbb{N},\ y_k \in \mathcal{C}_j \quad \text{and} \quad \lim_{k\to+\infty} y_k = \bar{y}^*_{i,j}.$$

We finally obtain

$$\begin{aligned} 0 &= \rho v_j(y_k) - (b y_k - r_j D_j) v'_j(y_k) - \frac{\sigma^2 y_k^2}{2} v''_j(y_k) \\ &= \rho v_j(\bar{y}^*_{i,j}) - (b\bar{y}^*_{i,j} - r_j D_j) v'_j(\bar{y}^*_{i,j}). \end{aligned}$$

Using $v_j(\bar{y}^*_{i,j}) = v_i(y^*_{i,j})$ and $v'_j(\bar{y}^*_{i,j}) = v'_i(y^*_{i,j}) = 1$ (from Lemma 4.1 and $y^*_{i,j} \in \mathcal{D}_i$), we may obtain the desired results and conclude the proof. □

We now establish an important result in determining the description of the switching regions. The following Theorem states that it is never optimal to expand its operation, i.e. to make investment, through debt financing, should it result in a lower "drift" ($(b - r_i)D_i$) regime. However, when the firm's value is low, i.e. with a relatively high bankruptcy risk, it may be optimal to make some divestment, i.e. sell parts of the company, and use the proceedings to lower its debt outstanding, even if it results in a regime with lower "drift". In other words, to lower the firm's bankruptcy risk, one should try to decrease its volatility, i.e. the diffusion coefficient. In our model, this clearly means making some debt repayment in order to lower the firm's volatility, i.e. σX_t.

Theorem 4.1 *Let $i, j \in \mathbb{I}_N$ such that $(b - r_j)D_j > (b - r_i)D_i$. We have the following results:*

1) $x^*_j \notin \mathcal{S}_{j,i}$ and $\mathring{\mathcal{D}}_j = (x^*_j,\ +\infty)$.

2) $\mathring{\mathcal{S}}_{j,i} \subset (D_j + g,\ x^*_j)$. Furthermore, if $D_j < D_i$, then $\mathring{\mathcal{S}}_{j,i} = \emptyset$.

Proof:
1). Since $(b - r_j)D_j > (b - r_i)D_i$, we have $\Delta_{j,i} < 0$. It follows from part *ii)* of Corollary 4.1 that $x^*_j \notin \mathcal{S}_{j,i}$. Let $y \in \mathring{\mathcal{D}}_j$. There exists $\varepsilon > 0$ such that $(y - \varepsilon,\ y + \varepsilon) \subset \mathring{\mathcal{D}}_j$. For $x \in (y - \varepsilon,\ y + \varepsilon)$, we have

$$0 \le -\mathcal{A}_j v_j(x) = \rho v_j(x) - (bx - r_j D_j).$$

Hence, $\rho v_j(x) \ge bx - r_j D_j$ and using Remark 4.1 and Lemma 4.3 –(i), we may obtain $y \ge x^*_j$.

2). Let us assume that there exists $y \in \mathring{\mathcal{S}}_{j,i}$. we first need to prove that $y < x^*_j$.

Let's assume that $y \geq x_j^*$. From Lemma 4.4, we know that $[y, \ +\infty) \subset \mathcal{S}_{j,i}$. We set $s_{j,i}^* = \inf \mathring{\mathcal{S}}_{j,i} \cap \mathcal{D}_j$. As $x_j^* \notin \mathcal{S}_j$, we have $x_j^* \leq s_{j,i}^*$. On the other hand, it is easy to see that $s_{j,i}^* + \delta_{j,i} - g = x_i^*$ and $x_i^* \notin \mathcal{S}_i$. We obtain

$$
\begin{aligned}
\rho v_j(s_{j,i}^*) &= \rho v_i(x_i^*) \\
&= b x_i^* - r_i D_i \\
&= b s_{j,i}^* - r_j D_j - (\Delta_{i,j} + bg) \\
&< b s_{j,i}^* - r_j D_j \\
&< \rho v_j(x_j^*).
\end{aligned}
$$

We may deduce that $x_j^* > s_{j,i}^*,$, which contradicts the fact that $x_j^* \leq s_{j,i}^*$, so $y < x_j^*$.

We now prove that if $D_j < D_i$, $\mathring{\mathcal{S}}_{j,i} = \emptyset$.

Assume that there exists $x \in \mathring{\mathcal{S}}_{j,i}$. From the first step, we know that $x \notin \mathcal{D}_j$. We deduce from Lemma 4.1 that $\bar{x} := x + \delta_{j,i} - g \in \mathcal{C}_i$ then we have

$$
\begin{aligned}
\frac{1}{2}\sigma^2 x^2 v_j''(x) + (bx - r_j D_j)\, v_j'(x) &\leq \rho v_j(x) \\
&= \rho v_i(\bar{x}) \\
&= \frac{1}{2}\sigma^2 \bar{x}^2 v_i''(\bar{x}) + (b\bar{x} - r_i D_i)\, v_i'(\bar{x}) \\
&= \frac{1}{2}\sigma^2 \bar{x}^2 v_j''(x) + (b\bar{x} - r_i D_i)\, v_j'(x).
\end{aligned}
$$

Combining these equations, we get

$$
0 \leq \frac{\sigma^2}{2}(\bar{x}^2 - x^2) v_j''(x) - (\Delta_{i,j} + bg) v_j'(x).
$$

As $\bar{x}^2 - x^2 \geq 0$ (using $D_j < D_i$ and Assumption (2.5)), $(\Delta_{i,j} + bg)v_j'(x) > 0$ and $v_j'' < 0$ on $[\frac{r_j D_j}{b}, x_j^*)$, we necessarily have $x \in (D_j + g, \ \frac{r_j D_j}{b} \wedge x_j^*)$. Therefore, if $b \geq r_j$, $\mathcal{S}_{j,i} = \emptyset$.
Now, we assume that $b < r_j$ and $\mathring{\mathcal{S}}_{j,i} \neq \emptyset$. Let $S_{j,i} = \sup \mathring{\mathcal{S}}_{j,i}$ and if we set $\bar{S}_{j,i} := S_{j,i} + \delta_{j,i} - g$, it follows that

$$
0 \leq \frac{\sigma^2}{2}(\bar{S}_{j,i}^2 - S_{j,i}^2) v_j''(S_{j,i}^-) - (\Delta_{i,j} + bg) v_j'(S_{j,i}).
$$

Hence, we have

$$
\begin{aligned}
0 &\leq \rho v_j(S_{j,i}) - \left(b\bar{S}_{j,i} - r_i D_i + \frac{\bar{S}_{j,i}^2}{\bar{S}_{j,i}^2 - S_{j,i}^2}(\Delta_{i,j} + bg) \right) v_j'(S_{j,i}) \\
&\leq \rho v_j(S_{j,i}) - \left(b S_{j,i} - r_j D_j + \frac{{S_{j,i}}^2}{\bar{S}_{j,i}^2 - S_{j,i}^2}(\Delta_{i,j} + bg) \right) v_j'(S_{j,i}). \qquad (4.2)
\end{aligned}
$$

On the other hand, we have $(S_{j,i}, x_j^*) \subset \mathcal{C}_j$ so

$$
0 \leq \frac{\sigma^2 S_{j,i}^2}{2} v_j''(S_{j,i}^+) - \frac{{S_{j,i}}^2}{\bar{S}_{j,i}^2 - S_{j,i}^2}(\Delta_{i,j} + bg) v_j'(S_{j,i}).
$$

