Numerical approximation for a portfolio optimization problem under liquidity risk and costs

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Abstract

This paper concerns with numerical resolution of an impulse control problem under state constraints arising from optimal portfolio selection under liquidity risk and price impact. We show that the value function could be obtained as the limit of an iterative procedure where each step is an optimal stopping problem and the reward function is related to the impulse operator. Given the dimension of our problem and the complexity of its solvency region, we use a numerical approximation algorithm based on quantization procedure instead of finite difference methods to calculate the value function, the transaction and no-transaction regions. We also focus on the convergence of our numerical scheme, in particular, we show that it satisfies monotonicity, stability and consistency properties. We further enrich our studies with some numerical results for the optimal transaction strategy.

Keywords: Impulse control problem, Optimal transaction strategy, Quantization method, Viscosity solution.

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1 Introduction

This paper concerns with a control problem of portfolio optimization under liquidity risk and price impact. We consider the liquidity framework studied in Ly Vath, Mnif and Pham (2007). Under the impact of liquidity risk, prices are pushed up when the investor buys stock shares and moved down when he sells stock shares. Transactions incur some fixed costs and, therefore, are allowed only in discrete times. The investor maximizes his expected utility of terminal liquidation wealth under solvency constraints. In Ly Vath, Mnif and Pham (2007), the authors formulate this problem as an impulse control problem under state constraints and undertake the theoretical studies of the problem. They show that the value function is characterized as the unique constrained viscosity solution to the associated Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJBQVI). In this paper, we investigate numerical aspects of the problem. Our main objective is to provide a numerical algorithm as well as numerical results of the optimal transaction strategy.

Hamilton-Jacobi-Bellman equations are usually solved by using numerical methods based on finite difference methods. The Howard algorithm, which consists in computing two sequences, ie the optimal strategy and the value function, is often used for the resolution of these types of equation. From Barles and Souganidis (1995), we know that a monotone, stable and consistent scheme insures the convergence of the algorithm to the unique viscosity solution of the HJBQVI. Chancelier, Øksendal, and Sulem (2001) use the Howard algorithm to solve numerically a bi-dimensional HJBQVI related to a problem of optimal consumption and portfolio with both fixed and proportional transaction costs. They solve the problem in a bounded domain and they assumed zero Neumann boundary conditions on the localized boundary. However, this finite difference approach has two main limitations. First, it is only suitable to solve HJB equations when the solvency region has a simple shape such as $\mathbb{R}^n_+$ or when its boundaries are straight. A second more critical limitation of this approach concerns the state dimension of the problem. For large dimension as in our case, we have to use probabilistic algorithms as this approach is no longer suitable. Indeed, our associated HJBQVI has, in addition to time variable, three variables (x, y, and p, respectively the cash holding, the stock holding, and the stock share price) as well a very complex solvency region.

Guilbaud, Mnif and Pham (2013) give some numerical methods to solve an impulse control problem arising from optimal portfolio liquidation with bid-ask spread and market price impact. In the latter paper, the authors are able to provide an explicit backward numerical scheme for the time discretization of the dynamic programming QVI by taking advantage of the lag variable tracking the time interval between trades. This lag feature does not exist in our model, as such, the same technique may not be used. In our study, we give an alternative and efficient approach using an iterative method to estimate our value function.

Korn (1998) studies the problem of portfolio optimization with strictly positive transaction costs and impulse control and presents a sequence of optimal stopping problems where the reward function is expressed in terms of the impulse operator. He proves the convergence of the sequence of optimal stopping problems towards the value function of the initial problem. Chancelier, Øksendal and Sulem (2001) suggest an iterative method to solve the impulse control problem. They consider an auxiliary value function where the number of transactions is bounded by a positive number.

In this article, we prove that both iterative methods coincide. We study numerically our problem by reducing the impulse control problem to an iterative sequence of optimal stopping problems. Then, we introduce a numerical approximation algorithm for each optimal stopping problem based on quantization numerical procedures. Our numerical approach, called "value-iteration algorithm", could be adapted to every shape of the solvency region and we don't need to assume some artificial boundary conditions.

The paper is organized as follows. In section 2, we recall the model settings of our problem and its theoretical results. In section 3, we show that the value function could be obtained as the limit of an iterative procedure where each step is an optimal stopping problem and the reward function is related to the impulse operator. In sections 4 and 5, we provide a numerical method based on Quantization calculus and give numerical results for the optimal transaction strategy. We also focus on the convergence of our numerical scheme, in particular, we show that it satisfies monotonicity, stability and consistency properties. Here, the stability is proved in a general case where the utility function is not bounded unlike the proof in Guilbaud, Mnif and Pham (2013). Finally, in the last section, we further enrich our studies with some numerical illustrations.

2 Problem formulation

2.1 The model of the portfolio optimization

This section presents the details of the model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ supporting an one-dimensional Brownian motion W on a finite horizon $[0, T]$, $T < \infty$. We consider a continuous time financial market model consisting of a money market account yielding a constant interest rate $r \geq 0$ and a risky asset (or stock) of price process $P = (P_t)$. We denote by X_t the amount of money (or cash holdings) and by Y_t the number of shares in the stock held by the investor at time t.

Liquidity constraints. We assume that the investor can only trade discretely on $[0, T)$. This is modelled through an impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$, where the non-decreasing stopping times $\tau_1 \leq \ldots \tau_n \leq \ldots < T$ represent the intervention times of the investor and ζ_n, $n \geq 1$, are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $\mathbb{R}$ and represent the number of stock purchased if $\zeta_n \geq 0$ or sold if $\zeta_n < 0$ at these times. The sequence (τ_n, ζ_n) may be a priori finite or infinite. The dynamics of Y are then given by :

$$Y_s = Y_{\tau_n}, \qquad \tau_n \leq s < \tau_{n+1} \tag{2.1}$$

$$Y_{\tau_{n+1}} = Y_{\tau_n} + \zeta_{n+1} \tag{2.2}$$

Notice that we do not allow trade at the terminal date T, which is the liquidation date.

Price impact. The large investor affects the price of the risky stock P by his purchases and sales : the stock price goes up when the trader buys and goes down when he sells and the impact is increasing with the size of the order. We introduce a positive price impact function $Q(\zeta, p)$ which indicates the post-trade price when the large investor trades a position of ζ shares of stock at a pre-trade price p. In absence of price impact, we have $Q(\zeta, p) = p$. Here, we have $Q(0, p) = p$ meaning that no trading incurs no impact and Q is nondecreasing in ζ with $Q(\zeta, p) \geq$ (resp. $\leq$) p for $\zeta \geq$ (resp. $\leq$) 0. Actually, in the rest of the paper, we consider a price impact function in the form

$$Q(\zeta, p) \quad = \quad pe^{\lambda\zeta} \qquad \text{where } \lambda > 0 \tag{2.3}$$

The proportionality factor $e^{\lambda\zeta}$ represents the price increase (resp. discount) due to the ζ shares bought (resp. sold). The positive constant λ measures the fact that larger trades generate larger quantity impact, everything else constant. This form of price impact function is consistent with both the asymmetric information and inventory motives in the market microstructure literature (see Kyle (1985)).

We then model the dynamics of the price impact as follows. In the absence of trading, the price process is governed by

$$dP_s \quad = \quad P_s(bds + \sigma dW_s), \qquad \tau_n \leq s < \tau_{n+1} \tag{2.4}$$

where b, σ are constants with $\sigma > 0$. When a discrete trading $\Delta Y_s := Y_s - Y_{s^-} = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$, the price jumps to $P_s = Q(\Delta Y_s, P_{s^-})$, ie

$$P_{\tau_{n+1}} \;=\; Q(\zeta_{n+1}, P_{\tau_{n+1}^-}) \tag{2.5}$$

Notice that with this modelling of price impact, the price process P is always strictly positive, ie valued in $\mathbb{R}_+^* = (0, \infty)$.

Cash holdings. We denote by $\theta(\zeta, p)$ the cost function, which indicates the amount for a (large) investor to buy or sell ζ shares of stock when the pre-trade price is p :

$$\theta(\zeta, p) \quad = \quad \zeta Q(\zeta, p)$$

In absence of transaction, the process X grows deterministically at exponential rate r :

$$dX_s \quad = \quad rX_s ds, \qquad \tau_n \leq s < \tau_{n+1} \tag{2.6}$$

When a discrete trading $\Delta Y_s = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$ with pre-trade price $P_{s^-} = P_{\tau_{n+1}^-}$, we assume that in addition to the amount of stocks $\theta(\Delta Y_s, P_{s^-}) = \theta(\zeta_{n+1}, P_{\tau_{n+1}^-})$, there is a fixed cost $k > 0$ to be paid. This results in a variation of cash holdings by $\Delta X_s := X_s - X_{s^-} = -\theta(\Delta Y_s, P_{s^-}) - k$, ie

$$X_{\tau_{n+1}} \quad = \quad X_{\tau_{n+1}^-} - \theta(\zeta_{n+1}, P_{\tau_{n+1}^-}) - k \tag{2.7}$$

The assumption that any trading incurs a fixed cost of money to be paid will rule out continuous trading, ie optimally, the sequence (τ_n, ζ_n) is not degenerate in the sense that for all n, $\tau_n < \tau_{n+1}$ and $\zeta_n \neq 0$ a.s. A similar modelling of fixed transaction costs is considered in Morton and Pliska (1995) and Korn (1998).

Liquidation value and solvency constraint. The solvency constraint is a key issue in any portfolio/consumption choice problem. The point is to define in an economically meaningful way the portfolio value of a position in cash and stocks. In our context, we introduce the liquidation function $\ell(y,p)$ representing the value that an investor would obtained by liquidating immediately his stock position y by a single block trade, when the pre-trade price is p. It is given by :

$$\ell(y,p) \quad = \quad -\theta(-y,p)$$

If the agent has the amount x in the bank account, the number of shares y of stocks at the pre-trade price p, ie a state value $z=(x,y,p)$, his net wealth or liquidation value is given by :

$$L(z) \quad = \quad \max[L_0(z), L_1(z)]\mathbb{1}_{y\geq 0} + L_0(z)\mathbb{1}_{y<0} \tag{2.8}$$

where

$$L_0(z) \;=\; x + \ell(y,p) - k, \qquad L_1(z) \;=\; x$$

The interpretation is the following. $L_0(z)$ corresponds to the net wealth of the agent when he liquidates his position in stock. Moreover, if he has a long position in stock, ie $y \geq 0$, he can also choose to bin his stock shares, by keeping only his cash amount, which leads to a net wealth $L_1(z)$. This last possibility may be advantageous, ie $L_1(z) \geq L_0(z)$, due to the fixed cost k. Hence, globally, his net wealth is given by (2.8). In the absence of liquidity risk, ie $\lambda = 0$, and fixed transaction cost, ie $k = 0$, we recover the usual definition of wealth $L(z) = x + py$. Our definition (2.8) of liquidation value is also consistent with the one in transaction cost models where portfolio value is measured after stock position is liquidated and rebalanced in cash, see eg Cvitanic and Karatzas (1996) and Oksendal and Sulem (2002). Another alternative would be to measure the portfolio value separately in cash and stock as in Deelstra, Pham and Touzi (2002) for transaction cost models. This study would lead to multidimensional utility functions and is left for future research.

We then naturally introduce the liquidation solvency region :

$$\mathcal{S} \quad = \quad \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: L(z) > 0\big\}$$

and we denote its boundary and its closure by

$$\partial\mathcal{S} \;=\; \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: L(z) = 0\big\} \quad \text{and} \quad \bar{\mathcal{S}} \;=\; \mathcal{S}\cup\partial\mathcal{S}$$

The boundary of the solvency region may then be explicited as follows :

$$\partial\mathcal{S} \quad = \quad \partial_\ell^-\mathcal{S}\cup\partial^y\mathcal{S}\cup\partial_0^x\mathcal{S}\cup\partial_1^x\mathcal{S}\cup\partial_2^x\mathcal{S}\cup\partial_\ell^+\mathcal{S}$$

where

$$\begin{aligned}
\partial_\ell^-\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: x + \ell(y,p) = k,\; y \leq 0\big\} \\
\partial^y\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: 0 \leq x < k,\; y = 0\big\} \\
\partial_0^x\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: x = 0,\; y > 0,\; p < k\lambda e\big\} \\
\partial_1^x\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: x = 0,\; 0 < y < y_1(p)),\; p \geq k\lambda e\big\} \\
\partial_2^x\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: x = 0,\; y > y_2(p),\; p \geq k\lambda e\big\} \\
\partial_\ell^+\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \;: x + \ell(y,p) = k,\; y_1(p) \leq y \leq y_2(p),\; p \geq k\lambda e\big\}
\end{aligned}$$

In the sequel, we also introduce the corner lines in $\partial \mathcal{S}$:

$$\begin{aligned} D_0 &= \{(0,0)\} \times \mathbb{R}_+^* \subset \partial^y \mathcal{S}, & D_k &= \{(k,0)\} \times \mathbb{R}_+^* \subset \partial_\ell^- \mathcal{S} \\ C_1 &= \{(0, y_1(p), p) \ : p \in \mathbb{R}_+^*\} \subset \partial_\ell^+ \mathcal{S}, & C_2 &= \{(0, y_2(p), p) \ : p \in \mathbb{R}_+^*\} \subset \partial_\ell^+ \mathcal{S} \end{aligned}$$

Admissible controls. Given $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$ and an initial state $Z_{t^-} = z$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$ is admissible if the process $Z_s = (X_s, Y_s, P_s)$ given by (2.1)-(2.2)-(2.4)-(2.5)-(2.6)-(2.7) (with the convention $\tau_0 = t$) lies in $\bar{\mathcal{S}}$ for all $s \in [t,T]$. We denote by $\mathcal{A}(t,z)$ the set of all such policies. We shall see later that this set of admissible controls is nonempty for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$. In the sequel, for $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$, we also denote $Z_s^{0,t,z} = (X_s^{0,t,x}, y, P_s^{0,t,p})$, $t \leq s \leq T$, the state process when no transaction (ie no impulse control) is applied between t and T, ie the solution to :

$$dZ_s^0 = \begin{pmatrix} rX_s^0 \\ 0 \\ bP_s^0 \end{pmatrix} ds + \begin{pmatrix} 0 \\ 0 \\ \sigma P_s^0 \end{pmatrix} dW_s \tag{2.9}$$

starting from z at time t.

