Conditional propagation of chaos for mean field systems of interacting neurons

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1 Introduction
   - Point process
   - Exchangeability
   - Modeling of neural network

2 Model
   - Definitions of the systems
   - Well-posedness of the limit system

3 Propagation of chaos
   - Martingale problem
   - Convergence of $\left(\mu^N\right)_N$
A point process $Z$ is:

- a random countable set of $\mathbb{R}_+: Z = \{ T_i : i \in \mathbb{N} \}$
- a random point measure on $\mathbb{R}_+: Z = \sum_{i \in \mathbb{N}} \delta_{T_i}$
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A process $\lambda$ is the stochastic intensity of $Z$ if:

$$\forall 0 \leq a < b, \mathbb{E}[Z([a, b])|\mathcal{F}_a] = \mathbb{E}\left[\int_a^b \lambda_t \, dt \bigg| \mathcal{F}_a\right]$$
Poisson measure

$E$ measurable space
**Poisson measure**

\[ E \text{ measurable space} \]

\[ \pi \text{ Poisson measure on } E : \text{random point measure that satisfies} \]

\[ \forall A, \pi(A) \] is Poisson variable,
\[ \forall A_1, \ldots, A_n \] disjoint, \((\pi(A_1), \ldots, \pi(A_n))\) independent.

Intensity of \( \pi \):
\[ \mu(A) = E[\pi(A)] \]
\( \mu \) characterizes the law of \( \pi \).
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\[ \pi \text{ Poisson measure on } \mathbb{R}_+ \times \mathbb{R}_+ \text{ with intensity } dt \cdot dz \]
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\( \lambda \) predictable and positive process
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\( \lambda \) predictable and positive process

\[ Z(A) = \int_{A \times \mathbb{R}_+} 1_{\{z \leq \lambda(t)\}} d\pi(t, z) \]
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\[ \lambda \text{ predictable and positive process} \]
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Then: \( \lambda \) is the stochastic intensity of \( Z \)
Definition

A system of r.v. \((X_i)_{i \in I}\) is exchangeable if:

for all finite permutation \(\sigma\), \(L((X_i)_{i \in I}) = L((X_{\sigma(i)})_{i \in I})\)
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Basic example: i.i.d. \(\Rightarrow\) exchangeable
Exchangeable system

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**Theorem (de Finetti’s theorem)**

Let \((X_i)_{i \in I}\) infinite and exchangeable. Then there exists a random measure \(\mu\) such that, conditionally on \(\mu\) the system \((X_i)_{i \in I}\) is i.i.d. \(\mu\)–distributed
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- \(\mu\) is unique a.s.
- \(\mu\) is the directing measure of \((X_i)_{i \in I}\)
Modeling in neuroscience

Neural activity = Set of spike times
Modeling in neuroscience

Neural activity  $=$  Set of spike times
                $=$  Point process (i.e. random set of $\mathbb{R}_+$)
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Network of $N$ neurons:

$Z^{N,i} = \text{set of spike times of neuron } i$

= point process with intensity $f(X^{N,i}_{t-})$

$X^{N,i} = \text{potential of neuron } i$
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Network of $N$ neurons:

$Z^N,i =$ set of spike times of neuron $i$

= point process with intensity $f(X^N,i)$

$X^N,i =$ potential of neuron $i$

Here, $X^N,i$ solves an SDE directed by $(Z^N,j)_{1 \leq j \leq N}$
Mean field limit

\( N \)-particle system:

\[
Z^{N,i}_t = \int_0^t \int_0^{\infty} 1 \{ z \leq f(X^{N,i}_s) \} \, d\pi^i(s, z)
\]

\[
dX^{N,i}_t = b(X^{N,i}_t) \, dt + \sum_{j=1}^{N} \int_0^{\infty} u^{ji}(t)1 \{ z \leq f(X^{N,j}_t) \} \, d\pi^j(t, z)
\]

\( \pi^j \) iid Poisson measures with intensity \( dt \cdot dz \)
Mean field limit

$N$–particle system:

- $Z_{t}^{N,i} = \int_{0}^{t} \int_{0}^{\infty} 1\{z \leq f(X_{s-}^{N,i})\} d\pi^{i}(s, z)$

- $dX_{t}^{N,i} = b(X_{t}^{N,i}) dt + \sum_{j=1}^{N} \int_{0}^{\infty} u^{ji}(t) 1\{z \leq f(X_{t-}^{N,j})\} d\pi^{j}(t, z)$

$\pi^{j}$ iid Poisson measures with intensity $dt \cdot dz$

Study the limit $N \to \infty \implies$ rescale the sum:
Mean field limit

$N$–particle system :

- $Z_{t}^{N,i} = \int_{0}^{t} \int_{0}^{\infty} 1\{z \leq f(X_{s_{-}}^{N,i})\} d\pi^{i}(s, z)$

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Study the limit $N \to \infty$ $\Rightarrow$ rescale the sum :

- linear scaling $N^{-1}$ (LLN) :
  
  [Delattre et al. (2016)] (Hawkes process, $u^{ji}(t) = 1$),
  [Chevallier et al. (2017)] ($u^{ji}(t) = w(j, i)$)
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Study the limit \( N \to \infty \implies \) rescale the sum :