Especially, we have $v_j''(S_{j,i}^+) > 0$. Moreover, v_j is a $\mathcal{C}^2$ function and $v_j' > 1$ on $(S_{j,i}, x_j^*)$, it follows that there exists $y \in (S_{j,i}, x_j^*)$ such that $v_j''(y) = 0$ since $v'(x_j^*) = 1$. We set $y_j = \inf\{y \in (S_{j,i}, x_j^*) : \; v_j''(y) \leq 0\}$. As $v_j'' \leq 0$ on $[\frac{r_j D_j}{b}, +\infty)$, we know that $y_j \leq \frac{r_j D_j}{b}$. We have $v_j''(y_j) = 0$ and $y_j \in \mathcal{C}_j$, so we can assert that $h(y_j) = 0$ where we have set

$$h(x) = (bx - r_j D_j)v_j'(x) - \rho v_j(x).$$

On $(S_{j,i}, y_j)$, we have $v_j'' > 0$ so h is decreasing. Indeed, we have

$$h'(x) = (bx - r_j D_j)v_j''(x) - (\rho - b)v_j'(x) \leq 0.$$

Finally, this proves that $\rho v_j(S_{j,i}) \leq (bS_{j,i} - r_j D_j)v_j'(S_{j,i})$. Reporting this in the inequality (4.2), we get

$$0 \leq -\frac{{S_{j,i}}^2}{\bar{S}_{j,i}^2 - S_{j,i}^2}(\Delta_{i,j} + bg)v_j'(S_{j,i}).$$

This is impossible as $\bar{S}_{j,i}^2 > S_{j,i}^2$, $\Delta_{i,j} + bg > 0$ and $v_j'(S_{j,i}) \geq 1$. In conclusion, $\mathring{\mathcal{S}}_{j,i} = \emptyset$. $\square$
We now turn to an important corollary.

Corollary 4.2 *Let $m \in \mathbb{I}_N$ such that $(b - r_m)D_m = \max_{i \in \mathbb{I}_N}(b - r_i)D_i$.*

1) $x_m^* \notin \mathcal{S}_m$ and $\mathring{\mathcal{D}}_m = (x_m^*, \; +\infty)$.

2) For all $i \in \mathbb{I}_N - \{m\}$, we have:

- *i) If $D_m < D_i$, $\mathring{\mathcal{S}}_{m,i} = \emptyset$.*
- *ii) If $D_i < D_m$, $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g, \; x_m^*)$. Furthermore, if $b \geq r_i$, then $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g, \; (a_i^* + \delta_{i,m} + g) \wedge x_m^*)$, where a_i^* is the unique solution of the equation $\rho v_i(x) = (bx - r_i D_i)v_i'(x)$. We further have $a_i^* \neq x_i^*$.*

Proof:
The only point left to show is **2.ii)**. We now assume that there exists $i \in \mathbb{I}_N - \{m\}$ such that $D_i < D_m$, $b \geq r_i$ and $\mathring{\mathcal{S}}_{m,i} \neq \emptyset$

We prove that the equation $\rho v_i(x) = (bx - r_i D_i)v_i'(x)$ admits a unique solution a_i^* and prove that $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g, \; a_i^* + \delta_{m,i} - g)$.
Let $x \in \mathring{\mathcal{S}}_{m,i}$. It follows from the first step that $\mathring{\mathcal{S}}_{m,i} \cap \mathcal{D}_m = \emptyset$. Hence, from Lemma 4.1, we have $\underline{x} := x + \delta_{m,i} - g \in \mathcal{C}_i$. We obtain

$$\begin{aligned} 0 &\geq \frac{\sigma^2 x^2}{2} v_i''(\underline{x}) + (bx - r_m D_m)v_i'(\underline{x}) - \rho v_i(\underline{x}) \\ &= \mathcal{A}_i v_i(\underline{x}) + \frac{\sigma^2}{2}(x^2 - \underline{x}^2)v_i''(\underline{x}) + (\Delta_{i,m} + bg)v_i'(\underline{x}) \\ &= -\frac{x^2 - \underline{x}^2}{\underline{x}^2} H_i(\underline{x}), \end{aligned}$$

where we have set

$$H_i(x) = \left(bx - r_i D_i - \frac{x^2}{(x + \delta_{i,m} + g)^2 - x^2}(\Delta_{i,m} + bg) \right) v_i'(x) - \rho v_i(x).$$

Hence, we have

$$\mathring{\mathcal{S}}_{m,i} \subset \{x \in (D_m + g,\ +\infty):\ H_i(\underline{x}) \geq 0\} \subset \{x \in (D_m + g,\ +\infty):\ G_i(x + \delta_{m,i} - g) \leq 0\},$$

where we set $G_i(y) = \rho v_i(y) - (by - r_i D_i) v_i'(y)$.
We notice that, for all $y \in (D_i, +\infty)$, $G_i(y) \geq \frac{\sigma^2 y^2}{2} v_i''(y)$. Recalling our assumption that $b \geq r_i$, it follows that

$$G_i'(y) \geq (\rho - b) v_i'(y) - \frac{2(by - r_i D_i)}{\sigma^2 y^2} G_i(y) > -\frac{2(by - r_i D_i)}{\sigma^2 y^2} G_i(y).$$

As G_i is continuous on $(D_i, +\infty)$ and $G_i(D_i) < 0$, it implies that the equation $G_i(y) = 0$ admits a unique solution which will be denoted by a_i^*. Therefore, we have $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g, a_i^* + \delta_{i,m} + g)$. Furthermore, from Corollary 4.1, we either have $G(x_i^*) = 0$ or $G(x_i^*) > 0$. As such, we deduced that $a_i^* \in (D_i, x_i^*)$.

□

We now turn to the following results ordering the left-boundaries $(x_i^*)_{i \in \mathbb{I}_N}$ of the dividend regions $(\mathcal{D}_i)_{i \in \mathbb{I}_N}$.

Proposition 4.1 *Consider $i, j \in \mathbb{I}_N$, such that $(b - r_i)D_i < (b - r_j)D_j$. We always have $x_i^* + \delta_{i,j} - g \leq x_j^*$ unless there exists a regime k such that $(b - r_j)D_j < (b - r_k)D_k$ and $x_i^* \in \mathcal{S}_{i,k}$, then we have $x_j^* - \delta_{i,j} + g < x_i^* < x_k^* - \delta_{i,k} + g$.*

Proof: First, we assume that $x_i^* \notin \mathcal{S}_i$. From Lemma 4.3, we know that $\rho v_i(x_i^*) = b x_i^* - r_i D_i$. On the other hand, we have

$$\begin{aligned}
\rho v_i(x_j^* - (\delta_{i,j} - g)) &\geq \rho v_j(x_j^*) \\
&\geq b x_j^* - r_j D_j \\
&\geq b(x_j^* - (\delta_{i,j} - g)) - r_i D_i + \Delta_{i,j} - bg \\
&> b(x_j^* - (\delta_{i,j} - g)) - r_i D_i.
\end{aligned}$$

Hence, we have $x_i^* + \delta_{i,j} - g < x_j^*$.
Now, we assume that there exists $k \in \mathbb{I}_N - \{i\}$ such that $x_i^* \in \mathcal{S}_{i,k}$. If $k = j$, we have $x_i^* + \delta_{i,j} - g = x_j^*$. If $k \neq j$, we have $x_i^* + \delta_{i,k} - g = x_k^*$ and

$$\rho v_i(x_i^*) = b x_i^* - r_i D_i + \Delta_{i,k} - bg = b(x_i^* + \delta_{i,k} - g) - r_k D_k.$$

On the other hand, we have

$$\begin{aligned}
\rho v_i(x_j^* - (\delta_{i,j} - g)) &\geq \rho v_j(x_j^*) \\
&= b x_j^* - r_j D_j \\
&= b(x_j^* - (\delta_{i,j} - g)) - r_i D_i + \Delta_{i,j} - bg \\
&= b\left(x_j^* - (\delta_{i,j} - g) + (\delta_{i,k} - g)\right) - r_k D_k + \Delta_{k,j}.
\end{aligned}$$

If $(b - r_j)D_j > (b - r_k)D_k$, i.e. $\Delta_{k,j} > 0$, then we have

$$\rho v_i(x_j^* - (\delta_{i,j} - g)) > b\left(x_j^* - (\delta_{i,j} - g) + (\delta_{i,k} - g)\right) - r_k D_k.$$

Hence, we have $x_i^* + \delta_{i,j} - g < x_j^*$.
However, in the case that $(b - r_j)D_j < (b - r_k)D_k$, then

$$\rho v_i(x_j^* - (\delta_{i,j} - g)) \quad < \quad b\left(x_j^* - (\delta_{i,j} - g) + (\delta_{i,k} - g)\right) - r_k D_k.$$

Hence, we have $x_i^* + \delta_{i,j} - g > x_j^*$. □

5 The two regime-case

Before investigating the two-regime case, for the sake of completeness, we give the results in the case where there is no regime change, i.e. the firm's debt level remains constant.