Investment problem. We consider an utility function U from $\mathbb{R}_+$ into $\mathbb{R}$, strictly increasing, concave and w.l.o.g. $U(0) = 0$, and s.t. there exist $K \geq 0$, $\gamma \in [0,1)$:

$$U(w) \leq K w^\gamma, \quad \forall w \geq 0 \tag{2.10}$$

We denote U_L the function defined on $\bar{\mathcal{S}}$ by :

$$U_L(z) = U(L(z))$$

We study the problem of maximizing the expected utility from terminal liquidation wealth and we then consider the value function :

$$v(t,z) = \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[e^{-r(T-t)} U_L(Z_T)\right], \quad (t,z) \in [0,T] \times \bar{\mathcal{S}} \tag{2.11}$$

2.2 Viscosity solution of the associated Quasi-variational Hamilton-Jacobi-Bellman inequality

The HJB quasi-variational inequality satisfied by the value function (2.11) is as follows:

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v \ , \ v - \mathcal{H}v\right] = 0, \quad \text{on} \quad [0,T) \times \mathcal{S} \tag{2.12}$$

where $\mathcal{L}$ as the infinitesimal generator associated to the system (2.9) corresponding to a no-trading period :

$$\mathcal{L}\varphi = rx\frac{\partial \varphi}{\partial x} + bp\frac{\partial \varphi}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2} - r\varphi$$

$\mathcal{H}$ is the impulse operator defined by

$$\mathcal{H}\varphi(t,z) \quad = \quad \sup_{\zeta \in \mathcal{C}(z)} \varphi(t, \Gamma(z,\zeta)), \quad (t,z) \in [0,T] \times \bar{\mathcal{S}}$$

Γ is the impulse transaction function defined from $\bar{\mathcal{S}} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$:

$$\Gamma(z,\zeta) \quad = \quad (x - \theta(\zeta,p) - k, y + \zeta, Q(\zeta,p)), \;\; z = (x,y,p) \in \bar{\mathcal{S}}, \; \zeta \in \mathbb{R}$$

and $\mathcal{C}(z)$ the set of admissible transactions :

$$\mathcal{C}(z) \quad = \quad \left\{\zeta \in \mathbb{R} \; : \Gamma(z,\zeta) \; \in \; \bar{\mathcal{S}}\right\} \; = \; \{\zeta \in \mathbb{R} \; : L(\Gamma(z,\zeta)) \; \geq \; 0\}$$

We related the value function (2.11) and the associated HJB quasi-variational inequality (2.12) by means of constrained viscosity solutions. The definition of viscosity solutions is given as follows:

Definition 2.1 *(i) Let $\mathcal{O} \subset \bar{\mathcal{S}}$. A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a viscosity subsolution (resp. supersolution) of (2.12) in $[0,T) \times \mathcal{O}$ if for all $(\bar{t},\bar{z}) \in [0,T) \times \mathcal{O}$ and $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ s.t. $(u^* - \varphi)(\bar{t},\bar{z}) = 0$ (resp. $(u_* - \varphi)(\bar{t},\bar{z}) = 0$) and $(\bar{t},\bar{z})$ is a maximum of $u^* - \varphi$ (resp. minimum of $u_* - \varphi$) on $[0,T) \times \mathcal{O}$, we have*

$$\min\left[-\frac{\partial\varphi}{\partial t}(\bar{t},\bar{z}) - \mathcal{L}\varphi(\bar{t},\bar{z}), u^*(\bar{t},\bar{z}) - \mathcal{H}u^*(\bar{t},\bar{z})\right] \quad \leq \quad 0 \tag{2.13}$$

$$(\;\textit{resp.} \quad \geq \quad 0) \tag{2.14}$$

(ii) A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a constrained viscosity solution of (2.12) in $[0,T) \times \mathcal{S}$ if u is a viscosity subsolution of (2.12) in $[0,T) \times \bar{\mathcal{S}}$ and a viscosity supersolution of (2.12) in $[0,T) \times \mathcal{S}$.

In Ly Vath, Mnif and Pham (2007), the following characterization was obtained

Theorem 2.1 *The value function v is continuous on $[0,T) \times \mathcal{S}$ and is the unique (in $[0,T) \times \mathcal{S}$) constrained viscosity solution to (2.12) satisfying the boundary and terminal condition :*

$$\lim_{\substack{(t',z') \to (t,z) \\ z' \in \mathcal{S}}} v(t',z') \quad = \quad 0, \qquad \forall (t,z) \in [0,T) \times D_0 \tag{2.15}$$

$$\lim_{\substack{(t,z') \to (T,z) \\ t < T, z' \in \mathcal{S}}} v(t,z') \quad = \quad \max[U_L(z), \mathcal{H}U_L(z)], \qquad \forall z \in \bar{\mathcal{S}} \tag{2.16}$$

and the growth condition :

$$|v(t,z)| \quad \leq \quad K\left(1 + \left(x + \frac{p}{\lambda}\right)\right)^\gamma, \quad \forall (t,z) \in [0,T) \times \mathcal{S} \tag{2.17}$$

for some positive constant $K < \infty$.

Remark 2.1 *In Ly Vath, Mnif and Pham (2007), the authors have also shown that the value function lies in the set of functions satisfying the growth condition :*

$$\mathcal{G}_\gamma([0,T] \times \bar{S}) = \left\{ v : [0,T] \times \bar{S} \to \mathbb{R}; \sup_{[0,T] \times \bar{S}} \frac{\mid v(t,z) \mid}{1 + (x + \frac{p}{\lambda})^\gamma} < \infty \right\} \tag{2.18}$$

For simplifying notation and when there is no ambiguity, this set will be noted $\mathcal{G}_\gamma$.

3 Convergence of the iterative scheme

We first introduce the following subsets of $\mathcal{A}(t,z)$, the set of the admissible impulse control strategies :

$$\mathcal{A}_n(t,z) \;:=\; \{\alpha \;=\; (\tau_k,\xi_k)_{k=0,\ldots,n} \in \mathcal{A}(t,z)\}$$

and the corresponding value function v_n, which describes the value function when the investor is allowed to trade at most n times:

$$v_n(t,z) \;:=\; \sup_{\alpha\in\mathcal{A}_n(t,z)} \mathbb{E}[e^{-r(T-t)}U_L(Z_T)] \quad (t,z)\in[0,T]\times\overline{\mathcal{S}} \tag{3.1}$$

For $t\in[0,T]$ and $z=(x,y,p)\in\overline{\mathcal{S}}$, if $x,\ y$ are both nonnegative, we clearly have $L(Z_s^{0,t,z})\geq 0$, and so $\mathcal{A}_0(t,z)$ is nonempty. Otherwise, if $x<0,\ y\geq 0$ or $x\geq 0,\ y<0$, due to the diffusion term $P^{0,t,z}$, it is clear that the probability for $L(Z_s^{0,t,z})$ to be negative before time T, is strictly positive, so that $\mathcal{A}_0(t,z)$ is empty. Hence, the value function for $n=0$ is initialized to:

$$v_0(t,z) = \begin{cases} \mathbb{E}\left[e^{-r(T-t)}U_L(Z_T^{0,t,z})\right] & \text{if } x\geq 0,\ y\geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

We now show the convergence of the sequence of the value functions v_n towards our initial value function v.

Lemma 3.1 *For all* $(t,z)\in\mathcal{S}$

$$\lim_{n\to\infty} v_n(t,z) \;=\; v(t,z).$$

Proof. From the definition of $\mathcal{A}_n(t,z)$, we have:

$$\mathcal{A}_n(t,z) \;\subset\; \mathcal{A}_{n+1}(t,z) \;\subset\; \mathcal{A}(t,z)$$

As such,

$$v_n(t,z) \;\leq\; v_{n+1}(t,z) \;\leq\; v(t,z)$$

which gives the existence of the limit and the first inequality:

$$\lim_{n\to\infty} v_n(t,z) \;\leq\; v(t,z) \tag{3.2}$$

Given $\varepsilon>0$, from the definition of v, there exists an impulse control $\alpha \;=\; (\tau_1,\tau_2,...;\xi_1,\xi_2,...)\in\mathcal{A}(t,z)$ such that

$$\mathbb{E}[e^{-r(T-t)}U_L(Z_T^{\alpha})] \;\geq\; v(t,z)-\varepsilon \tag{3.3}$$

with Z^α diffusing under the impulse control α.
We now set the control

$$\alpha_n \;:=\; (\tau_1,\tau_2,...,\tau_{n-1},\underline{\tau};\xi_1,\xi_2,...,\xi_{n-1},y_{\tau_{n-1}})$$

where $\tau_{n-1} < \underline{\tau} < \min\{\tau_n, T\}$. We see that $\alpha_n \in \mathcal{A}_n(t,z)$ and consider the corresponding process $Z^{(\alpha_n)}$. Using Fatou lemma, we obtain:

$$\liminf_{n\to\infty} \mathbb{E}[e^{-r(T-t)}U_L(Z_T^{(\alpha_n)})] \;\geq\; \mathbb{E}[\liminf_{n\to\infty} e^{-r(T-t)}U_L(Z_T^{(\alpha_n)})] \;=\; \mathbb{E}[e^{-r(T-t)}U_L(Z_T^{\alpha})] \quad (3.4)$$

Using (3.3) and (3.4), we obtain

$$\liminf_{n\to\infty} v_n(t,z) \;\geq\; \liminf_{n\to\infty} \mathbb{E}[e^{-r(T-t)}U_L(Z_T^{(\alpha_n)})] \;\geq\; v(t,z) - \varepsilon$$

As we obtain the latter inequality with an arbitrary $\varepsilon > 0$, and combining with the relation (3.2), we obtain the desired result:

$$\lim_{n\to\infty} v_n(t,z) \;=\; v(t,z)$$

□

Theorem 3.1 *We define $\varphi_n(t,z)$ iteratively as a sequence of optimal stopping problems:*

$$\begin{aligned} \varphi_{n+1}(t,z) &= \sup_{\tau\in\mathcal{S}_{t,T}} \mathbb{E}\left[e^{-r(\tau-t)}\mathcal{H}\varphi_n(\tau, Z_\tau^{0,t,z})\right] \\ \varphi_0(t,z) &= v_0(t,z) \end{aligned}$$

where $\mathcal{S}_{t,T}$ is the set of stopping times in $[t,T]$. Then

$$\varphi_n(t,z) \;=\; v_n(t,z)$$

Remark 3.1 Theorem 3.1 together with Lemma 3.1 show that

$$\lim_{n\to\infty} \varphi_n(t,z) \;=\; v(t,z), \quad (t,z) \in [0,T] \times \mathcal{S}$$

so that the iteration scheme for φ_n provides an approximation for v.

