Conditional propagation of chaos for mean field systems of interacting neurons

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Introduction

The aim of this projet is to study the convergence of a particle system that modelizes a neural network, as the number of neurons goes to infinity. Before stating the formal model, let us briefly explain its particularities. To begin with, the neural networks we consider are mean field systems, what means that the neurons of a given network have all the same characteristics. In our model, it implies, for instance, that the spike rates of the neurons are identically distributed. The convergence we prove for this model can be called conditional propagation of chaos. A classical propagation of chaos for particle system means that, in the limit system (i.e. when the number of particles is infinite) the particles are independent. In this model, we will see that, in the limit system, our neurons will share a common noise, whence they will not be truly independent, but only independent conditionally to this common noise.

The $N-$neurons network

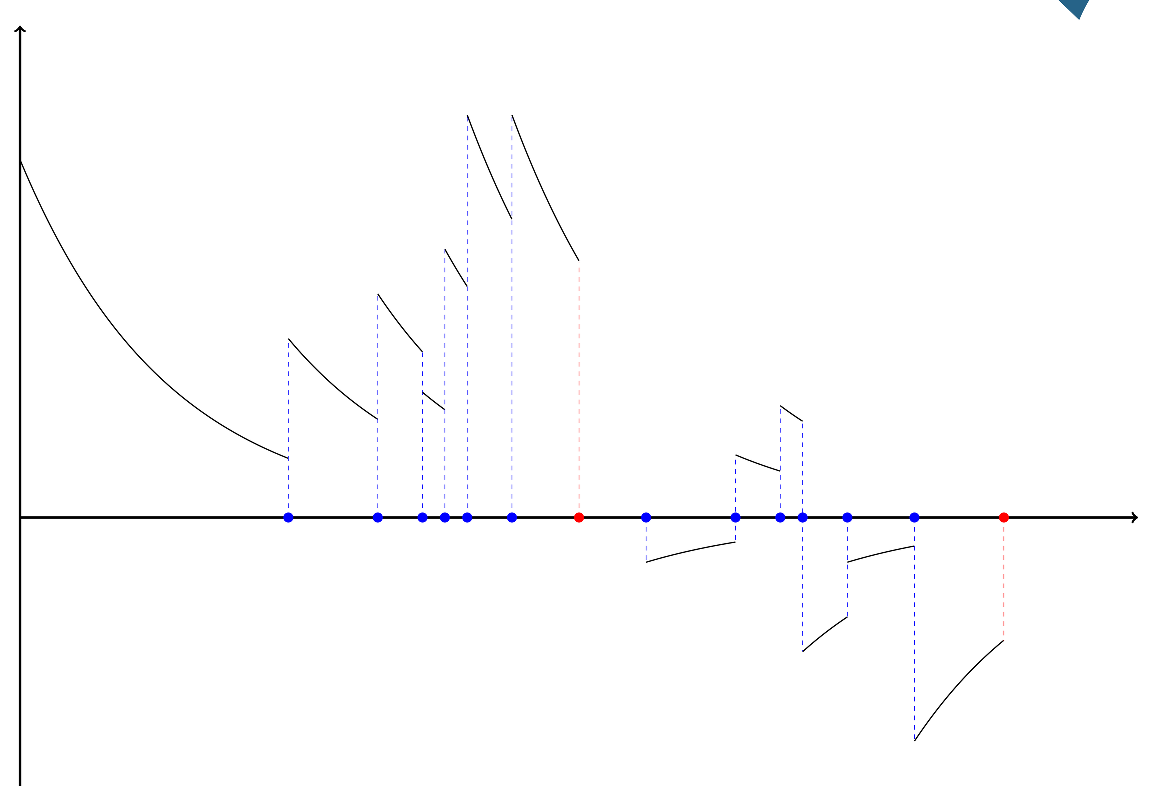
Let N be a positive integer. We consider a $N-$ dimensional system of stochastic processes $(X^{N,i})_{1\leq i\leq N}$ where $X^{N,i}$ represents the membrane potential of the $i-$th neuron of the network. The dynamics of the system can be described informally by:

- the $i-$th neuron emits spike at rate $f(X^{N,i}_{t-})$
- while no neurons emit spike in a time interval $[s,t]$, the evolution of $X^{N,i}$ is a (deterministic) negative exponential, modeling the loss of potential to its rest value 0
$$X^{N,i}_t = X^{N,i}_s e^{-\alpha(t-s)}$$
- if a neuron i emits a spike at time t, the potential of the other neurons receive an additional contribution of the form $U(t)/\sqrt{N}$ (where $U(t)$ is a centered random variable), and the potential of the neuron i jumps to the resting value 0
$$X^{N,i}_t = 0$$
$$X^{N,j}_t = X^{N,j}_{t-} + \frac{U(t)}{\sqrt{N}} \quad ,\forall j\neq i$$