Proposition 5.1 *The value function $\hat{V}$ is C^2 on (D_1, ∞). There exists $\hat{x} \geq D$ such that $\hat{\mathcal{C}} = (D, \hat{x})$, and $\hat{\mathcal{D}} = (\hat{x}, \infty)$. Furthermore, If $(b - rD) > -\rho P$, then, on $\hat{\mathcal{C}} = (D, \hat{x})$, $\hat{V}$ is the unique solution (in the classical sense) to*

$$\rho v - \mathcal{L} v = 0$$

and

$$\begin{aligned} \hat{V}(x) &= x - \hat{x} + \hat{V}(\hat{x}), \quad x \geq \hat{x} \\ \text{where } \hat{V}(\hat{x}) &= \frac{b\hat{x} - rD}{\rho}. \end{aligned}$$

If $(b - rD) \leq -\rho P$, then $\hat{x} = D$ and the optimal value function $V(x) = x - D - P$.

This result directly derives from Corollary 4.1. □

Throughout this section, we now assume that $N = 2$, in which case, we will get a complete description of the different regions. We will see that the most important parameter to consider is the so-called "drifts" $((b - r_i)D_i)_{i=1,2}$ and in particular their relative positions. To avoid cases with trivial solution, i.e. immediate consumption, we will assume that $-\rho P < (b - r_i)D_i, i = 1, 2$.
We now distinguish the two following cases: $(b - r_2)D_2 < (b - r_1)D_1$ and $(b - r_1)D_1 < (b - r_2)D_2$. Throughout Theorem 5.1 and Theorem 5.2, we provide a complete resolution to our problem in each case.

Theorem 5.1 *We assume that $(b - r_2)D_2 < (b - r_1)D_1$.*
We have

$$\mathcal{C}_1 = [D_1, x_1^*), \ \mathcal{D}_1 = [x_1^*, \ +\infty), \ \text{and } \mathring{\mathcal{S}}_1 = \emptyset \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

1) If $\mathcal{S}_2 = \emptyset$ then we have

$$\mathcal{C}_2 = [D_2, x_2^*), \ \text{and } \mathcal{D}_2 = [x_2^*, \ +\infty) \text{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2.$$

2) If $\mathcal{S}_2 \neq \emptyset$ then there exists y_2^* such that $\mathcal{S}_2 = [y_2^*, \ +\infty)$ and we distinguish two cases *a) If* $x_2^* + \delta_{2,1} - g < x_1^*$, *then* $y_2^* > x_2^*$, $y_2^* = x_1^* + \delta_{1,2} + g$ *and*

$$\mathcal{C}_2 = [D_2, x_2^*), \text{ and } \mathcal{D}_2 = [x_2^*, +\infty) \text{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$ *then* $y_2^* \le x_2^*$, $\rho v_2(x_2^*) = bx_2^* - r_2 D_2 + \Delta_{2,1} - bg$. *We define* a_2^* *as the solution of* $\rho v_2(a_2^*) = ba_2^* - r_2 D_2$ *and have two cases*

i) If $a_2^* \notin \mathcal{D}_2$, *we have*

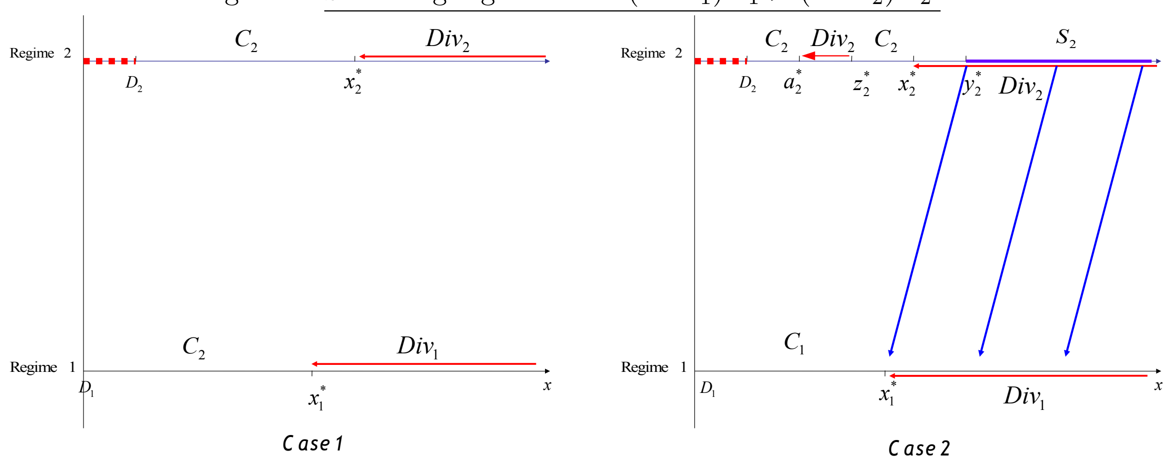
$$\mathcal{D}_2 = [x_2^*, +\infty) \quad \text{and} \quad \mathcal{C}_2 = [D_2, y_2^*).$$

ii) If $a_2^* \in \mathcal{D}_2$, *there exists* $z_2^* \in (a_2^*, y_2^*)$ *such that*

$$\mathcal{D}_2 = [a_2^*, z_2^*] \cup [x_2^*, +\infty) \quad \text{and} \quad \mathcal{C}_2 = [D_2, a_2^*) \cup (z_2^*, y_2^*).$$

Remark 5.1 Theorem 5.1 clearly states that it is never optimal to make growth investment through debt financing when it results in lower "drift" $(b - r_i)D_i$. However, when the firm value process exceeds the threshold, y_i^*, it may be optimal to switch to a lower debt regime should it result in a higher "drift" $(b - r_i)D_i$.

Figure 1: Switching regions: case $(b - r_1)D_1 > (b - r_2)D_2$.