Remark 3.2 The value function φ_n satisfies the system of variational inequalities, which can be solved by induction starting from φ_0:

$$\min\left[-\frac{\partial\varphi_{n+1}}{\partial t} - \mathcal{L}\varphi_{n+1}\;,\; \varphi_{n+1} - \mathcal{H}\varphi_n\right] \;=\; 0, \quad (t,z) \in [0,T) \times \mathcal{S}$$

together with the terminal condition:

$$\varphi_{n+1}(T,z) \;=\; \mathcal{H}\varphi_n(T,z)$$

Proof of Theorem 3.1. We show by induction that $v_n(t,z) = \varphi_n(t,z)$, for all n. First, we have $v_0 = \varphi_0$. Considering an impulse control strategy $\alpha_1 = (\tau,\xi) \in \mathcal{A}_1(t,z)$, we clearly have

$$\begin{aligned} \varphi_1(t,z) &\geq \mathbb{E}[e^{-r(\tau-t)}\mathcal{H}\varphi_0(\tau, Z_\tau^{0,t,z})] \\ &\geq \mathbb{E}[e^{-r(\tau-t)}\mathcal{H}v_0(\tau, Z_\tau^{0,t,z})] \end{aligned}$$

From the definition of the operator $\mathcal{H}$, we obtain

$$\varphi_1(t,z) \;\; \geq \;\; \mathbb{E}[e^{-r(\tau-t)} v_0(\tau, \Gamma(Z_\tau^{0,t,z}, \xi))], \quad \forall \alpha_1 = \; (\tau,\xi) \in \mathcal{A}_1(t,z) \tag{3.5}$$

Let $Z^{(\alpha_1)}$ be the diffusion of Z, starting at time t, with $Z_t^{(\alpha_1)} = z$, and evolving under the impulse control α_1. Relation (3.5) becomes:

$$\varphi_1(t,z) \; \geq \; \mathbb{E}[e^{-r(\tau-t)} v_0(\tau, Z_\tau^{(\alpha_1)})], \quad \forall\, \alpha_1 = (\tau,\xi) \; \in \; \mathcal{A}_1(t,z) \tag{3.6}$$

Given the arbitrariness of α_1 and by using the dynamic programming principle applied to $v_1(t,z)$, we obtain

$$\varphi_1(t,z) \; \geq \; v_1(t,z)$$

From the definition of φ_1, for a given $\varepsilon \; > \; 0$, there exists τ^* such that

$$\varphi_1(t,z) - \varepsilon \; \leq \; \mathbb{E}[e^{-r(\tau^*-t)} \mathcal{H}\varphi_0(\tau^*, Z_{\tau^*}^{0,t,z})] \tag{3.7}$$

From the compactness of the set of admissible transactions, there exists ξ^* such that

$$\begin{aligned} \varphi_1(t,z) - \varepsilon \quad & \leq \quad \mathbb{E}[e^{-r(\tau^*-t)} v_0(\tau^*, \Gamma(Z_{\tau^*}^{0,t,z}, \xi^*))] \\ & \leq \; \mathbb{E}[e^{-r(\tau^*-t)} v_0(\tau^*, Z_{\tau^*}^{(*)})] \end{aligned}$$

where $Z^{(*)}$ is the processus starting at time t, with $Z_t^{(*)} = z$, and evolving under the impulse control $\alpha^* := (\tau^*, \xi^*)$.
Using the dynamic programming principle applied on $v_1(t,z)$, we obtain

$$\varphi_1(t,z) - \varepsilon \quad \leq \quad v_1(t,z)$$

The latter inequality is satisfied for any value of $\varepsilon > 0$, as such, we have

$$\varphi_1(t,z) \; \leq \; v_1(t,z)$$

which leads to $\varphi_1(t,z) = v_1(t,z)$, for all $(t,z) \in [0,T) \times \mathcal{S}$.
By induction, assuming that for a given n, we have $\varphi_n(t,z) = v_n(t,z)$, we will prove that $\varphi_{n+1}(t,z) = v_{n+1}(t,z)$. By definition, we have for any $\alpha_{n+1} = (\tau_1, ..., \tau_{n+1}, \xi_1, ..., \xi_{n+1}) \in \mathcal{A}_{n+1}(t,z)$,

$$\begin{aligned} \varphi_{n+1}(t,z) \;\; & \geq \;\; \mathbb{E}[e^{-r(\tau_1-t)} \mathcal{H}\varphi_n(\tau_1, Z_{\tau_1}^{0,t,z})], \\ & \geq \;\; \mathbb{E}[e^{-r(\tau_1-t)} v_n(\tau_1, \Gamma(Z_{\tau_1}^{0,t,z}, \xi_1))] \\ & \geq \;\; \mathbb{E}[e^{-r(\tau_1-t)} v_n(\tau_1, Z_{\tau_1}^{(n+1)})] \end{aligned} \tag{3.8}$$

where $Z^{(n+1)}$ is the diffusion starting at time t, with $Z_t^{(n+1)} = z$ and evolves under the control α_{n+1}. Given the arbitrariness of the control α_{n+1} and by using the dynamic programming principle applied to v_{n+1}, relation (3.8) becomes:

$$\varphi_{n+1}(t,z) \quad \geq \quad v_{n+1}(t,z)$$

To prove the opposite inequality, we use the definition of φ_{n+1}. For any $\varepsilon > 0$, there exists τ^* such that

$$\begin{aligned}\varphi_{n+1}(t,z)-\varepsilon &\leq \mathbb{E}[e^{-r(\tau^*-t)}\mathcal{H}\varphi_n(\tau^*, Z^{0,t,z}_{\tau^*})] && (3.9)\\ &\leq \mathbb{E}[e^{-r(\tau^*-t)}\mathcal{H}v_n(\tau^*, Z^{0,t,z}_{\tau^*})] && (3.10)\end{aligned}$$

From the compactness of the set of admissible transactions, there also exists ξ^* such that

$$\mathcal{H}v_n(\tau^*, Z^{0,t,z}_{\tau^*}) \;=\; v_n(\tau^*, Z^{(\alpha^*)}_{\tau^*})$$

where $Z^{(\alpha^*)}$, the processus starting at time t, with $Z_t = z$, evolves under the impulse control $\alpha^* := (\tau^*, \xi^*)$. Using the dynamic programming principle applied on v_{n+1}, the relation (3.10) becomes

$$\begin{aligned}\varphi_{n+1}(t,z)-\varepsilon &\leq \mathbb{E}[e^{-r(\tau^*-t)}v_n(\tau^*, Z^{(\alpha^*)}_{\tau^*})]\\ &\leq v_{n+1}(t,z)\end{aligned}$$

The inequality is obtained for any given ε, this leads to the required inequality

$$\varphi_{n+1}(t,z) \;=\; v_{n+1}(t,z)$$

$\square$

4 Approximation Scheme and Numerical Algorithm

In this section, we introduce a numerical scheme that approximates the HJB-QVI continuous operator, defined in (2.12), by a discrete one. This discrete operator is meant to converge towards the continuous operator as the discretization step goes to zero.
For the rest of the paper, we suppose that $r = 0$.

4.1 Approximation Scheme

For a time step $h > 0$ on the interval $[0, T]$, let us consider the following approximation scheme:

$$S^h(t, z, v^h(t,z), v^h) = 0 \qquad (t,z) \in [0,T] \times \overline{S} \tag{4.1}$$

where $S^h : [0,T] \times \bar{S} \times \mathbb{R} \times \mathcal{G}_\gamma \to \mathbb{R}$ is defined by

$$S^h(t,z,g,\varphi) := \begin{cases} min\Big[g - \mathbb{E}[\varphi(t+h, Z^{0,t,z}_{t+h})], g - \mathcal{H}\varphi(t,z)\Big], & t \;\in [0, T-h] \\ min\Big[g - \mathbb{E}[\varphi(T, Z^{0,t,z}_{T})], g - \mathcal{H}\varphi(t,z)\Big], & t \;\in (T-h, T) \\ min\Big[g - U_L(z), g - \mathcal{H}U_L(z)\Big], & t \;= T \end{cases} \tag{4.2}$$

We recall that $Z^{0,t,z}$ stands for the state process starting from z at time t, and without any impulse control strategy. It is given by

$$Z^{0,t,z}_s = (X^{0,t,x}_s, y, P^{0,t,p}_s), \quad s \geq t$$

with $P^{0,t,p}$ solution of (2.4) starting from p at time t. Notice that (4.1) is formulated as a backward scheme for the solution v^h through:

$$v^h(T,z) = \max\Big[U_L(z), \mathcal{H}U_L(z)\Big] \tag{4.3}$$
$$v^h(t,z) = \max\Big[\mathbb{E}[v^h(t+h, Z^{0,t,z}_{t+h})], \mathcal{H}v^h(t,z)\Big], \quad 0 \le t \le T-h \tag{4.4}$$
$$v^h(t,z) = v^h(T-h,z), \qquad T-h < t < T$$

This approximation scheme seems a priori implicit due to the nonlocal obstacle term $\mathcal{H}$. This is typically the case in impulse control problems, and the usual way to circumvent this problem is to iterate the scheme by considering a sequence of optimal stopping problems:

$$v^{h,n+1}(T,z) = \max\Big[U_L(z), \mathcal{H}U_L(z)\Big] \tag{4.5}$$
$$v^{h,n+1}(t,z) = \max\Big[\mathbb{E}[v^{h,n+1}(t+h, Z^{0,t,z}_{t+h})], \mathcal{H}v^{h,n}(t,z)\Big], \quad 0 \le t \le T-h \tag{4.6}$$
$$v^{h,n+1}(t,z) = v^{h,n+1}(T-h,z), \qquad T-h < t < T$$

starting from $v^{h,0}(t,z) = \mathbb{E}[U_L(Z^{0,t,z}_T)]$.

In Section 5, we will show that the solution $v^{h,n}$ of this scheme converges towards v^h the solution of the scheme (4.3)-(4.4) as n goes to infinity and that v^h converges towards v solution of (2.12) when we take h goes to zero.

Notice that at this stage, this approximation scheme is not yet fully implementable since it requires an approximation method for the expectations arising in (4.2).

4.2 Time and Space Discretization

We consider a time step $h = T/m$, $m \in \mathbb{N}\backslash\{0\}$ and denote by $\mathbb{T}_m = \{t_i = ih, i = 0, ..., m\}$ the regular grid over the interval $[0,T]$. Thus, from the previous section, the time discretization of step h for the QVI (2.12) leads to the explicit backward scheme:

$$v^{h,n+1}(t_m,z) = \max\Big[U_L(z), \sup_{\zeta\in\mathcal{C}(z)} U_L(\Gamma(z,\zeta))\Big] \tag{4.7}$$
$$v^{h,n+1}(t_i,z) = \max\Big[\mathbb{E}[v^{h,n+1}(t_{i+1}, Z^{0,t_i,z}_{t_{i+1}})], \sup_{\zeta\in\mathcal{C}(z)} v^{h,n}(t_i,\Gamma(z,\zeta))\Big] \tag{4.8}$$

for $i = 0, ..., m-1$, $z = (x,y,p) \in \bar{S}$ and starting from $v^{h,0}(t) = \mathbb{E}[U_L(Z^{0,t,z}_{t_m})]$.

The above scheme involves nonlocal terms in the variable z for the solution v^h in relation with the supremum over $\zeta \in \mathcal{C}(z)$ and the expectations, and thus the practical implementation requires a discretization for the state variable z together with an interpolation.

Since $\bar{S}$ is unbounded, we first localize the domain by setting

$$\bar{S}_{loc} = \bar{S} \cap ([x_{min}, x_{max}] \times [y_{min}, y_{max}] \times [p_{min}, p_{max}])$$

where $x_{min} < 0 < x_{max} \in \mathbb{R}$, $y_{min} < 0 < y_{max} \in \mathbb{R}$ and $0 \leq p_{min} < p_{max} \in \mathbb{R}$ are fixed constants, and then we define the regular grid

$$\mathbb{Z}_l = \{z = (x, y, p) \in \mathbb{X}_l \times \mathbb{Y}_l \times \mathbb{P}_l; z \in \bar{S}_{loc}\}$$

where $\mathbb{X}_l$ is the uniform grid on $[x_{min}, x_{max}]$ of step $\frac{x_{max}-x_{min}}{l}$ ($l \in \mathbb{N}^*$), and similarly for $\mathbb{Y}_l$ and $\mathbb{P}_l$. For the rest of the paper we will consider that $p_{min} = 0$.

Similarly we define the regular grid of the admissible controls:

$$\mathcal{C}_{M,R}(z) = \{\zeta_i = \zeta_{min} + \frac{i}{M}(\zeta_{max} - \zeta_{min}); 0 \leq i \leq M / \Gamma(z, \zeta_i) \in \bar{S}_{loc}\}$$

where $\zeta_{min} < \zeta_{max} \in \mathbb{R}$ and $M \in \mathbb{N}^*$ are fixed constants.