Formally, this system satisfies the following stochastic differential equation
$$dX^{N,i}_t = -\alpha X^{N,i}_t dt - X^{N,i}_{t-}\int_0^\infty\int_{\mathbb{R}} \mathbb{1}_{\left\{z\leq f\left(X^{N,i}_{t-}\right)\right\}}\pi^i(dt,dz,du) + \frac{1}{\sqrt{N}}\sum_{\substack{j=1\\ j\neq i}}^N \int_0^\infty\int_{\mathbb{R}} u\mathbb{1}_{\left\{z\leq f\left(X^{N,j}_{t-}\right)\right\}}\pi^j(dt,dz,du)$$
where π^i are independent Poisson measures on $\mathbb{R}_+^2\times\mathbb{R}$ with intensity $dt\cdot dz\cdot d\nu(u)$ (where ν is the law of the variables $U(t)$ we introduced before)

As there are $N-1$ processes that can create jumps in each $X^{N,j}$, it is necessary to scale these jumps. A scaling N^{-1} would lead to a true propagation of chaos. Here, the scaling $N^{-1/2}$ leads to a common noise in the limit system, and to a conditional propagation of chaos. Let us note that, it is because of this particular scaling that we need to assume the random variables $U(t)$ to be centered.

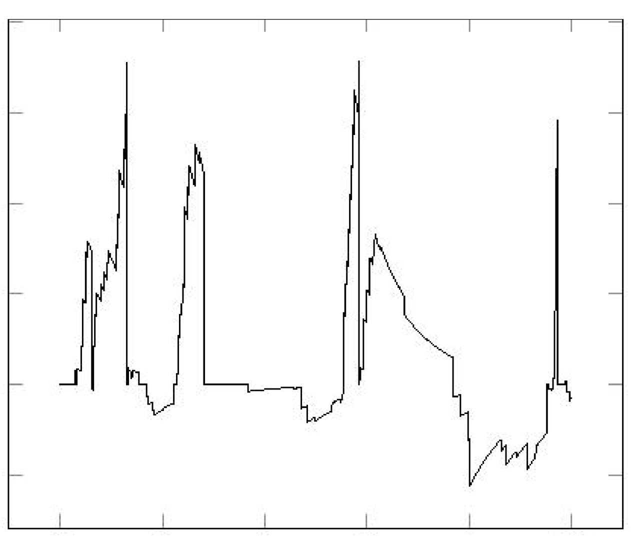
Dynamic of $X^{N,i}$



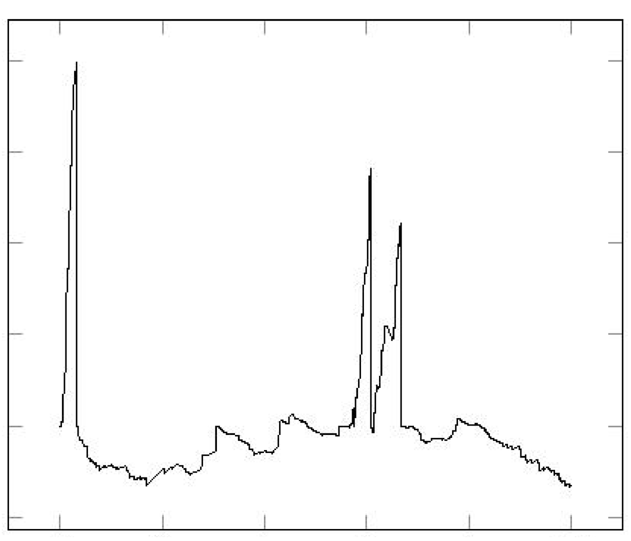
The limit network

Let us explain the form of the limit network with some heuristics. Looking closely at the equation of $X^{N,i}$, one can note that the only term that does depend on N is the term of the second line. Let us note M^N_t this term (to simplify we assume that the sum ranges over all $1\leq j\leq N$, including the index i)
$$M^N_t := \frac{1}{\sqrt{N}}\sum_{j=1}^N\int_0^t\int_0^\infty\int_{\mathbb{R}} u\mathbb{1}_{\left\{z\leq f\left(X^{N,j}_{s-}\right)\right\}}\pi^j(ds,dz,du)$$
If we find $\bar{M}_t$ the limit of M^N_t, the limit equation would be
$$d\bar{X}^i_t = -\alpha\bar{X}^i_t dt - \bar{X}^i_{t-}\int_0^\infty\int_{\mathbb{R}}\mathbb{1}_{\left\{z\leq f\left(\bar{X}^i_{t-}\right)\right\}}\pi^i(dt,dz,du) + d\bar{M}_t$$
Let us discuss the form of $\bar{M}_t$: it is the limit of the pure-jump local martingales M^N_t, and the height of the jumps of M^N_t vanish as N goes to infinity. This implies that $\bar{M}_t$ can be written as a stochastic integral w.r.t. a Brownian motion $(W_t)_{t\geq 0}$. In order to find the explicit form of $\bar{M}_t$, one has to find its quadratic variation, which is the limit of the quadratic variations of M^N_t