Proof: From Theorem 4.2, we have

$$\mathcal{D}_1 = [x_1^*, +\infty) \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1 \text{ and } \mathcal{S}_1 = \emptyset.$$

1) We assume that $\mathring{\mathcal{S}}_2 = \emptyset$. From Corollary 4.1, we know that $\rho v_2(x_2^*) = bx_2^* - r_2 D_2$. If there exists $x \in \mathring{\mathcal{D}}_2 \cap (D_2, x_2^*)$, we would have $0 \le \rho v_2(x) - (bx - r_2 D_2)$ but this is impossible for $x < x_2^*$. Hence we have

$$\mathcal{C}_2 = [D_2, x_2^*), \text{ and } \mathcal{D}_2 = [x_2^*, +\infty).$$

2) Now, we assume that $\mathring{\mathcal{S}}_2 \neq \emptyset$ and set $y_2^* = \inf \mathring{\mathcal{S}}_2$. We first prove that $\mathcal{S}_2 = [y_2^*, +\infty)$. We define the following function

$$V_2(x) = \begin{cases} v_2(x) & \text{if} \quad D_2 \le x < y_2^* \\ v_1(x + \delta_{2,1} - g) & \text{if} \quad y_2^* \le x. \end{cases}$$

V_2 is a $\mathcal{C}^1$ function on $[D_2, +\infty)$. We prove that V_2 is the solution of the variational inequality satisfied by v_2. We obviously have $V_2'(x) \geq 1$ and $V_2(x) \geq v_1(x + \delta_{2,1} - g)$. Moreover, we have

$$V_2(x + \delta_{1,2} - g) = \begin{cases} v_2(x + \delta_{1,2} - g) \leq v_1(x) & \text{if} \quad D_2 \leq x < y_2^* + \delta_{2,1} + g \\ v_1(x - 2g) \leq v_1(x) & \text{if} \quad y_2^* + \delta_{2,1} + g \leq x. \end{cases}$$

It remains to prove that $\mathcal{A}_2 V_2(x) \leq 0$ on $[y_2^*,\ +\infty)$. For $x \geq y_2^*$, we set $\underline{x} = x + \delta_{2,1} - g$ and we have

$$\begin{aligned} \mathcal{A}_2 V_2(x) &= \frac{\sigma^2 x^2}{2} v_1''(\underline{x}) + (bx - r_2 D_2) v_1'(\underline{x}) - \rho v_1(\underline{x}) \\ &= \mathcal{A}_1 v_1(\underline{x}) + \frac{\sigma^2}{2}(x^2 - \underline{x}^2) v_1''(\underline{x}) - (\Delta_{2,1} + bg) v_1'(\underline{x}) \\ &\leq \frac{\sigma^2}{2}(x^2 - \underline{x}^2) v_1''(\underline{x}). \end{aligned}$$

As $D_1 < D_2$, we have $x^2 > \underline{x}^2$. On the other hand, we have seen that v_1 is concave so we can assert that $\mathcal{A}_2 V_2(x) \leq 0$ on $[y_2^*,\ +\infty)$. This proves that $v_2 = V_2$ and especially that $\mathring{\mathcal{S}}_2 = (y_2^*,\ +\infty)$.

a) If $x_2^* + \delta_{2,1} - g < x_1^*$, then using Proposition 4.1, we have $y_2^* > x_2^*$ and $x_2^* \notin \mathcal{S}_2$, so it follows from Corollary 4.1 that $\rho v_2(x_2^*) = b x_2^* - r_2 D_2$. Moreover, we have $\mathring{\mathcal{D}}_2 = (x_2^*,\ +\infty)$ and from Lemma 4.4, we have $y_2^* = x_1^* + \delta_{1,2} + g$.

b) If $x_2^* + \delta_{2,1} - g = x_1^*$, then using Proposition 4.1, we have $y_2^* \leq x_2^*$. In this case, $x_2^* \in \mathcal{S}_2$ and it follows from Corollary 4.1 that $\rho v_2(x_2^*) = b x_2^* - r_2 D_2 + \Delta_{2,1} - bg$. We define a_2^* as the solution of $\rho v_2(a_2^*) = b a_2^* - r_2 D_2$ and distinguish two cases:

   i) If $a_2^* \notin \mathcal{D}_2$, it follows from Lemma 4.3 that

   $$\mathcal{D}_2 = [x_2^*,\ +\infty) \quad \text{and} \quad \mathcal{C}_2 = [D_2,\ x_2^*).$$

   ii) Finally, we assume that $a_2^* \in \mathcal{D}_2$. We set $z_2^* = \inf\{x \geq a_2^* :\ v_2' > 1\}$ and have

   $$\mathcal{D}_2 = [a_2^*,\ z_2^*] \cup [x_2^*,\ +\infty) \quad \text{and} \quad \mathcal{C}_2 = [D_2,\ a_2^*) \cup (z_2^*,\ x_2^*).$$

□

We now turn to the case where $(b - r_1)D_1 < (b - r_2)D_2$.

Theorem 5.2 *We assume that $(b - r_1)D_1 < (b - r_2)D_2$,*

1) we have

$$\begin{aligned} \mathcal{D}_2 &= [x_2^*,\ +\infty) \text{ where } \rho v_2(x_2^*) = b x_2^* - r_2 D_2 \\ \mathring{\mathcal{S}}_2 &= \emptyset \text{ or there exist } s_2^*, S_2^* \in (D_2 + g,\ x_2^*) \text{ such that } \mathring{\mathcal{S}}_2 = (s_2^*, S_2^*). \end{aligned}$$

2) If $\mathring{\mathcal{S}}_1 = \emptyset$ then we have

$$\mathcal{C}_1 = [D_1, x_1^*), \text{ and } \mathcal{D}_1 = [x_1^*, +\infty) \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

3) If $\mathring{\mathcal{S}}_1 \neq \emptyset$ there exists y_1^* such that $\mathring{\mathcal{S}}_1 = (y_1^*, +\infty)$

a) If $x_1^* + \delta_{1,2} - g < x_2^*$, then $y_1^* > x_1^*$, $y_1^* = x_2^* + \delta_{2,1} + g$ and

$$\mathcal{C}_1 = [D_1, x_1^*), \text{ and } \mathcal{D}_1 = [x_1^*, +\infty) \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$, then $y_1^* \leq x_1^*$, $\rho v_1(x_1^*) = bx_1^* - r_1 D_1 + \Delta_{1,2} - bg$. We define a_1^* as the solution of $\rho v_1(a_1^*) = ba_1^* - r_1 D_1$ and have two cases.

i) If $a_1^* \notin \mathcal{D}_1$, we have

$$\mathcal{D}_1 = [x_1^*, +\infty) \quad \text{and} \quad \mathcal{C}_1 = [D_1, y_1^*).$$

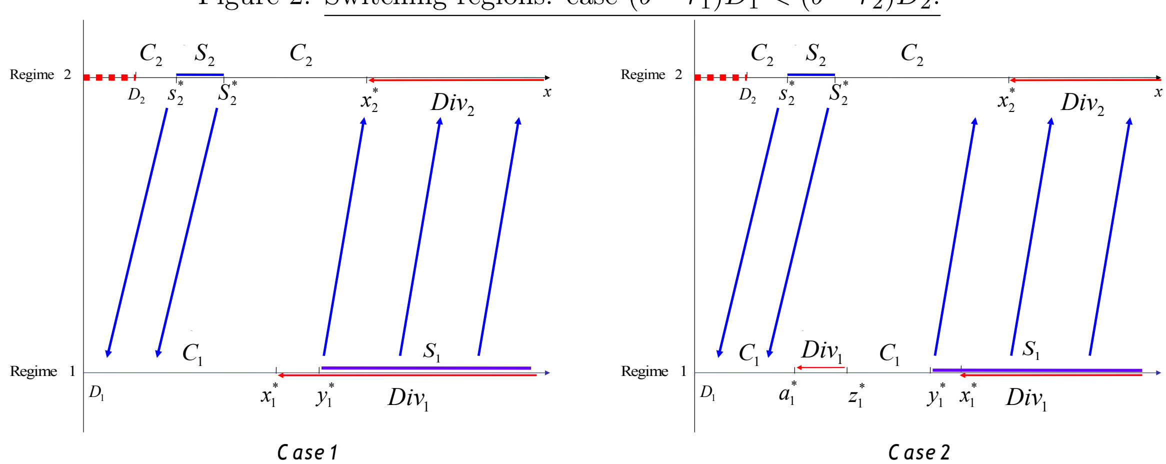
ii) If $a_1^* \in \mathcal{D}_1$, there exists $z_1^* \in (a_1^*, y_1^*)$ such that

$$\mathcal{D}_1 = [a_1^*, z_1^*] \cup [x_1^*, +\infty) \quad \text{and} \quad \mathcal{C}_1 = [D_1, a_1^*) \cup (z_1^*, y_1^*).$$

Remark 5.2 Theorem 5.2 states that when the firm's value is sufficiently high (above y_1^* threshold), it's optimal to switch to a higher-debt regime which operates at a higher drift $(b - r_i)D_i$, see figure 2, case 2. However, when the firm is too small, it may be optimal not to postpone dividend payment and to operate under as a medium size company (cash-reserve lower than the threshold a_1^*, as in figure 2, case 3), i.e. to distribute dividend, whenever the cash-reserve exceed the threshold a_1^*.