We also consider the following projection

$$\begin{aligned} \Pi_{[0,p_{max}]} : \mathbb{R}_+ &\rightarrow [0, p_{max}] \\ p &\rightarrow p\mathbb{1}_{[0,p_{max}]} + p_{max}\mathbb{1}_{]p_{max},+\infty[} \end{aligned}$$

Further more, we define R as

$$R := \min\Big(\mid x_{min} \mid, \mid x_{max} \mid, \mid y_{min} \mid, \mid y_{max} \mid, \mid p_{max} \mid \Big)$$

4.3 Functional quantization of the Brownian motion

We shall now describe the numerical procedure for computing the expectations arising in (4.8). Recalling that $Z_s^{0,t,z} = (X_s^{0,t,x}, y, P_s^{0,t,p})$, this involves only the expectation with respect to the price process, assumed here to follow a Black-Scholes model (2.4). We shall then use a Functional quantization of the Brownian motion W which is obtained by exploiting the Karhunen-Loève decomposition. It consists in truncating the decomposition at a fixed order $d(N)$ and quantizing the $\mathbb{R}^{d(N)}$-value Gaussian vector, which is constituted of the $d(N)$ first coordinates of the process on its Karhunen-Loève decomposition. Then, we get

$$W^i(t) = W(w_i, t) = \sum_{k=1}^{d(N)} \xi_k^i e_k(t) \quad 1 \leq i \leq N$$

where $\xi^i = (\xi_1^i, ..., \xi_{d(N)}^i) \in \mathbb{R}^{d(N)}$ quantize the $d(N)$-dimensional vector $\xi^N = (\xi_1, ..., \xi_{d(N)})$ and where $(W^1(t), ..., W^N(t))$ is an N-tuple of points in $\mathbb{R}^N$ quantizing the Brownian motion. In the case of a standard Brownian motion of dimension M (in our case $M = 1$) on $[0, T]$, the terms of the Karhunen-Loève's sequence are explicit, thus for $k \geq 1$:

$$e_k(t) = \sqrt{\frac{2}{T}} \sin\Big(\frac{\pi t}{T}(k - \frac{1}{2})\Big) \quad , \quad \xi_k \sim \mathcal{N}(0, \lambda_k \mathbf{I}_M) \quad , \quad \lambda_k = \left(\frac{T}{\pi(k - \frac{1}{2})}\right)^2$$

The Karhunen-Loève quantizer is a product quantizer based on the decomposition of the number N in a product of $d(N)$ elements. We denote by $W_{i_1..i_{d(N)}}^N(t)$ the quantizer of size

N obtained from the optimal decomposition of $N = N_1 \times N_2 \times ... \times N_{d(N)}$ which gives optimal $d(N)$ and (N_k) for a given N.

Thus, for the Brownian motion on $[0, T]$ we have

$$W^N_{i_1..i_{d(N)}}(t) = \sum_{k=1}^{d(N)} \sqrt{\lambda_k} x_{i_k} e_k(t) = \sum_{k=1}^{d(N)} \frac{\sqrt{2T}}{\pi(k-\frac{1}{2})} x_{i_k} \sin\left(\frac{\pi t}{T}(k - \frac{1}{2})\right)$$

$$1 \leq i_k \leq N_k \quad , \quad \prod_{k=1}^{d(N)} N_k = N$$

where (x_{i_k}) is the N_k quantizer of the standard normal distribution.

The weight associated to this quantizer is $\mathbb{P}_{i_1..i_{d(N)}} = \prod_{j=1}^{d(N)} \mathbb{P}(x_{i_j})$ which is the product of the weights associated to the quantization of the normal distribution.

The optimal grid (x_{i_n}) and the associated weights $\mathbb{P}_{i_1..i_{d(N)}}$ are downloaded from the website: http://www.quantize.maths-fi.com/downloads.

Our choice of using the functional quantization instead of the classical quantization is due to the fact that, to prove the stability property of the numerical scheme, we need to deal with processes. This is insured by the very definition of the Karhunen-Loève quantizer which is, unlike for the classical quantization where the quantizers are random variables, a process quantizer. The last quantizer is stationary and optimal in the sense of the L^2-norm as stated in Pags and Luschgy (2002) and Pags and Luschgy (2006).
Hence, the expectations arising in (4.8) are approximated by

$$\mathcal{E}^{N,R}[v^{h,n}(t, Z_t^{0,s,z})] := \sum_{i_1=1}^{N_1} .. \sum_{i_{d(N)}=1}^{N_{d(N)}} \mathbb{P}_{i_1..i_{d(N)}} v^{h,n}(t, Z_{N,R}^{0,s,z}(t)) \quad \forall\, s \leq t \tag{4.9}$$

where

$$Z_{N,R}^{0,s,z}(t) := \Big(x, y, \Pi_{[0,p_{max}]}(p \exp\big\{(b - \frac{\sigma^2}{2})(t-s) + \sigma W^N_{i_1..i_{d(N)}}(t-s)\big\})\Big)$$

Notations: For reading purposes, we recall the following notations :

h: the time step.
N: the number of points quantizing the brownian motion.
n: the index associated to the iterative scheme.
M: the number of admissible transactions.
R: the boundaries of the localized solvency region.

4.4 Numerical Algorithm

Once the expectations arising in (4.8) are replaced by their approximated from (4.9) and the domain is discretized and truncated. The scheme S^h defined in (4.2) becomes

$$S^{h,R,N,M}(t,z,g,\varphi) :=$$

$$\begin{cases} min\Big[g - \mathcal{E}^{N,R}[\varphi(t+h, Z^{0,t,z}_{t+h})], g - \mathcal{H}^{M,R}\varphi(t,z)\Big], & t \in [0, T-h] \\ min\Big[g - \mathcal{E}^{N,R}[\varphi(T, Z^{0,t,z}_{T})], g - \mathcal{H}^{M,R}\varphi(t,z)\Big], & t \in (T-h, T) \\ min\Big[g - U_L(z), g - \mathcal{H}^{M,R}U_L(z)\Big], & t = T \end{cases} \tag{4.10}$$

where $\mathcal{H}^{M,R}\varphi(t,z) = \sup_{\zeta \in \mathcal{C}_{M,R}(z)} \varphi(t, \Gamma(t,\zeta))$, $\forall z \in \mathbb{Z}_l$.

Thus, considering the iterative scheme defined in (4.5)-(4.6), we obtain the following backward scheme:

Our Numerical Algorithm

$$v^{h,n+1}(t_m, z) = \max\Big[U_L(z), \sup_{\zeta \in \mathcal{C}_{M,R}(z)} U_L(\Gamma(z,\zeta))\Big] \tag{4.11}$$

$$v^{h,n+1}(t_i, z) = \max\Big[\mathcal{E}^{N,R}[v^{h,n+1}(t_{i+1}, Z^{0,t_i,z}_{t_{i+1}})], \sup_{\zeta \in \mathcal{C}_{M,R}(z)} v^{h,n}(t_i, \Gamma(z,\zeta))\Big] \tag{4.12}$$

for $i = 0, ..., m-1$; $z = (x, y, p) \in \mathbb{Z}_l$ and starting from $v^{h,0}(t,z) = \mathcal{E}^{N,R}[U_L(Z^{0,t,z}_T)]$, which is explicit and fully implementable.

Remark 4.1 *The localization argument is efficient in the sense that it allows us to obtain the pointwise stability. Such argument was used in an earlier work of Barles et al. (1995). In fact the proof of the stability is not obvious when we use the growth condition interpolation (see inequality (2.17)) if $z \notin \mathbb{Z}_l$. A way around this would be to truncate the interpolation to the nearest neighbour in $\mathbb{Z}_l$, and then, we carry some numerical tests to show that the error is small if $\mathbb{Z}_l$ is large enough.*

The stopping criterion for this scheme is

$$\|v^{h,n+1} - v^{h,n}\|_\infty < \bar{\varepsilon}$$

where $\bar{\varepsilon}$ is a strictly positive constant.

Complexity of the algorithm. Due to the high dimension of the grid, the computation of the optimal policy on the entire grid has an expensive computational cost. Indeed, this grid contains $O(ml^3)$ points, and at each iteration n at each point $(t_i, z) \in \mathbb{T}_m \times \mathbb{Z}_l$, one has to compute:

- The approximation of conditional expectation $\mathcal{E}^{N,R}\big[v^{h,n+1}(t_i + h, Z^{0,t_i,z}_{t_i+h})\big]$ that costs $O(N)$ unitary operations.

- The approximation of the static supremum $\sup_{\zeta \in \mathcal{C}_{M,R}(z)} v^h(t_i, \Gamma(z,\zeta))$, together with its argument maximum, that costs $O(M)$ unitary operations when using linear search[1].
- The localization procedure and the interpolation procedure has constant computational cost $O(1)$ since we have to use the nearset neighbour in $\mathbb{Z}_l$ for teh localization and for the interplolation, we use the eight nearest neighbours of $z \in \mathbb{Z}_l$.

Therefore, we obtain a complexity of:

$$\text{Complexity} \quad = \quad O(ml^3 \max(N, M))$$

Actually, denoting by $K = \max(l, m, N, M)$, the complexity of the algorithm at each iteration n is $O(K^5)$. Yet, practical implementation of the algorithm can achieve quite better performance. The grid computation algorithm can be parallelized easily, which is a very desirable property when targeting an industrial application. Indeed, at each date t_i the computation of $\mathcal{E}^{N,R}\big[v^{h,n+1}(t_i + h, Z^{0,t_i,z}_{t_i+h})\big]$ and $\sup_{\zeta \in \mathcal{C}_{M,R}(z)} v^h(t_i, \Gamma(z,\zeta))$ can be done independently for each (x, y, p).
Finally, the complexity displayed above represents the amount of computations needed to build up the optimal policy. When targeting a live trading application, one can compute off-line and store optimal policies for a given set of market parameters, and when actually trading, one does only need to read (with constant cost) the optimal policy corresponding to current market state.

5 Convergence analysis

We now focus on the convergence (when we take h to zero and n, M, N, R to infinity) of the solution $v^{h,n}$ to (4.11)-(4.12) towards the value function v solution to our HJBQVI (2.12). Following Barles and Souganidis (1991), we have to show that the scheme $S^{h,R,N,M}$ in (4.10) satisfies monotonicity, stability and consistency properties. We will need the following notations

$$\mathcal{E}^N[v^{h,n}(t, Z^{0,s,z}_t)] := \sum_{i_1=1}^{N_1} .. \sum_{i_{d(N)}=1}^{N_{d(N)}} \mathbb{P}_{i_1..i_{d(N)}} v^{h,n}(t, Z^{0,s,z}_N(t)) \quad \forall\ s \le t$$

where

$$Z^{0,s,z}_N(t) := \Big(x, y, p \exp\big\{(b - \frac{\sigma^2}{2})(t-s) + \sigma W^N_{i_1..i_{d(N)}}(t-s)\big\}\Big)$$

[1]Note that the supremum computation can be improved by the use of dichotomy-based search instead of linear search if we are able to use a concavity argument on $\zeta \mapsto v(t, \Gamma(x,y,p,\zeta))$ which would lead to a complexity of $O(\ln(M))$. From numerical experiments, this dichotomy search method leads to acceptable results.

Proposition 5.1 *(Monotonicity)*
For all $h > 0$, $(t,z) \in [0,T] \times \bar{S}$, $g \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{G}_\gamma$ s.t. $\varphi \leq \psi$ we have

$$S^{h,R,N,M}(t,z,g,\varphi) \geq S^{h,R,N,M}(t,z,g,\psi)$$

Proof. It follows directly from the definition (4.10) of the scheme.

□

Proposition 5.2 *(Consistency)*
We suppose that $N = \exp\big(\frac{1}{h^q}\big)$ s.t. $q > 2$.
(i) For all $(t,z) \in [0,T) \times \bar{S}$ and Lipschitz function $\phi \in C^{1,2}([0,T) \times \bar{S})$ we have

$$\limsup_{\substack{(h,t',z')\to(0,t,z)\\(M,N,R)\to+\infty}} \quad \min\left\{\frac{\phi(t',z') - \mathcal{E}^{N,R}[\phi(t'+h, Z^{0,t',z'}_{t'+h})]}{h}, \Big(\phi(t',z') - \mathcal{H}^{M,R}\phi(t',z')\Big)\right\}$$

$$\leq \min\Big\{\big(-\frac{\partial\phi}{\partial t} - \mathcal{L}\phi\big)(t,z), \big(\phi(t,z) - \mathcal{H}\phi(t,z)\big)\Big\}$$

and

$$\liminf_{\substack{(h,t',z')\to(0,t,z)\\(M,N,R)\to+\infty}} \quad \min\left\{\frac{\phi(t',z') - \mathcal{E}^{N,R}[\phi(t'+h, Z^{0,t',z'}_{t'+h})]}{h}, \Big(\phi(t',z') - \mathcal{H}^{M,R}\phi(t',z')\Big)\right\}$$

$$\geq \min\Big\{\big(-\frac{\partial\phi}{\partial t} - \mathcal{L}\phi\big)(t,z), \big(\phi(t,z) - \mathcal{H}\phi(t,z)\big)\Big\}$$

(ii) For all $z \in \bar{S}$ and Lipschitz function $\phi \in C^{1,2}([0,T] \times \bar{S})$ we have

$$\limsup_{\substack{(h,t',z')\to(0,T,z)\\(M,N,R)\to+\infty}} \quad \min\left\{\phi(t',z') - U_L(z'), \Big(\phi(t',z') - \mathcal{H}^{M,R}U_L(z')\Big)\right\}$$

$$\leq \min\Big\{\phi(T,z) - U_L(z), \big(\phi(T,z) - \mathcal{H}U_L(z)\big)\Big\}$$

and

$$\liminf_{\substack{(h,t',z')\to(0,T,z)\\(M,N,R)\to+\infty}} \quad \min\left\{\phi(t',z') - U_L(z'), \Big(\phi(t',z') - \mathcal{H}^{M,R}U_L(z')\Big)\right\}$$

$$\geq \min\Big\{\phi(T,z) - U_L(z), \big(\phi(T,z) - \mathcal{H}U_L(z)\big)\Big\}$$

Proof. We only show the first inequality since the others may be obtained similarly. We consider the following steps:

Step 1. First, we have to show that

$$\Big|\mathcal{E}^N[\phi(t'+h, Z^{0,t',z'}_{t'+h})] - \mathbb{E}[\phi(t'+h, Z^{0,t',z'}_{t'+h})]\Big| \leq \quad h \;\; \varepsilon(h)$$

where ε is strictly positive function, such that $\varepsilon(h) \to 0$ when $h \to 0$.