$$\mathbb{E}\left[[M^N]_t\right] = \frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^t\int_0^\infty\int_{\mathbb{R}} u^2\mathbb{1}_{\left\{z\leq f\left(X^{N,j}_{s-}\right)\right\}}d\pi^j(s,z,u)\right]$$
$$= \frac{1}{N}\sum_{j=1}^N\int_0^t\int_{\mathbb{R}} u^2\mathbb{E}f\left(X^{N,j}_s\right)d\nu(u)ds = \sigma^2\int_0^t\mathbb{E}\mu^N_s(f)ds$$
where $\sigma^2 := \int_{\mathbb{R}} u^2 d\nu(u)$ and $\mu^N_t := N^{-1}\sum_{j=1}^N\delta_{X^{N,j}_t}$ is the empirical measure of the system $(X^{N,j}_t)_{1\leq j\leq N}$. Then, denoting by $\bar{\mu}^N_t := N^{-1}\sum_{j=1}^N\delta_{\bar{X}^j}$ the empirical measure of the limit system, and admitting that $\bar{\mu}^N$ converges to some (random) measure μ, we know that M_t can be written as $M_t = \sigma\int_0^t\mu_s(f)^{1/2}dW_s$, whence the limit equation is
$$d\bar{X}^i_t = -\alpha\bar{X}^i_t dt - \bar{X}^i_{t-}\int_0^\infty\int_{\mathbb{R}}\mathbb{1}_{\left\{z\leq f\left(\bar{X}^i_{t-}\right)\right\}}\pi^i(dt,dz,du) + \sigma\mu_t(f)^{1/2}dW_t,$$
where W is a standard Brownian motion and μ is the limit of the empirical measure of $(\bar{X}^j)_{1\leq j\leq N}$.
The last thing to do is to guess the explicit form of μ. As a limit of empirical measures of exchangeable systems, μ is necessarily the directing measure of $(\bar{X}^j)_{j\geq 1}$: conditionally on μ, the variables $\bar{X}^j$ are i.i.d. with law μ. But, one can note that, conditionally on W, the variables $\bar{X}^j$ are i.i.d., and this implies that $\mu_t = \mathcal{L}(\bar{X}^i_t|W)$. Hence, we can rewrite the limit equation as
$$d\bar{X}^i_t = -\alpha\bar{X}^i_t dt - \bar{X}^i_{t-}\int_0^\infty\int_{\mathbb{R}}\mathbb{1}_{\left\{z\leq f\left(\bar{X}^i_{t-}\right)\right\}}\pi^i(dt,dz,du) + \sigma\mathbb{E}\left[f\left(\bar{X}^i_t\right)|W\right]^{1/2}dW_t$$

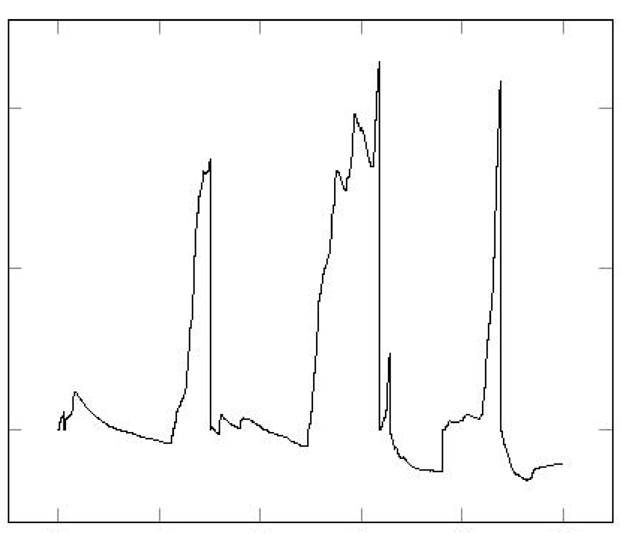
Simulation of $X^{10,1}$



Simulation of $X^{100,1}$



Simulation of $X^{1000,1}$



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