However, one should not switch to a lower drift regime unless it lowers the firm's bankruptcy risk. It may happen, when the value firm dangerously approaches bankruptcy threshold, i.e. when its cash reserve stands between s_2^* and S_2^*.

Figure 2: Switching regions: case $(b - r_1)D_1 < (b - r_2)D_2$.



Proof: Throughout the proof, for $x \in \mathbb{R}$, we set $\bar{x} = x + \delta_{1,2} - g$ and $\underline{x} = x + \delta_{2,1} - g$. Notice that we have $\underline{x} < x < \bar{x}$.

1.) From Theorem 4.2 we have

$$\begin{aligned} \mathcal{D}_2 &= [x_2^*,\ +\infty) \text{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2 \\ \mathring{\mathcal{S}}_2 &\subset (D_2 + g,\ (a_1^* + \delta_{2,1} - g) \wedge x_2^*), \end{aligned}$$

where a_1^* is the unique solution of the equation $\rho v_1(x) = (bx - r_1 D_1)v_1'(x)$.
Assume that $\mathring{\mathcal{S}}_2 \neq \emptyset$. We set $s_2^* = \inf \mathring{\mathcal{S}}_2$ and $S_2^* = \sup \mathring{\mathcal{S}}_2$. Now we prove that $\mathring{\mathcal{S}}_2 = (s_2^*,\ S_2^*)$.
On $[D_2,\ +\infty)$, we define the following function:

$$V_2(x) = \begin{cases} v_2(x) & \text{if} \quad x < s_2^* \\ v_1(x + \delta_{2,1} - g) & \text{if} \quad s_2^* \leq x \leq S_2^* \\ v_2(x) & \text{if} \quad x > S_2^*. \end{cases}$$

V_2 is a continuous function on $[D_2,\ +\infty)$ and it is easy to see that $V_2' \geq 1$. For all $x \in [D_2,\ +\infty)$, we have $V_2(x) \geq v_1(x+\delta_{2,1}-g)$ and for $x+\delta_{1,2}-g \in [s_2^*,\ S_2^*]$, $V_2(x+\delta_{1,2}-g) = v_1(x-2g) < v_1(x)$. We now prove that $\mathcal{A}_2 V_2 \leq 0$. Let $x \in [s_2^*,\ S_2^*]$, we have

$$\begin{aligned} \mathcal{A}_2 V_2(x) &= \mathcal{A}_1 v_1(\underline{x}) + \frac{x^2 - \underline{x}^2}{\underline{x}^2}\left(\frac{\sigma^2 \underline{x}^2}{2} v_1''(\underline{x}) + \frac{\underline{x}^2}{x^2 - \underline{x}^2}(\Delta_{1,2} + bg)v_1'(\underline{x})\right) \\ &= -\frac{x^2 - \underline{x}^2}{\underline{x}^2} H_1(\underline{x}). \end{aligned}$$

We recall that

$$H_1(x) = \left(bx - r_1 D_1 - \frac{x^2}{(x + \delta_{1,2} + g)^2 - x^2}(\Delta_{1,2} + bg)\right) v_1'(x) - \rho v_1(x).$$

We have seen in the proof of Theorem 4.2 that $\mathring{\mathcal{S}}_2 \subset \{x \in (D_m, +\infty) :\ H_1(\underline{x}) \geq 0\}$. Especially, we have $H_1(\underline{S_2^*}) \geq 0$, with $\underline{S_2^*} \leq a_1^*$. Now, we prove that H_1 is decreasing on $(D_1,\ a_1^*)$. As H_1 is continuous, this will lead to $\mathcal{A}_2 V_2 \leq 0$ and allows us to assert that $v_2 = V_2$ and especially that $\mathring{\mathcal{S}}_2 = (s_2^*,\ S_2^*)$.
We may rewrite H_1:

$$H_1(x) = U_1(x) - G_1(x),$$

where

$$\begin{aligned} G_1(x) &= \rho v_1(x) - (bx - r_1 D_1)V_1'(x) \\ U_1(x) &= -\frac{x^2}{(x + \delta_{1,2} + g)^2 - x^2}(\Delta_{1,2} + bg)v_1'(x). \end{aligned}$$

From the proof of Theorem 4.2, we have G_1 is strictly non-decreasing on (D_1, a_1^*). Furthermore, a straight study of the function U_1 and recalling that on $(D_1, a_1^*]$, $v_1''(x) \leq 0$, we may obtain that U_1 is non-increasing. As such, H_1 is strictly non-increasing.

2.) If $\mathring{\mathcal{S}}_1 = \emptyset$, then $x_1^* \notin \mathcal{S}_1$. Using the arguments from 1.) of Theorem 4.2, we may obtain

$$\mathcal{C}_1 = [D_1, x_1^*), \text{ and } \mathcal{D}_1 = [x_1^*,\ +\infty) \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

3.) We now assume that $\mathring{\mathcal{S}}_1 \neq \emptyset$. We set $y_1^* = \inf \mathring{\mathcal{S}}_1$ and prove that $\mathring{\mathcal{S}}_1 = (y_1^*,\ +\infty)$.

On $[D_1,\ +\infty)$, we define the following function:

$$V_1(x) = \begin{cases} v_1(x) & \text{if} \quad x < y_1^* \\ v_2(x + \delta_{1,2} - g) & \text{if} \quad y_1^* \leq x. \end{cases}$$

V_1 is a $\mathcal{C}^1$ function on $[D_1,\ +\infty)$ and it is easy to see that $V_1' \geq 1$. For all $x \in [D_1,\ +\infty)$, we have $V_1(x) \geq v_2(x + \delta_{1,2} - g)$ and for $x \geq y_1^*$, $V_1(x + \delta_{2,1} - g) = v_2(x - 2g) < v_2(x)$. We now prove that $\mathcal{A}_1 V_1 \leq 0$. Let $x \in [y_1^*,\ +\infty)$, we have

$$\begin{aligned} \mathcal{A}_1 V_1(x) &= \mathcal{A}_2 v_2(\bar{x}) - \frac{(\bar{x}^2 - x^2)\sigma^2}{2} v_2''(\bar{x}) - \Delta_{1,2}^- v_2'(\bar{x}) \\ &\leq -\frac{(\bar{x}^2 - x^2)\sigma^2}{2} v_2''(\bar{x}) - \Delta_{1,2}^- v_2'(\bar{x}). \end{aligned}$$