We consider

$$
\begin{aligned}
\mathcal{I} &:= \left|\mathcal{E}^N[\phi(t^{'}+h, Z_{t^{'}+h}^{0,t^{'},z^{'}})] - \mathbb{E}[\phi(t^{'}+h, Z_{t^{'}+h}^{0,t^{'},z^{'}})]\right| \\
&= \Big|\mathbb{E}[\phi\Big(t^{'}+h, x^{'}, y^{'}, p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W_{i_1..i_{d(N)}}^N(h)\big\}\Big)] \\
&\qquad -\mathbb{E}[\phi\Big(t^{'}+h, x^{'}, y^{'}, p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W(h)\big\}\Big)]\Big| \\
&\leq \mathbb{E}\Big[\Big|[\phi]_{lip} p^{'}\exp(b-\frac{\sigma^2}{2})h\Big| \times \Big|\exp\big(\sigma W_{i_1..i_{d(N)}}^N(h)\big) - \exp\big(\sigma W(h)\big)\Big|\Big]. \ (\phi \text{ lipschitz})
\end{aligned}
$$

Using the fact that $\dfrac{|\exp x_1 - \exp x_2|}{|x_1 - x_2|} \leq \exp x_1 + \exp x_2$, we obtain

$$
\mathcal{I} \leq \Big|[\phi]_{lip} p^{'}\exp\{(b-\frac{\sigma^2}{2})h\}\Big| \mathbb{E}\Big[\Big(\exp\big(\sigma W_{i_1..i_{d(N)}}^N(h)\big) + \exp\big(\sigma W(h)\big)\Big)\Big|W_{i_1..i_{d(N)}}^N(h) - W(h)\Big|\Big]
$$

Using Cauchy-Schwarz inequality, we may obtain

$$
\begin{aligned}
\mathcal{I} &\leq [\phi]_{lip}\ p^{'}\ e^{\{(b-\frac{\sigma^2}{2})h\}} \big\|W_{i_1..i_{d(N)}}^N(h) - W(h)\big\|_{L^2} \\
&\qquad \times \big\|\exp\big(\sigma W_{i_1..i_{d(N)}}^N(h)\big) + \exp\big(\sigma W(h)\big)\big\|_{L^2} \\
&\leq [\phi]_{lip}\ p^{'}\ e^{\{(b-\frac{\sigma^2}{2})h\}} \big\|W_{i_1..i_{d(N)}}^N(h) - W(h)\big\|_{L^2} \\
&\qquad \times \Big(\big\|\exp\big(\sigma W_{i_1..i_{d(N)}}^N(h)\big)\big\|_{L^2} + \big\|\exp\big(\sigma W(h)\big)\big\|_{L^2}\Big) \qquad (5.1)
\end{aligned}
$$

Noticing that $f : x \mapsto \exp(2\sigma x)$ is a convex function and $W_{i_1..i_{d(N)}}^N$ a stationary quantizer, we may use the Jensen inequality to obtain

$$
\begin{aligned}
\big\|\exp\big(\sigma W_{i_1..i_{d(N)}}^N(h)\big)\big\|_{L^2}^2 &= \mathbb{E}[f\big(W_{i_1..i_{d(N)}}^N(h)\big)] = \mathbb{E}[f\big(\mathbb{E}\big[W(h)/W_{i_1..i_{d(N)}}^N(h)\big]\big)] \\
&\leq \mathbb{E}[\big(\mathbb{E}\Big[f\Big(W(h)\Big)/W_{i_1..i_{d(N)}}^N(h)\Big]\big)] = \mathbb{E}\Big[f\Big(W(h)\Big)\Big] \\
&= \mathbb{E}\Big[\exp\big(2\sigma W(h)\big)\Big] = \exp\big(2\sigma^2 h\big)
\end{aligned}
$$

On the other hand, by the Zador theorem Pags and Luschgy (2002), we have

$$
\begin{aligned}
\big\|W_{i_1..i_{d(N)}}^N(h) - W(h)\big\|_{L^2} &= \sqrt{\mathbb{E}\Big[\Big|W_{i_1..i_{d(N)}}^N(h) - W(h)\Big|^2\Big]} \\
&= O\Big(\frac{1}{(\log N)^{\frac{1}{2}}}\Big)
\end{aligned}
$$

Since $N = \exp\frac{1}{h^q} \quad s.t. \quad q > 2$, we obtain

$$
\big\|W_{i_1..i_{d(N)}}^N(h) - W(h)\big\|_{L^2} \leq h\ \ \varepsilon(h),
$$

where ε represents a strictly positive function such that $\varepsilon(h) \to 0$ when $h \to 0$.
Hence, the inequality (5.1) becomes

$$
\left|\mathcal{E}^N[\phi(t^{'}+h, Z_{t^{'}+h}^{0,t^{'},z^{'}})] - \mathbb{E}[\phi(t^{'}+h, Z_{t^{'}+h}^{0,t^{'},z^{'}})]\right| \leq 2[\phi]_{lip}\ p^{'}\ h\exp\Big((b-\frac{\sigma^2}{2})h + \sigma^2 h\Big)\ \ \varepsilon(h) \qquad (5.2)
$$

The last quantity goes to zero when h goes to zero and we have

$$\Big|\mathcal{E}^N[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})] - \mathbb{E}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})]\Big| \leq \quad h \ \ \varepsilon(h) \tag{5.3}$$

Step 2. We now show that $\Big|\mathcal{E}^{N,R}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})] - \mathcal{E}^N[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})]\Big| \leq \quad h \ \ \varepsilon(h)$.
We have

$$\begin{aligned}
& \Big|\mathcal{E}^{N,R}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})] - \mathcal{E}^N[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})]\Big| \\
= \quad & \Big|\mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, \Pi_{[0,p_{max}]}(p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W^N_{i_1..i_{d(N)}}(h)\big\})\Big)\big] \\
& \quad - \quad \mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W^N_{i_1..i_{d(N)}}(h)\big\}\Big)\big]\Big| \\
\leq \quad & A \ + \ B \ + \ C
\end{aligned}$$

where

$$\begin{aligned}
A \ := \ & \Big|\mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, \Pi_{[0,p_{max}]}(p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W^N_{i_1..i_{d(N)}}(h)\big\})\Big)\big] \\
& \quad - \quad \mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, \Pi_{[0,p_{max}]}(p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W(h)\big\})\Big)\big]\Big| \\
B \ := \ & \Big|\mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, \Pi_{[0,p_{max}]}(p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W(h)\big\})\Big)\big] \\
& \quad - \quad \mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W(h)\big\}\Big)\big]\Big| \\
C \ := \ & \Big|\mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W^N_{i_1..i_{d(N)}}(h)\big\}\Big)\big] \\
& \quad - \quad \mathbb{E}\big[\phi\Big(t^{'}+h, x^{'}, y^{'}, p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h + \sigma W(h)\big\}\Big)\big]\Big|
\end{aligned}$$

Using the Jensen Inequality and the Zador theorem, we obtain

$$\begin{aligned}
A \ \leq \ & [\phi]_{lip}[\Pi_{[0,p_{max}]}]_{lip} \ p^{'}e^{\{(b-\frac{\sigma^2}{2})h\}} \ \mathbb{E}\big[\Big|\exp\big(\sigma W^N_{i_1..i_{d(N)}}(h)\big) - \exp\big(\sigma W(h)\big)\Big|\big] \\
\leq \ & [\phi]_{lip}[\Pi_{[0,p_{max}]}]_{lip} \ p^{'}e^{\{(b-\frac{\sigma^2}{2})h\}} \ \mathbb{E}\big[\Big|\exp\big(\sigma W^N_{i_1..i_{d(N)}}(h)\big) + \exp\big(\sigma W(h)\big)\Big|.\Big|W^N_{i_1..i_{d(N)}}(h) - W(h)\Big|\big] \\
\leq \ & [\phi]_{lip}[\Pi_{[0,p_{max}]}]_{lip} \ p^{'} \ e^{\{(b-\frac{\sigma^2}{2})h\}}\big\|W^N_{i_1..i_{d(N)}}(h) - W(h)\big\|_{L^2} \\
& \quad \times \quad \big\|\exp\big(\sigma W^N_{i_1..i_{d(N)}}(h)\big) + \exp\big(\sigma W(h)\big)\big\|_{L^2} \\
\leq \ & [\phi]_{lip}[\Pi_{[0,p_{max}]}]_{lip} \ p^{'} \ e^{\{(b-\frac{\sigma^2}{2})h\}}\big\|W^N_{i_1..i_{d(N)}}(h) - W(h)\big\|_{L^2} \\
& \quad \times \quad \Big(\big\|\exp\big(\sigma W^N_{i_1..i_{d(N)}}(h)\big)\big\|_{L^2} + \ \big\|\exp\big(\sigma W(h)\big)\big\|_{L^2}\Big) \\
\leq \ & 2[\phi]_{lip}[\Pi_{[0,p_{max}]}]_{lip} \ p^{'} \ h\exp\Big((b-\frac{\sigma^2}{2})h + \sigma^2 h\Big) \ \varepsilon(h)
\end{aligned} \tag{5.4}$$

From relation (5.2), we also have

$$B \quad \leq \quad 2[\phi]_{lip} \ p^{'} \ h\exp\Big((b-\frac{\sigma^2}{2})h + \sigma^2 h\Big) \ \varepsilon(h) \tag{5.5}$$

Let us now turn to the last term

$$\begin{aligned}
C &= \Big|\mathbb{E}\big[\phi\Big(t^{'}+h,x^{'},y^{'},\Pi_{[0,p_{max}]}(p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h+\sigma W(h)\big\})\Big)\big] \\
&\qquad - \quad \mathbb{E}\big[\phi\Big(t^{'}+h,x^{'},y^{'},p^{'}\exp\big\{(b-\frac{\sigma^2}{2})h+\sigma W(h)\big\}\Big)\big]\Big| \\
&\leq [\phi]_{lip}\mathbb{E}\Bigg[\,p^{'}e^{\left((b-\frac{\sigma^2}{2})h+\sigma W(h)\right)}\mathbb{1}_{\{p^{'}e^{\left((b-\frac{\sigma^2}{2})h+\sigma W(h)\right)}>p_{max}\}}\Bigg] \\
&\leq [\phi]_{lip}e^{(b-\frac{\sigma^2}{2})h}p^{'}\sqrt{\mathbb{E}[\exp\Big(2\sigma W(h)\Big)]}\sqrt{\mathbb{P}[p^{'}\exp\big((b-\frac{\sigma^2}{2})h+\sigma W(h)\big)>p_{max}]} \\
&= [\phi]_{lip}e^{(b-\frac{\sigma^2}{2})h}p^{'}\sqrt{\mathbb{E}[\exp\Big(2\sigma W(h)\Big)]}\sqrt{\mathbb{P}[p^{'}\exp\big((b-\frac{\sigma^2}{2})h+\sigma\sqrt{h}U\big)>p_{max}]} \\
&= [\phi]_{lip}e^{(b-\frac{\sigma^2}{2})h}p^{'}\sqrt{\mathbb{E}[\exp\Big(2\sigma W(h)\Big)]}\sqrt{\mathbb{P}[U>d]}
\end{aligned}$$

where $U\sim\mathcal{N}(0,1)$ and $d := \dfrac{\log\left(\frac{p_{max}}{p^{'}e^{(b-\frac{\sigma^2}{2})h}}\right)}{\sigma\sqrt{h}}$.
We have

$$\mathbb{P}[U>d] \;=\; 1-\mathbb{P}[U\leq d]\sim\frac{\varphi(d)}{d}\quad when\quad d\to+\infty$$

with φ is the probability density function of the standard normal distribution and that

$$d \;=\; \frac{\log\left(\frac{p_{max}}{p^{'}e^{(b-\frac{\sigma^2}{2})h}}\right)}{\sigma\sqrt{h}}\to+\infty\quad when\quad R\to+\infty$$

and

$$\frac{\varphi(d)}{d} \;=\; \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\Bigg(\frac{\log\left(\frac{p_{max}}{p^{'}e^{(b-\frac{\sigma^2}{2})h}}\right)}{\sigma\sqrt{h}}\Bigg)^2\Big)\frac{\sigma\sqrt{h}}{\log\left(\frac{p_{max}}{p^{'}e^{(b-\frac{\sigma^2}{2})h}}\right)}$$