If $\bar{x} \in \mathcal{D}_2$, we obviously have $\mathcal{A}_1 V_1(x) \leq 0$. Assume that $\bar{x} \in \mathcal{C}_2$, then we have

$$\mathcal{A}_1 V_1(x) = \frac{\bar{x}^2 - x^2}{\bar{x}^2} H_2(\bar{x}).$$

As H_2 is decreasing, we have $H_2(\bar{x}) \leq H_2(y_1^*) \leq 0$ so $\mathcal{A}_1 V_1(x) \leq 0$. Finally, we assume that $\bar{x} \in \mathring{\mathcal{S}}_2$. In this case, we have

$$\begin{aligned} \frac{(x-g)^2}{\bar{x}^2 - x^2} \mathcal{A}_1 V_1(x) &\leq -\frac{\sigma^2 (x-g)^2}{2} v_1''(x-g) - \frac{(x-g)^2}{\bar{x}^2 - x^2} \Delta_{1,2}^- v_1'(x-g) \\ &= -\rho v_1(x-g) + \left(b(x-g) - r_1 D_1 - \frac{(x-g)^2}{\bar{x}^2 - x^2} \Delta_{1,2}^- \right) v_1'(x-g) \\ &= -\rho v_2(\bar{x}) + \left(b\bar{x} - r_2 D_2 - \Delta_{1,2} - \frac{(x-g)^2}{\bar{x}^2 - x^2} \Delta_{1,2}^- \right) v_2'(\bar{x}) \\ &= H_2(\bar{x}) + \left(\frac{\bar{x}^2 - (x-g)^2}{\bar{x}^2 - x^2} \Delta_{1,2}^- - \Delta_{1,2} \right) v_2'(\bar{x}) \\ &\leq \left(\frac{\bar{x}^2 - (x-g)^2}{\bar{x}^2 - x^2} \Delta_{1,2}^- - \Delta_{1,2} \right) v_2'(\bar{x}). \end{aligned}$$

However, we have

$$\begin{aligned} (\bar{x}^2 - (x-g)^2)\Delta_{1,2}^- - (\bar{x}^2 - x^2)\Delta_{1,2} &= -bg(\bar{x}^2 - x^2) + g(2x - g))(\Delta_{1,2} - bg) \\ &= g\left(2(r_1 D_1 - r_2 D_2)x - g\Delta_{1,2}^- - b(\delta_{1,2} - g)^2 \right) \\ &\leq 0. \end{aligned}$$

Therefore, $\mathcal{A}_1 V_1 \leq 0$ on $(D_1,\ +\infty)$. This allows us to assert that $v_1 = V_1$ and especially that $\mathring{\mathcal{S}}_1 = (y_1^*,\ +\infty)$.

a) If $x_1^* + \delta_{1,2} - g < x_2^*$, then using Proposition 4.1, we have $y_1^* > x_1^*$ and $x_1^* \notin \mathcal{S}_1$. So it follows from Corollary 4.1 that $\rho v_1(x_1^*) = bx_1^* - r_1 D_1$. Moreover, we have $\mathring{\mathcal{D}}_1 = (x_1^*,\ +\infty)$ and from Lemma 4.4, we have $y_1^* + \delta_{1,2} - g = x_2^*$.

b) If $x_1^*+\delta_{1,2}-g=x_2^*$, then using Proposition 4.1, we have $y_1^* \leq x_1^*$. In this case, $x_1^* \in \mathcal{S}_1$ and it follows from Corollary 4.1 that $\rho v_1(x_1^*) = bx_1^* - r_1 D_1 + \Delta_{1,2} - bg$. We define a_1^* as the solution of $\rho v_1(a_1^*) = ba_1^* - r_1 D_1$ and distinguish two cases:

   i) If $a_1^* \notin \mathcal{D}_1$, it follows from Lemma 4.3 that

$$\mathcal{D}_1 = [x_1^*,\ +\infty) \quad \text{and} \quad \mathcal{C}_1 = [D_1,\ x_1^*).$$

   ii) Finally, we assume that $a_1^* \in \mathcal{D}_1$. We set $z_1^* = \inf\{x \geq a_1^* :\ v_1' > 1\}$ and have

$$\mathcal{D}_1 = [a_1^*,\ z_1^*] \cup [x_1^*,\ +\infty) \quad \text{and} \quad \mathcal{C}_1 = [D_1,\ a_1^*) \cup (z_1^*,\ x_1^*).$$

□

Remark 5.3 The arguments used to obtain the above results in Theorems 5.1 and 5.2 in the two-regime problem may also apply to higher regime problems although the required analysis involved would be much lengthier and depends on many more parameters. It is particularly the case when we reconsider our initial multi-switching problem with a slight but realistic change to our initial model: we only allow the firm to change, i.e. increase or repay, its debt level to the one immediately above or below. This latter case may be subject to further studies in the future, but we may already obtain:

- *The elementary decomposition of the switching regions becomes* $\mathcal{S}_i = \mathcal{S}_{i,i-1} \cup \mathcal{S}_{i,i+1}$. *The system of variational inequalities becomes :*

$$\min\left[-\mathcal{A}_i v_i(x)\ ,\ v_i'(x)-1\ ,\ v_i(x) - \max_{j=i-1,i+1} v_j(x+(j-i)D-g)\right] \geq 0,\ x > D_i \quad (5.1)$$

- *With the exception of part i) of Lemma 4.1, all the other results still hold. For results obtained by using part i) of Lemma 4.1, it suffices to slightly modify the existing proofs.*

- *The complete solution to our modified problem may be obtained by applying iteratively the results from Theorem 5.1 and 5.2.*

Some numerical illustrations:

Below are some numerical analysis on value functions as defined in equation (2.7) versus the real value for shareholders as defined in equation (2.8) for different values of P, see Figure 3.

Finally, Figure 4 shows the contribution of the management team in creating value for shareholders for different values of P.

Figure 3: Optimal values for managers (v_i) Vs shareholders' value (u_i) for increasing penalty P.

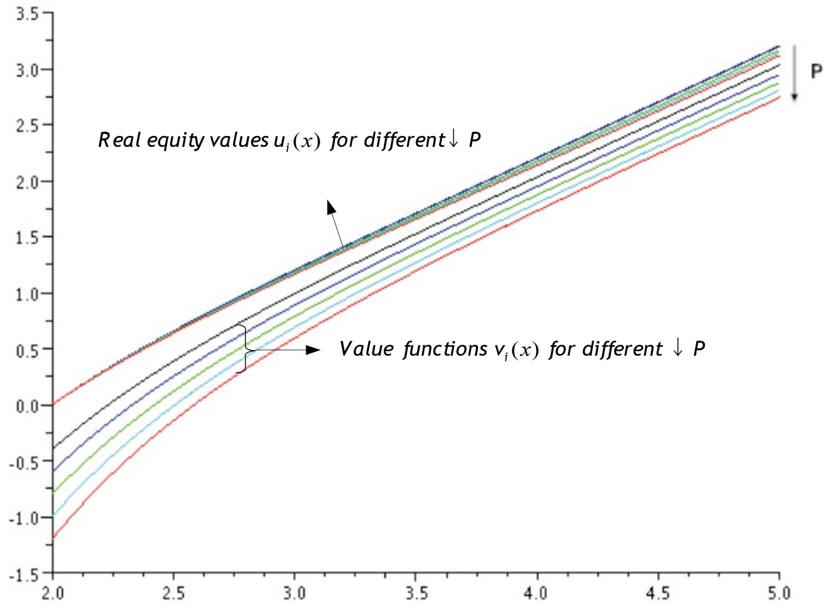
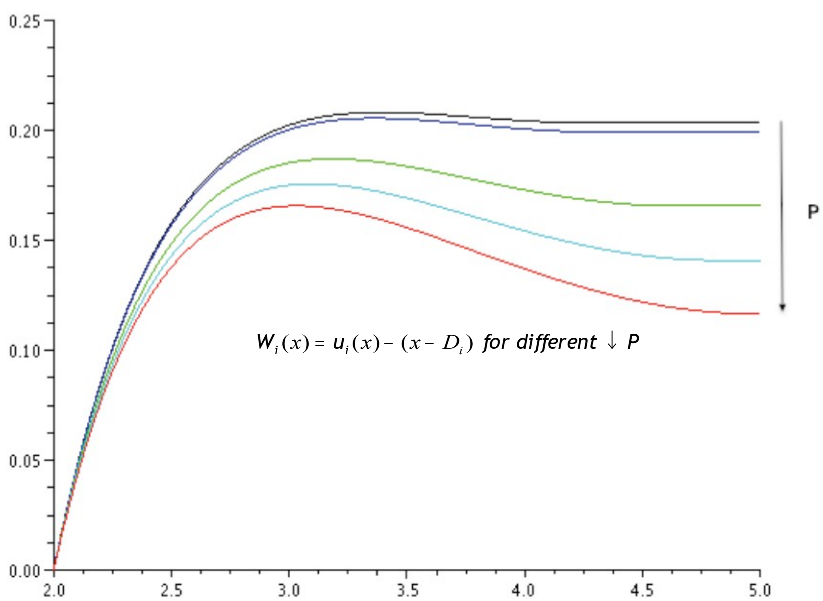


Figure 4: Excess shareholders' values Vs immediate consumption.