Therefore, we may obtain

$$C\leq[\phi]_{lip}e^{(b-\frac{\sigma^2}{2})h}p^{'}\sqrt{\mathbb{E}[\exp\Big(2\sigma W(h)\Big)]}\sqrt{\frac{\varphi(d)}{d}} \tag{5.6}$$

By sending h to 0, we obtain

$$\frac{\varphi(d)}{h^{2+\eta}d}\longrightarrow 0,\;\;\text{for all }\eta>0$$

which implies

$$\frac{\varphi(d)}{d}\leq h^2\,\varepsilon(h)$$

Combining relations (5.4), (5.5) and (5.6) we obtain that

$$\Big|\mathcal{E}^{N,R}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})] - \mathcal{E}^{N}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})]\Big| \leq \quad h \;\; \varepsilon(h) \tag{5.7}$$

Finally, we conclude from (5.3) and (5.7) that

$$\Big|\mathcal{E}^{N,R}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})] - \mathbb{E}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})]\Big| \leq \quad h \;\; \varepsilon(h) \tag{5.8}$$

Step 3. We prove now that

$$\lim_{h\to 0} \frac{\phi(t,z) - \mathbb{E}[\phi(t+h, Z^{0,t,z}_{t+h})]}{h} = -\frac{\partial \phi}{\partial t}(t,z) - \mathcal{L}\phi(t,z)$$

By applying Itô's formula to $\phi(s, Z^{0,t,z}_s)$ between t and $t+h$, we obtain

$$\begin{aligned}\phi(t+h, Z^{0,t,z}_{t+h}) - \phi(t,z) &= \int_t^{t+h} \Big(\frac{\partial \phi}{\partial t}(u, Z^{0,t,z}_u) + \mathcal{L}\phi(u, Z^{0,t,z}_u)\Big) du \\ &+ \int_t^{t+h} \frac{\partial \phi}{\partial p}(u, Z^{0,t,z}_u) \sigma P^{0,t,p}_u dW_u\end{aligned}$$

Dividing the last equality by h and applying the expectations, we obtain

$$\frac{\mathbb{E}[\phi(t+h, Z^{0,t,z}_{t+h})] - \phi(t,z)}{h} = \frac{1}{h}\mathbb{E}[\int_t^{t+h} \Big(\frac{\partial \phi}{\partial t}(u, Z^{0,t,z}_u) + \mathcal{L}\phi(u, Z^{0,t,z}_u)\Big) du]$$

Thus, when h goes to 0, we have that

$$\lim_{h\to 0} \frac{\phi(t,z) - \mathbb{E}[\phi(t+h, Z^{0,t,z}_{t+h})]}{h} \leq -\frac{\partial \phi}{\partial t}(t,z) - \mathcal{L}\phi(t,z)$$

Combining the above inequality with the inequality (5.8), we may obtain

$$\limsup_{\substack{(h,t^{'},z^{'})\to(0,t,z) \\ (M,N,R)\to+\infty}} \frac{\phi(t^{'},z^{'}) - \mathcal{E}^{N,R}[\phi(t^{'}+h, Z^{0,t^{'},z^{'}}_{t^{'}+h})]}{h} \leq \Big(-\frac{\partial \phi}{\partial t} - \mathcal{L}\phi\Big)(t,z) \tag{5.9}$$

Finally, noticing that $\overline{\bigcup_{M,R=1}^{+\infty} \mathcal{C}_{M,R}(z)} = \mathcal{C}(z)$, $\mathcal{C}(z)$ a compact set, ϕ and Γ continuous functions, we may obtain

$$\lim_{M,R\to+\infty} \sup_{\zeta\in\mathcal{C}_{M,R}(z)} \phi(t, \Gamma(z,\zeta)) = \sup_{\zeta\in\mathcal{C}(z)} \phi(t, \Gamma(z,\zeta))$$

We may therefore derive the following inequality

$$\limsup_{\substack{(h,t^{'},z^{'})\to(0,t,z) \\ (M,N,R)\to+\infty}} \Big(\phi(t^{'},z^{'}) - \mathcal{H}^{M,R}\phi(t^{'},z^{'})\Big) \leq \phi(t,z) - \mathcal{H}\phi(t,z) \tag{5.10}$$

which we combine to (5.9) to complete our proof of consistency property.

$\square$

Remark 5.1 *To prove the pointwise consistency property, we need to take* $N = \exp\left(\frac{1}{h^q}\right)$, *where* $q > 2$. *In fact, for functional quantization, the Zador theorem Pags and Luschgy (2002) states that*

$$\begin{aligned}\left\|W^N_{i_1..i_{d(N)}}(h) - W(h)\right\|_{L^2} &= \sqrt{\mathbb{E}\left[\left|W^N_{i_1..i_{d(N)}}(h) - W(h)\right|^2\right]} \\ &= O\Big(\frac{1}{(\log N)^{\frac{1}{2}}}\Big) = O(h^{\frac{q}{2}})\end{aligned}$$

However, numerical experiments show that we have convergence when $N = 96$.

We now turn to the stability property of the discrete scheme $S^{h,R,N,M}$. However, before doing so, we need to obtain some preliminary results. We first introduce the set of the admissible discrete impulse control strategies

$$\begin{aligned}\mathcal{A}^{h,M,R}(t,z) \quad &= \quad \Big\{\alpha = (\tau_n, \zeta_n); \tau_n \text{ are } \widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)\text{-stopping times valued in } S^h_{t,T} \\ &\text{and} \quad \zeta_n \quad \text{are } \widehat{\mathcal{F}}_{\tau_n}\text{-measurable functions valued in } \mathcal{C}_{M,R}(z)\Big\}, \quad \forall (t,z) \in \mathbb{T}_m \times \bar{S}_{loc}\end{aligned}$$

Here $\widehat{\mathbb{F}}$ denotes the natural filtration of the quantized brownian motion $W^N_{i_1..i_{d(N)}}$ and $S^h_{t,T}$ is the set of stopping times taking values in $\Big\{\tau = t + ih \; ; \; i \in \{0,..,m\} \textit{ and } h = \frac{T-t}{m}\Big\}$

Then, we define the following subsets of $\mathcal{A}^{h,M,R}(t,z)$

$$\mathcal{A}^{h,M,R}_n(t,z) := \Big\{\alpha = (\tau_k, \zeta_k)_{k=0,..,n} \in \mathcal{A}^{h,M,R}(t,z)\Big\}$$

and the corresponding discrete value function

$$v^{h,R,N,M}_n(t,z) = \sup_{\alpha \in \mathcal{A}^{h,M,R}_n(t,z)} \mathcal{E}^{N,R}[U_L(Z_{N,R}(T))]; \quad \forall \, (t,z) \in \mathbb{T}_m \times \bar{S}_{loc} \tag{5.11}$$

$$v^{h,R,N,M}_0(t,z) = \begin{cases} \mathcal{E}^{N,R}\left[U_L(Z^{0,t,z}_T)\right] & \text{if } x \geq 0, \; y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

We now turn to our preliminary results.

Lemma 5.1 *For all* $(t,z) \in \mathbb{T}_m \times \bar{S}_{loc}$

$$\lim_{n \to +\infty} v^{h,R,N,M}_n(t,z) = v^{h,R,N,M}(t,z)$$

where $v^{h,R,N,M}$ *is the solution of the discrete HJB inequality without considering an iterative scheme.*

Proof. The proof of this Lemma follows the same arguments used in Lemma 3.1.

□

Proposition 5.3 *We define $\varphi_n^{h,R,N,M}$ iteratively as a sequence of optimal stopping problems*

$$\begin{aligned}\varphi_{n+1}^{h,R,N,M}(t,z) &= \sup_{\tau\in S_{t,T}^h} \mathcal{E}^{N,R}[\mathcal{H}^{M,R}\varphi_n^{h,R,N,M}(\tau, Z_{N,R}^{0,t,z}(\tau))] \quad \forall\,(t,z)\in\mathbb{T}_m\times\bar{S}_{loc}\\ \varphi_0^{h,R,N,M}(t,z) &= v_0^{h,R,N,M}(t,z) \quad \forall\,(t,z)\in\mathbb{T}_m\times\bar{S}_{loc}\end{aligned}$$

Then, for all $(t,z)\in\mathbb{T}_m\times\bar{S}_{loc}$

$$\varphi_n^{h,R,N,M}(t,z) = v_n^{h,R,N,M}(t,z)$$

Proof. The proof of this Proposition follows the same arguments used in Theorem 3.1.

□

Corollary 5.1 *For all $(t,z)\in\mathbb{T}_m\times\bar{S}_{loc}$*

$$\lim_{n\to+\infty}\varphi_n^{h,R,N,M}(t,z) = v^{h,R,N,M}(t,z)$$

Proof. This result is obtained by combining Lemma 5.1 and Proposition 5.3.

□

Proposition 5.4 *(Stability)*
For all $h>0$, there exists a unique solution $v_n^{h,R,N,M}\in\mathcal{G}_\gamma([0,T]\times\bar{S})$ to (4.1) and the sequence $(v_n^{h,R,N,M})_h$ is uniformly bounded in $\mathcal{G}_\gamma([0,T]\times\bar{S})$ ie there exists $w\in\mathcal{G}_\gamma([0,T]\times\bar{S})$ s.t. $|v_n^{h,R,N,M}|\leq|w|$ for all $h>0$.

Proof. The uniqueness of a solution $\in\mathcal{G}_\gamma([0,T]\times\bar{S})$ to (4.1) follows from the explicit backward scheme (4.11)-(4.12). The optimal portfolio liquidation problem associated to the discrete impulse control problem is defined via its discrete value function by

$$v_n^{h,R,N,M}(t,z) = \sup_{\alpha\in\mathcal{A}_n^{h,M,R}(t,z)} \mathbb{E}[U_L(Z_{N,R}(T))] \qquad \forall\,(t,z)\in\mathbb{T}_m\times\bar{S}_{loc}.$$

This numerical scheme is well defined. Let's show that $v_n^{h,R,N,M}$ is bounded independently from h,R,M and N. Indeed, using the assumption (2.10) and recalling that the domain is truncated, we have that

$$\begin{aligned}\mathbb{E}[U_L(Z_{N,R}(T))] &\leq \mathbb{E}[U_L(Z_N(T))] \qquad \forall\,\alpha\in\mathcal{A}_n^{h,M,R}(t,z)\\ &\leq K\mathbb{E}[\Big(L(Z_N(T))\Big)^\gamma] \qquad \forall\,\alpha\in\mathcal{A}_n^{h,M,R}(t,z)\end{aligned}$$

where Z_N is the controlled quantized state process associated to $\alpha\in\mathcal{A}_n^{h,M,R}(t,z)$.
Recalling from Ly Vath, Mnif and Pham (2007) that

$$\sup_{\alpha\in\mathcal{A}(t,z)} L(Z_s) \leq \sup_{\alpha\in\mathcal{A}(t,z)} \bar{L}(Z_s) \leq \bar{L}(Z_s^{0,t,z}) = X_s^{0,t,x} + \frac{P_s^{0,t,p}}{\lambda}; \quad \forall\, z=(x,y,p)\in\bar{S}$$

where

$$\bar{L}(z) := x + \frac{p}{\lambda}$$

and noticing that $\mathcal{A}^{h,M,R}(t,z) \subset \mathcal{A}(t,z)$ and Z_N is a quantizer process, we obtain

$$\mathbb{E}[U_L(Z_{N,R}(T))] \quad \leq \quad K\mathbb{E}[\Big(\bar{L}(Z_N^{0,t,z}(T))\Big)^{\gamma}]; \qquad \forall\, \alpha \in \mathcal{A}_n^{h,M,R}(t,z)$$

The use of functional quantization is crucial to derive the last inequality.
From the arbitrariness of α and recalling that $W^N_{i_1..i_{d(N)}}$ is a stationary quantizer, we obtain

$$\begin{aligned}
v^{h,R,N,M}(t,z) \quad &\leq \quad K\mathbb{E}[\Big(\bar{L}(Z_N^{0,t,z}(T))\Big)^{\gamma}] \\
&\leq \quad K\Big(\mathbb{E}[\bar{L}(Z_N^{0,t,z}(T))]\Big)^{\gamma} \\
&\leq \quad K\Big(x + \mathbb{E}[\frac{P_N^{0,t,p}(T)}{\lambda}]\Big)^{\gamma} \\
&\leq \quad K\Big(x + \mathbb{E}[\frac{P_T^{0,t,p}}{\lambda}]\Big)^{\gamma} \\
&\leq \quad K(x + \frac{p}{\lambda}e^{(b-\frac{\sigma^2}{2}+\sigma)(T-t)})^{\gamma} \\
&\leq \quad w(t,z)
\end{aligned}$$

where $w \in \mathcal{G}_\gamma([0,T]\times\bar{S})$ is a function independent from h, R, M and N.