Appendix A : Proof of Theorem 3.1

Proof of the uniqueness property.
Suppose u_i, $i \in I_N$, are continuous viscosity subsolutions to the system of variational inequalities on (D_i, ∞), and w_i, $i \in I_N$, continuous viscosity supersolutions to the system of variational inequalities on (D_i, ∞), satisfying the boundary conditions $u_i(D_i^+) \leq w_i(D_i^+)$, $i \in I_N$, and the linear growth condition :

$$|u_i(x)| + |w_i(x)| \leq C_1 + C_2 x, \quad \forall x \in (0, \infty), \; i \in I_N, \tag{A.1}$$

for some positive constants C_1 and C_2. We want to prove that

$$u_i \leq w_i, \quad \text{ on } \quad (D_i, \infty), \; i \in \mathbb{I}_N.$$

<u>*Step 1*</u>. We first construct strict supersolutions to the system with suitable perturbations of w_i, $i \in \mathbb{I}_N$ and any $j \neq i$. We set

$$h_i(x) = A_i + Bx, \quad x \geq D_i,$$

where

$$\begin{aligned} A_i &= D_N - D_i + 1 + \sup_{i \in \mathbb{I}_N} \left| w_i(D_i^+) \right| \\ B &= 2C_2 + 2. \end{aligned}$$

We then define for all $\gamma \in (0,1)$, the continuous functions on (D_i, ∞) by :

$$w_i^\gamma = (1-\gamma) w_i + \gamma h_i, \quad i \in \mathbb{I}_N.$$

We then see that for all $\gamma \in (0,1), i \in \mathbb{I}_N$:

$$\begin{aligned} w_i^\gamma(x) - w_j^\gamma(x + D_j - D_i - g) &= (1-\gamma)\left[w_i(x) - w_j(x + D_j - D_i - g)\right] \\ &+ \gamma\left[h_i(x) - h_j(x + D_j - D_i - g)\right], \end{aligned} \tag{A.2}$$

A straightforward calculation gives us

$$w_i^\gamma(x) - w_j^\gamma(x + D_j - D_i - g) = \gamma g, \quad i, j \in \mathbb{I}_N, \; i \neq j. \tag{A.3}$$

As such we obtain

$$w_i^\gamma(x) - \max_{j \neq i} w_j^\gamma(x + D_j - D_i - g) \geq \gamma g, \quad i \in \mathbb{I}_N. \tag{A.4}$$

Furthermore, we also easily obtain

$$h_i'(x) - 1 = B - 1 > 1. \tag{A.5}$$

A straight calculation will also provide us with the last required inequality, i.e.

$$\rho h_i(x) - \mathcal{L}_i h_i(x) \geq \min_{i \in \mathbb{I}_N} (\rho + r_i D_i) > 0, \tag{A.6}$$

where $\min\limits_{i\in\mathbb{I}_N}(\rho + r_i D_i) > \rho$.

Combining (A.4), (A.5), and (A.6), this shows that w_i^γ is a strict supersolution of the system : for $i \in \mathbb{I}_N$, we have on (D_i, ∞)

$$\min\left[\rho w_i^\gamma(x) - \mathcal{L}_i w_i^\gamma(x),\ w_i^{\gamma\prime}(x) - 1,\ w_i^\gamma(x) - \max_{j\neq i} w_j^\gamma(x + D_j - D_i - g)\right] \geq \delta > 0, \tag{A.7}$$

where $\delta = \gamma \min\{1, g, \rho\}$.

<u>*Step 2*</u>. In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0,1)$:

$$\max_{i\in\mathbb{I}_N} \sup_{(D_i,+\infty)} (u_i - w_i^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0. We argue by contradiction and suppose that there exist some $\gamma \in (0,1)$ and $i \in \mathbb{I}_N$, s.t.

$$\theta := \max_{j\in\mathbb{I}_N} \sup_{(D_j,+\infty)} (u_j - w_j^\gamma) = \sup_{(D_i,+\infty)} (u_i - w_i^\gamma) > 0. \tag{A.8}$$

Notice that $u_i(x) - w_i^\gamma(x)$ goes to $-\infty$ when x goes to infinity. We also have $\lim\limits_{x\to D_i^+} u_i(x) - \lim\limits_{x\to D_i^+} w_i^\gamma(x) \leq \gamma(\lim\limits_{x\to D_i^+} w_i(x) - h_i(D_i)) \leq 0$. Hence, by continuity of the functions u_i and w_i^γ, there exists $x_0 \in (D_i, \infty)$ s.t.

$$\theta = u_i(x_0) - w_i^\gamma(x_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned}
\Phi_\varepsilon(x,y) &= u_i(x) - w_i^\gamma(y) - \phi_\varepsilon(x,y), \\
\phi_\varepsilon(x,y) &= \frac{1}{4}|x - x_0|^4 + \frac{1}{2\varepsilon}|x - y|^2,
\end{aligned}$$

for all $x, y \in (D_i, \infty)$. By standard arguments in comparison principle, the function Φ_ε attains a maximum in $(x_\varepsilon, y_\varepsilon) \in (D_i, \infty)^2$, which converges (up to a subsequence) to (x_0, x_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{A.9}$$

Applying Theorem 3.2 in [6], we get the existence of M_ε, $N_\varepsilon \in \mathbb{R}$ such that:

$$\begin{aligned}
(p_\varepsilon, M_\varepsilon) &\in J^{2,+} u_i(x_\varepsilon), \\
(q_\varepsilon, N_\varepsilon) &\in J^{2,-} w_i^\gamma(y_\varepsilon),
\end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & -N_\varepsilon \end{pmatrix} \leq D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \varepsilon(D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon))^2, \tag{A.10}$$

where

$$
\begin{aligned}
p_\varepsilon &= D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3, \\
q_\varepsilon &= -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \\
D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) &= \begin{pmatrix} 3(x_\varepsilon - x_0)^2 + \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix}.
\end{aligned}
$$

By writing the viscosity subsolution property of u_i and the strict viscosity supersolution property (A.7) of w_i^γ, we have the following inequalities:

$$
\begin{aligned}
&\min\Big\{\rho u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right)(bx_\varepsilon - r_iD_i) - \frac{1}{2}\sigma^2 x_\varepsilon^2 M_\varepsilon, \\
&\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) - 1,\ u_i(x_\varepsilon) - \max_{j\neq i} u_j(x_\varepsilon + D_j - D_i - g)\Big\} \quad \leq \quad 0, \quad \text{(A.11)} \\
&\min\Big\{\rho w_i^\gamma(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)(by_\varepsilon - r_iD_i) - \frac{1}{2}\sigma^2 y_\varepsilon^2 N_\varepsilon,\ \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 1, \\
&\qquad w_i^\gamma(y_\varepsilon) - \max_{j\neq i} w_j^\gamma(y_\varepsilon + D_j - D_i - g)\Big\} \quad \geq \quad \delta. \quad \text{(A.12)}
\end{aligned}
$$

We then distinguish the following three cases :

$\star$ *Case 1 :* $u_i(x_\varepsilon) - \max_{j\neq i} u_j(x_\varepsilon + D_j - D_i - g) \leq 0$ in (A.11).