$\square$

We now have proved that our numerical scheme verifies monotonicity, consistency and stability properties. Before proving the convergence of the discrete value function towards the real value function, we need to introduce the following proposition.

Proposition 5.5 *(i) A locally bounded function v is a viscosity supersolution of (2.12) on $[0,T)\times S$ if and only if for all $(\bar{t},\bar{z}) \in [0,T)\times S$ and for all $\phi \in C^{1,2}(N(\bar{t},\bar{z}))$ ($N(x)$:neighborhood of x) such that $v_* - \phi$ has a global minimum at $(\bar{t},\bar{z})$ and $(v_*-\phi)(\bar{t},\bar{z}) = 0$, we have*

$$\min\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{z}) - \mathcal{L}\phi(\bar{t},\bar{z}), \phi(\bar{t},\bar{z}) - \mathcal{H}\phi(\bar{t},\bar{z})\right] \quad \geq \quad 0$$

(ii) A locally bounded function v is a viscosity subsolution of (2.12) on $[0,T)\times\bar{S}$ if and only if for all $(\bar{t},\bar{z}) \in [0,T)\times\bar{S}$ and for all $\phi \in C^{1,2}(N(\bar{t},\bar{z}))$ such that $v^* - \phi$ has a global maximum at $(\bar{t},\bar{z})$ and $(v^*-\phi)(\bar{t},\bar{z}) = 0$, we have

$$\min\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{z}) - \mathcal{L}\phi(\bar{t},\bar{z}), \phi(\bar{t},\bar{z}) - \mathcal{H}\phi(\bar{t},\bar{z})\right] \quad \leq \quad 0$$

(iii) The function v is said to be a viscosity solution of (2.12), if it is both sub- and super-solution of (2.12).

$$\left(\quad f^*(x) = \limsup_{y\to x} f(y) \quad \textit{and} \quad f_*(x) = \liminf_{y\to x} f(y) \quad \right)$$

Proof. We will only show (ii) since (i) is derived similarly.
1.) Suppose that v is a viscosity subsolution of (2.12) on $[0,T)\times\bar{S}$.
If $(-\dfrac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z})\leq 0)$ then we have immediately that

$$\min\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z}),\phi(\bar{t},\bar{z})-\mathcal{H}\phi(\bar{t},\bar{z})\right] \;\leq\; 0$$

If $(v^*(\bar{t},\bar{z})-\mathcal{H}v^*(\bar{t},\bar{z})\leq 0)$, since $v^*(\bar{t},\bar{z})=\phi(\bar{t},\bar{z})$ and $v^*<\phi$ on $[0,T)\times\bar{S}\setminus\{\bar{t},\bar{z}\}$ implying that $\mathcal{H}v^*(\bar{t},\bar{z})\leq\mathcal{H}\phi(\bar{t},\bar{z})$, we obtain that $\phi(\bar{t},\bar{z})-\mathcal{H}\phi(\bar{t},\bar{z})\leq 0$ which means that

$$\min\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z}),\phi(\bar{t},\bar{z})-\mathcal{H}\phi(\bar{t},\bar{z})\right] \;\leq\; 0$$

2.) Suppose now that we have $\min\left[-\dfrac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z}),\phi(\bar{t},\bar{z})-\mathcal{H}\phi(\bar{t},\bar{z})\right]\leq 0$ where $(\bar{t},\bar{z})\in[0,T)\times\bar{S}$ and $\phi\in C^{1,2}(N(\bar{t},\bar{z}))$ such that $(v^*-\phi)(\bar{t},\bar{z})=\max(v^*-\phi)=0$.
Let us consider two strictly positive real numbers ξ_1 and ξ_2 and define

$$\phi_{\xi_1,\xi_2}(t,z):=\begin{cases}\phi(t,z) & if\ (t,z)\in B\big((\bar{t},\bar{z}),\xi_1\big)\\ v^*(t,z)+\xi_2 & if\ (t,z)\notin B\big((\bar{t},\bar{z}),\xi_1\big)\end{cases}$$

Hence, we have that $\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})=\phi(\bar{t},\bar{z})=v^*(\bar{t},\bar{z})$ and so by hypothesis we obtain

$$\min\left[-\frac{\partial\phi_{\xi_1,\xi_2}}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z}),\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})-\mathcal{H}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})\right]\leq 0$$

So if $-\dfrac{\partial\phi_{\xi_1,\xi_2}}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})\leq 0$, then we have that $-\dfrac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z})\leq 0$, and it is obvious that

$$\min\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z}),v^*(\bar{t},\bar{z})-\mathcal{H}v^*(\bar{t},\bar{z})\right]\leq 0$$

Else, if $\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})-\mathcal{H}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})\leq 0$, we obtain

$$\begin{aligned}v^*(\bar{t},\bar{z})=\phi_{\xi_1,\xi_2}(\bar{t},\bar{z}) &\leq \mathcal{H}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})\\ &\leq \limsup_{\xi_1,\xi_2\to 0}\mathcal{H}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})\\ &\leq \mathcal{H}\limsup_{\xi_1,\xi_2\to 0}\phi_{\xi_1,\xi_2}(\bar{t},\bar{z})\\ &\quad\text{(see Lemma 5.1 Ly Vath, Mnif and Pham (2007) for the proof)}\\ &= \mathcal{H}v^*(\bar{t},\bar{z})\end{aligned}$$

Hence, $v^*(\bar{t},\bar{z})-\mathcal{H}v^*(\bar{t},\bar{z})\leq 0$ and we obtain

$$\min\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{z})-\mathcal{L}\phi(\bar{t},\bar{z}),v^*(\bar{t},\bar{z})-\mathcal{H}v^*(\bar{t},\bar{z})\right]\leq 0$$

which completes the proof.

□

Theorem 5.1 *(Convergence)*
For all $(t,z) \in [0,T) \times S$ *we have that*

$$\lim_{\substack{(t^{'},z^{'})\to(t,z)\\(h,M,N,R)\to(0,+\infty)\\(t',z')\in\mathbb{T}_m\times\mathbb{Z}_l}} v^{h,M,N,R}(t',z') = v(t,z)$$

where $v^{h,R,N,M}$ *is the solution of the discrete HJB inequality without considering an iterative scheme and* v *is the solution of* (2.12).

Here, $(h,M,N,R) \to (0,+\infty)$ stands for $h \to 0$, $M \to +\infty$, $N \to +\infty$ and $R \to +\infty$. This notation will be used in the following proof.

Proof. Let $\overline{v}$ and $\underline{v}$ be defined by

$$\overline{v}(t,z) = \limsup_{\substack{(t^{'},z^{'})\to(t,z)\\(h,M,N,R)\to(0,+\infty)}} v^{h,M,N,R}(t^{'},z^{'}) \quad \textit{and} \quad \underline{v}(t,z) = \liminf_{\substack{(t^{'},z^{'})\to(t,z)\\(h,M,N,R)\to(0,+\infty)}} v^{h,M,N,R}(t^{'},z^{'}) \quad (5.12)$$

We suppose that $\overline{v}$ and $\underline{v}$ are respectively sub- and supersolution of (2.12). Assume for the moment that this claim is true, then, since $\overline{v}$ and $\underline{v}$ are respectively upper semi-continuous (usc) and lower semi-continuous (lsc), the comparison principle proved in Ly Vath, Mnif and Pham (2007) yields $\overline{v} \leq \underline{v}$ on $[0,T] \times S$. Although the opposite inequality is obvious by the very definition of $\overline{v}$ and $\underline{v}$, hence $\overline{v} = \underline{v} \equiv v$ is the unique continuous solution of (2.12) which combined with (5.12) implies the uniform convergence of $v^{h,M,N,R}$ to v.

Next we prove the above claim. Here we only consider the $\overline{v}$ case, since the argument for $\underline{v}$ is identical. To this end, let $(\bar{t},\bar{z})$ be a global maximum of $\overline{v} - \varphi$ on $[0,T] \times \bar{S}$ for some $\varphi \in C^{1,2}([0,T] \times \bar{S})$. Without any loss of generality, we may assume that $(\bar{t},\bar{z})$ is a strict global maximum such that $\overline{v}(t,z) - \varphi(t,z) \leq \overline{v}(\bar{t},\bar{z}) - \varphi(\bar{t},\bar{z}) = 0$ in $[0,T] \times \bar{S}$. We suppose also that $\varphi \geq C(1 + (x + \frac{p}{\lambda}))^{\gamma} + 1$ outside the ball $B((\bar{t},\bar{z}),r)$ where C is a positive constant and $r > 0$. From the stability property, having that $v^{h,M,N,R} \leq w$ s.t $w \in \mathcal{G}_{\gamma}$, we obtain that $v^{h,M,N,R} - \varphi \leq -1$, which yields $\overline{v} - \varphi \leq -1$ outside $B((\bar{t},\bar{z}),r)$. Notice here that C was chosen in such a way that $w \leq C(1 + (x + \frac{p}{\lambda}))^{\gamma}$.

We also have that $0 \geq (\overline{v} - \varphi)(t,z)$ for all $(t,z) \in [0,T] \times \bar{S}$. Since $\overline{v} - \varphi$ is upper semi-continuous, there exists $0 < r^{'} < r$ such that

$$\begin{aligned} 0 &\geq (\overline{v} - \varphi)(t,z) \geq -1, \quad \forall\, (t,z) \in B((\bar{t},\bar{z}),r^{'}) \\ 0 &\geq v^{h,M,N,R}(t,z) - \varphi(t,z) \geq -1, \quad \forall\, (t,z) \in B((\bar{t},\bar{z}),r^{'}) \end{aligned}$$

For convenience, we denote $\rho := (h,M,N,R)$. Let (t_ρ,z_ρ) be the maximum of $v^\rho - \varphi$ over the closed ball $B((\bar{t},\bar{z}),r^{'})$ which is a global maximum on $[0,T] \times \bar{S}$ and let $(\hat{t},\hat{z})$ be the limit of the subsequence also denoted (t_ρ,z_ρ), which exists due to the boundedness of

(t_ρ, z_ρ). So, by the definition of (t_ρ, z_ρ) and $(\bar{t}, \bar{z})$ we may obtain

$$
\begin{aligned}
0 = (\bar{v} - \varphi)(\bar{t}, \bar{z}) &= \limsup_{\substack{\rho \to (0,+\infty) \\ (t',z') \to (\bar{t},\bar{z}) \\ (t',z') \in B((\bar{t},\bar{z}), r')}} (v^\rho - \varphi)(t', z') \\
&\leq \limsup_{\substack{\rho \to (0,+\infty) \\ (t',z') \to (t^\rho,z^\rho) \\ (t',z') \in B((\bar{t},\bar{z}), r')}} (v^\rho - \varphi)(t', z') \\
&= \limsup_{\substack{\rho \to (0,+\infty) \\ (t',z') \to (\hat{t},\hat{z}) \\ (t',z') \in B((\bar{t},\bar{z}), r')}} (v^\rho - \varphi)(t', z') \\
&= (\bar{v} - \varphi)(\hat{t}, \hat{z}) \leq 0
\end{aligned}
$$

which means that $(\overline{v} - \varphi)(\hat{t}, \hat{z}) = 0$. On the other hand, noticing that $(\overline{v} - \varphi)(\bar{t}, \bar{z}) = 0$ and that $(\bar{t}, \bar{z})$ is a strict global maximum, then $\bar{t} = \hat{t}$ and $\bar{z} = \hat{z}$.
Hence, we have shown that there exists a subsequence (t_ρ, z_ρ) which is a global maximum on $[0, T] \times \bar{S}$ of $v^\rho - \varphi$, such that

$$
(t_\rho, z_\rho) \longrightarrow (\bar{t}, \bar{z}) \quad when \quad \rho \longrightarrow (0, +\infty)
$$

Since $\varphi \in C^{1,2}([0, T] \times \bar{S})$ and $v^{h,M,N,R} \leq w$, we have then

$$
\overline{v}(\bar{t}, \bar{z}) = \varphi(\bar{t}, \bar{z}) = \lim_{\rho \to (0,+\infty)} \varphi(t_\rho, z_\rho) = \lim_{\rho \to (0,+\infty)} v^\rho(t_\rho, z_\rho)
$$

We define

$$
\xi_\rho := v^\rho(t_\rho, z_\rho) - \varphi(t_\rho, z_\rho).
$$

Having that (t_ρ, z_ρ) is a global maximum we have that $(v^\rho - \varphi)(t, z) \leq (v^\rho - \varphi)(t_\rho, z_\rho) = \xi_\rho$ and so $v^\rho(t, z) \leq \varphi(t, z) + \xi_\rho$ for all $(t, z) \in [0, T] \times \bar{S}$. Hence, by the monotonicity of the numerical scheme S^ρ defined in (4.10) we obtain

$$
\begin{aligned}
0 = \frac{S^\rho\Big(t_\rho, z_\rho, v^\rho(t_\rho, z_\rho), v^\rho\Big)}{h} &\geq \frac{S^\rho\Big(t_\rho, z_\rho, v^\rho(t_\rho, z_\rho), \varphi + \xi_\rho\Big)}{h} \\
&= \frac{S^\rho\Big(t_\rho, z_\rho, \varphi(t_\rho, z_\rho) + \xi_\rho, \varphi + \xi_\rho\Big)}{h}
\end{aligned}
$$

Taking limits in the last inequality and using the consistency of S^ρ we get

$$
\begin{aligned}
0 &\geq \liminf_{\rho \to (0,+\infty)} \frac{S^\rho\Big(t_\rho, z_\rho, \varphi(t_\rho, z_\rho) + \xi_\rho, \varphi + \xi_\rho\Big)}{h} \\
&\geq \liminf_{\substack{(t,z,\xi) \to (\bar{t},\bar{z},0) \\ \rho \to (0,+\infty)}} \frac{S^\rho\Big(t, z, \varphi(t, z) + \xi, \varphi + \xi\Big)}{h} \\
&\geq \min\left[-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{z}) - \mathcal{L}\varphi(\bar{t}, \bar{z}), \varphi(\bar{t}, \bar{z}) - \mathcal{H}\varphi(\bar{t}, \bar{z})\right]
\end{aligned}
$$

The last inequality combined to the Proposition 5.5 shows that $\overline{v}$ is a subsolution of (2.12). $\square$

6 Numerical Results

In this section, we present numerical results, such as the value function and the optimal transaction strategy, obtained by applying the numerical procedure (4.11)-(4.12) described in Section 4. The parameter values used to perform the numerical tests are shown in the below Table 1 :

Parameter	Value	Parameter	Value
Maturity	1 year	x_{min}	-100
λ	5.00E(-07)	x_{max}	200
γ	0.5	y_{min}	-4
σ	0.25	y_{max}	20
b	0.1	p_{min}	0
k	1	p_{max}	50
		l	20
		m	40
		M	100
		N	96
		$\bar{\varepsilon}$	10^{-3}

Table 1 : *Parameters*

Using the above parameters, about 8 minutes are necessary to do the whole computation using Intel® Core 2 Duo at 2.00 Ghz CPU with 2.96 Go of RAM. The convergence is obtained after 13 iterations of the iterative scheme.

In the three below figures, we plot our numerical results (the value function and the associated optimal trading strategies for two different values of price impact λ) for $l = 40$.

Figure 1: The shape of the optimal transaction strategy (different regions).
We plot the shape of the optimal trading strategy (the policy) sliced in the plane (x, y) for a fixed (t, p). We can see four regions : a Buy region (denoted B on the graph), a Sell region (denoted S on the graph), a No-Transaction region (denoted N.T. on the graph) and a region where we are outside the domain $\bar{S}$ (denoted O.D. on the graph).
The financial interpretations of these results are as follows: when x is big and y is small, the investor has enough cash to buy shares of the risky asset in order to profit from an increased exposure. When y is large and x is small, the investor has to reduce exposure to the risky asset as well as to match the terminal liquidation constraint.

Figure 2: The Value function.
We plot the shape of the value function sliced in the plane (x, y) for a fixed (t, p). This figure is a typical pattern of the value function.

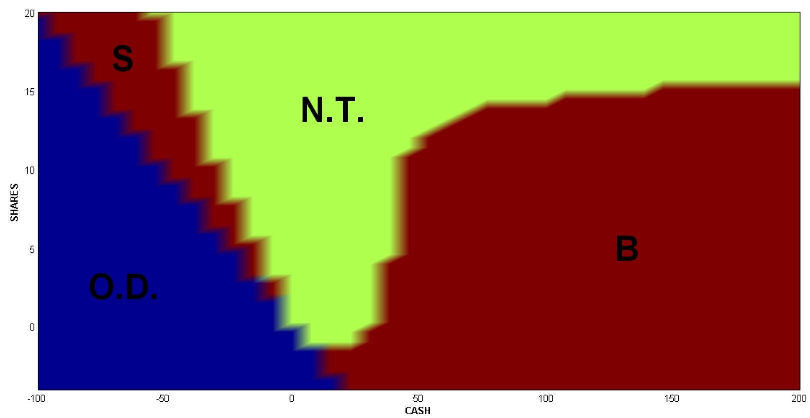

Figure 1: *The optimal policy sliced in XY*

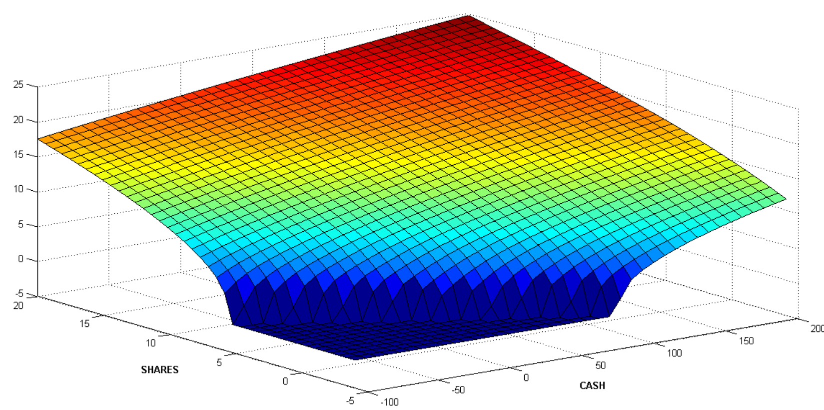

Figure 2: *Value function sliced in XY*

Figure 3: The shape of the optimal transaction strategy with a higher price impact (λ).
We plot the shape of the optimal consumption strategy sliced in the plane (x, y) for a fixed (t, p) and for a much higher price impact ($\lambda = 5.00E(-03)$) than in figure 1. We can see that the no trade region is larger than the one in figure 1 which is natural. Indeed, trading activities should be kept to a minimum when the liquidity price impact is high.

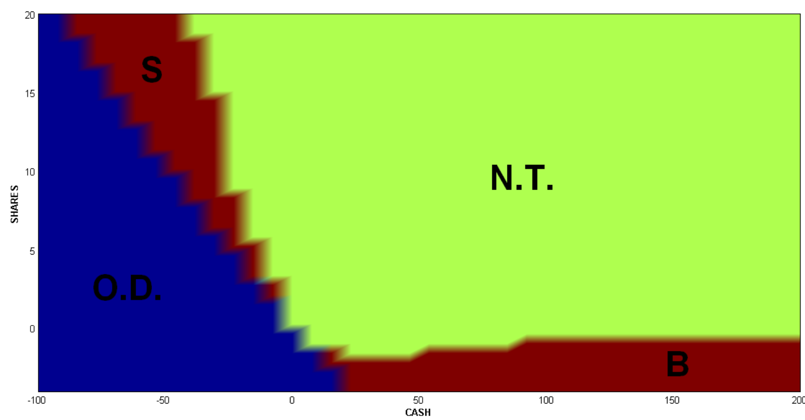

Figure 3: *Shape of the optimal policy for $\lambda = 5.00E(-03)$.*

Tables 2, 3, 4 and 5: Convergence analysis in function of N, M and R

In Table 2 and 3, we show some values of the value function for different values of N and M at a time $t = 0.05$, and for fixed nodes of the grid $z_1 = (x_1, y_1, p_1) = (42.10, 7.36, 23.68)$ and $z_2 = (x_2, y_2, p_2) = (121.052, 13.68, 36.84)$.

N	96	200
$v(t, z_1)$	15.8841	15.8807
$v(t, z_2)$	26.3895	26.3867

Table 2 : *Values of the value function for different values of N and z.*

M	200	250
$v(t, z_1)$	15.8825	15.8849
$v(t, z_2)$	26.3990	26.4011

Table 3 : *Values of the value function for different values of M and z.*

In Table 4, we show the number of iterations that the algorithm needs to converge when we vary the parameters N and M. Here, we choose the value of the discretization step in time and the number l of the grid's nodes as in Table 1.

	M=200	M=250	N=96	N=200
Number of iterations	13 Iterations	13 Iterations	13 Iterations	13 Iterations

Table 4 : *Number of iterations for different values of N and M.*

In Table 5, we show some values of the value function for two different values R_1 and R_2 of R (ie the boundaries) at a time $t = 0.05$, and for fixed nodes of the grid $z_3 = (x_3, y_3, p_3) = (57.89, 8.63, 36.84)$ and $z_4 = (x_4, y_4, p_4) = (184.21, 18.73, 21.05)$. Here, we choose the value of the discretization step in time, the number l of the grid's nodes and R_1 as in Table 1 and we choose R_2 as follows:

$$R_2 = \min\Big(\mid x_{min} = -257.90 \mid, \mid x_{max} = 342.10 \mid, \mid y_{min} = -16.63 \mid, \mid y_{max} = 31.36 \mid, \mid p_{max} = 100 \mid \Big)$$

R	R_1	R_2
$v(t, z_3)$	20.0038	19.9966
$v(t, z_4)$	24.5429	24.5340

Table 5 : *Values of the value function for different values of R and z.*

Figures 4 and 5: Relative error analysis of the value function for different values of N and M

In figure 4 (resp. 5), we plot the shape of the relative error between the value function computed for $N = 96$ (resp. $M = 200$) and the value function computed for $N = 200$ (resp. $M = 250$). The error is sliced in the plane (x, y) for a fixed (t, p). Figures 4 and 5 show that the convergence of the discrete value function, solution of our numerical algorithm (4.11)-(4.12), towards the real value function, solution of the HJB (2.12), can be obtained with only 96 points of quantization and 200 nodes of the discrete set of admissible strategies.

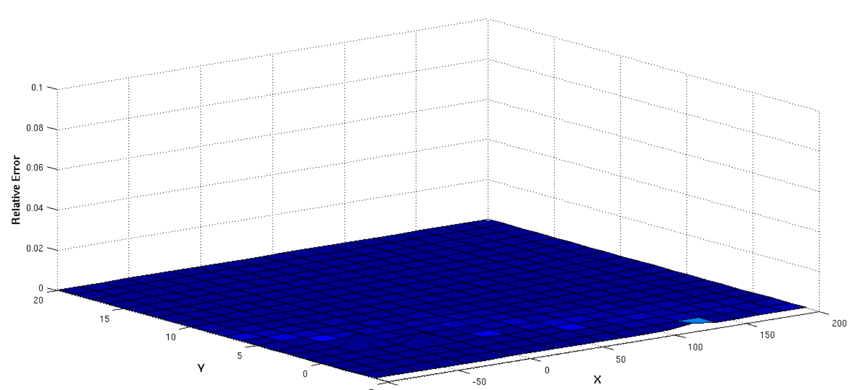

Figure 4: *Relative error of the value function computed when $N = 96$ Vs. $N = 200$.*

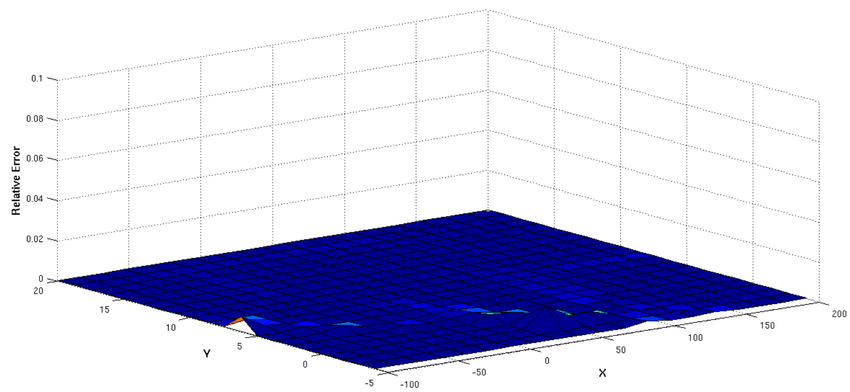

Figure 5: *Relative error of the value function when* $M = 200$ *Vs.* $M = 250$.

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