From the continuity of u_i and by sending $\varepsilon \to 0$, this implies

$$
u_i(x_0) \leq \max_{j\neq i} u_j(x_0 + D_j - D_i - g). \tag{A.13}
$$

On the other hand, from (A.12), we also have

$$
w_i^\gamma(y_\varepsilon) - \max_{j\neq i} w_j^\gamma(y_\varepsilon + D_j - D_i - g) \geq \delta,
$$

which implies, by sending $\varepsilon \to 0$ and using the continuity of w_i :

$$
w_i^\gamma(x_0) \geq \max_{j\neq i} w_j^\gamma(x_0 + D_j - D_i - g) + \delta. \tag{A.14}
$$

Combining (A.13) and (A.14), we obtain

$$
\begin{aligned}
\theta = u_i(x_0) - w_i^\gamma(x_0) &\leq \max_{j\neq i} u_j(x_0 + D_j - D_i - g) - \max_{j\neq i} w_j^\gamma(x_0 + D_j - D_i - g) - \delta, \\
&\leq \max_{j\neq i}\Big\{u_j(x_0 + D_j - D_i - g) - w_j^\gamma(x_0 + D_j - D_i - g)\Big\} - \delta, \\
&\leq \theta - \delta,
\end{aligned}
$$

which is a contradiction.

$\star$ *Case 2 :* $\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) - 1 \leq 0$ in (A.11)

Notice that by (A.12), we have

$$
\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 1 \geq \delta,
$$

which implies in this case

$$(x_\varepsilon - x_0)^3 \le -\delta.$$

By sending ε to zero, we obtain again a contradiction.

$\star$ *Case 3 :* $\rho u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right)(bx_\varepsilon - r_i D_i) - \frac{1}{2}\sigma^2 x_\varepsilon^2 M_\varepsilon \le 0$ in (A.11)
From (A.12), we have

$$\rho w_i^\gamma(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)(by_\varepsilon - r_i D_i) - \frac{1}{2}\sigma^2 y_\varepsilon^2 N_\varepsilon \ge \delta,$$

which implies in this case

$$\begin{aligned}\rho\left(u_i(x_\varepsilon) - w_i^\gamma(y_\varepsilon)\right) \quad \le \quad & \frac{b}{\varepsilon}(x_\varepsilon - y_\varepsilon)^2 + (bx_\varepsilon - r_i D_i)(x_\varepsilon - x_0)^3 \\ & + \frac{1}{2}\sigma^2(x_\varepsilon^2 M_\varepsilon - y_\varepsilon^2 N_\varepsilon) - \delta, \qquad \text{(A.15)}\end{aligned}$$

Using (A.10), we obtain an upper bound of $\frac{1}{2}\sigma^2(x_\varepsilon^2 M_\varepsilon - y_\varepsilon^2 N_\varepsilon)$ which may be plugged into (A.15). This yields an upper bound of $\rho\left(u_i(x_\varepsilon) - w_i^\gamma(y_\varepsilon)\right)$ which goes to $-\delta$ when we send ε to zero.

Using the continuity of u_i and w_i^γ, we obtain the required contradiction : $\rho\theta \le -\delta < 0$. This ends the proof. $\square$

References

[1] Bayraktar E. and M. Egami (2010) : “On the One-Dimensional Optimal Switching Problem", Mathematics of Operations Research, 2010, 35 (1), 140-159.

[2] Brekke K. and B. Oksendal (1994) : “Optimal switching in an economic activity under uncertainty", *SIAM J. Cont. Optim.*, **32**, 1021-1036.

[3] Brennan M. and E. Schwartz (1985) : “Evaluating natural resource extraction", J. Business, 58, 135-137.

[4] Carmona R. and M. Ludkovski (2010) : “Valuation of Energy Storage: An Optimal Switching Approach”, Quantitative Finance, 10(4), 359-374, 2010.

[5] Choulli T. , Taksar M. and X.Y. Zhou (2003) : “A diffusion model for optimal dividend distribution for a company with constraints on risk control", *SIAM J. Cont. Optim.*, **41**, 1946-1979.

[6] Crandall M., Ishii H. and P.L. Lions (1992) : “User’s guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.*, **27**, 1-67.

[7] Décamps J.P. and S. Villeneuve (2007) : “Optimal dividend policy and growth option", *Finance and Stochastics*, **11**, 3-27.

[8] Dai,M., Q. Zhang, and Q. Zhu (2010): “Trend Following Trading under a Regime Switching Model", SIAM Journal on Financial Mathematics 1, 780-810.

[9] Djehiche B., S. Hamadène and A. Popier (2009) : “A finite horizon optimal switching problem", SIAM J. Control and Optimization, Volume 48, Issue 4, pp. 2751-2770.

[10] Dixit A. and R. Pindick (1994) : Investment under uncertainty, Princeton University Press.

[11] Duckworth K. and M. Zervos (2001) : “A model for investment decisions with switching costs", *Annals of Applied Probability*, **11**, 239-250.

[12] Guo X. and H. Pham (2005) : “Optimal partially reversible investment with entry decision and general production function", *Stochastic Processes and their Applications*, **115**, 705-736.

[13] Guo, X. and Tomecek, P. (2008) : “Connections between singular control and optimal switching", *SIAM J. Control Optim.*, **47(1)**: 421-443.

[14] Hamadène S. and M. Jeanblanc (2007) : “On the Starting and Stopping Problem: Application in reversible investments”, *Mathematics of Operation Research*, vol. 32 no. 1 182-192.

[15] Hu Y. and S. Tang (2010) : “Multi-dimensional BSDE with oblique reflection and optimal switching", Prob. Theory and Related Fields, 147(1-2) 89-121. .

[16] Jeanblanc M. and A. Shiryaev (1995) : “Optimization of the flow of dividends", *Russian Math. Survey*, **50**, 257-277.

[17] I. Karatzas, D. Ocone, H. Wang and M. Zervos (2000) : “Finite-Fuel Singular Control with Discretionary Stopping", *Stochastics and Stochastics Reports*, **71**, pp.1-50

[18] Lon P. C. and M. Zervos (2011) :‘A model for optimally advertising and launching a product”, Mathematics of Operations Research, to appear.

[19] Ly Vath V. and H. Pham (2007) : “Explicit solution to an optimal switching problem in the two-regime case", *SIAM J. Cont. Optim.*, **46**, 395-426.

[20] Ly Vath V., H. Pham and S. Villeneuve (2008) : “A mixed singular/switching control problem for a dividend policy with reversible technology investment", *Annals of Applied Probability*, 2008, 18, pp. 1164-1200.

[21] Pham H. (2007) : “On the smooth-fit property for one-dimensional optimal switching problem", Séminaire de Probabilités, vol XL, 187-201.

[22] Pham H. (2009) : “Continuous-time Stochastic Control and Optimization with Financial Applications", Stochastic Modelling and Applied Probability, Vol. 61, 232p.

[23] Pham H., V. Ly Vath, and X.Y. Zhou (2009) : “Optimal switching over multiple regimes", SIAM Journal on Control and Optimization, 48, 2217-2253.

[24] Zervos M., T. Johnson, and F. Alazemi (2011) :"Buy-low and sell-high investment strategies", to appear in *Mathematical Finance*.