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- diffusive scaling \( N^{-1/2} \) (CLT) :
  
  [E. et al. (2019)] random and centered \( u^{ji}(s) \)
Linear scaling

\[ dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{j=1, j\neq i}^{N} \int_{0}^{\infty} 1\{z \leq f(X_t^{N,j})\} d\pi^j(t, z) \]

\[ - \int_{0}^{\infty} X_t^{N,i} 1\{z \leq f(X_t^{N,i})\} d\pi^i(t, z) \]

\[ \pi^j \text{ iid Poisson measures with intensity } dt \cdot dz \]
Linear scaling

\[ dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{j=1}^{N} \int_{0}^{\infty} 1\{z \leq f(X_t^{N,j})\} d\pi^j(t, z) + \int_{0}^{t} X_{t-}^{N,i} 1\{z \leq f(X_{t-}^{N,i})\} d\pi^i(t, z) \]

\[ \pi^j \text{ iid Poisson measures with intensity } dt \cdot dz \]

Interpretation:

- **drift**: \(-\alpha x\) models an exponential loss of the potential
- **small jump of order** \(N^{-1}\): the effect of spike of one neuron to the potential of the others
- **reset jump**: the effect of spike of one neuron to its potential

\[ \text{[De Masi et al. (2015)] and [Fournier & Locherbach (2016)]} \]

\[ \text{Generalization to McKean-Vlasov frame [Andreis et al. (2018)]} \]

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Linear scaling

\[ dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{j=1}^{N} \int_{0}^{\infty} 1\{z \leq f(X_t^{N,j})\} d\pi^j(t, z) \]

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- **drift**: \(-\alpha x\) models an exponential loss of the potential
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Diffusive scaling

\[ dX_t^{N,i} = -\alpha X_t^{N,i} \, dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\{z \leq f(X_t^{N,j})\}} \, d\pi^j(t, z, u) \]

\[ - \int_0^\infty \int_{\mathbb{R}} X_t^{N,i} \mathbb{1}_{\{z \leq f(X_t^{N,i})\}} \, d\pi^i(t, z, u) \]

\( \pi^j \) iid Poisson measures with intensity \( dt \cdot dz \cdot d\nu(u) \)

\( \nu \) probability measure on \( \mathbb{R} \) centered with \( \int_{\mathbb{R}} |u|^3 d\nu(u) < \infty \)

\( \sigma^2 = \int_{\mathbb{R}} u^2 d\nu(u) \)
Diffusive scaling

\[
\begin{align*}
    dX_t^{N,i} &= -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{0}^{\infty} \int_{\mathbb{R}} u 1\{z \leq f(X_t^{N,j})\} \, d\pi^j(t, z, u) \\
    &\quad - \int_{0}^{\infty} \int_{\mathbb{R}} X_{t-}^{N,i} 1\{z \leq f(X_t^{N,i})\} \, d\pi^i(t, z, u)
\end{align*}
\]

\(\pi^j\) iid Poisson measures with intensity \(dt \cdot dz \cdot d\nu(u)\)
\(\nu\) probability measure on \(\mathbb{R}\) centered with \(\int_{\mathbb{R}} |u|^3 d\nu(u) < \infty\)
\(\sigma^2 = \int_{\mathbb{R}} u^2 d\nu(u)\)

Dynamic of \(X^{N,i}\):
- \(X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}\) if the system does not jump in \([s, t]\)
- \(X_t^{N,i} = X_{t-}^{N,i} + \frac{U}{\sqrt{N}}\) if a neuron \(j \neq i\) emits a spike at \(t\)
- \(X_t^{N,i} = 0\) if neuron \(i\) emits a spike at \(t\)
Limit system: heuristic (1)

\[ dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j=1 \atop j \neq i}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1\{z \leq f(X_t^{N,j})\} d\pi^j(t, z, u) \]

\[ -X_{t^-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1\{z \leq f(X_t^{N,i})\} d\pi^i(t, z, u) \]
Limit system: heuristic (1)

\[ dX^N_{t,i} = -\alpha X^N_{t,i} dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}\{z \leq f(X^N_{t,j})\} d\pi^j(t, z, u) \]

\[ -X^N_{t-} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}\{z \leq f(X^N_{t-})\} d\pi^i(t, z, u) \]
Limit system: heuristic (1)

\[
dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_t^{N,j})\}} d\pi^j(t, z, u)
\]

\[
- X_t^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1 \{z \leq f(X_t^{N,i})\} d\pi^i(t, z, u)
\]

\[
M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_s^{N,j})\}} d\pi^j(s, z, u)
\]
Limit system : heuristic (1)

\[ dX^N_{t,i} = -\alpha X^N_{t,i} \, dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{\mathbb{R}^+ \times \mathbb{R}} u1\{z \leq f(X^N_{t,j})\} \, d\pi^j(t, z, u) \]

\[ - X^N_{t-} \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(X^N_{t-})\} \, d\pi^i(t, z, u) \]

\[ M^N_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R}^+ \times \mathbb{R}} u1\{z \leq f(X^N_{s,j})\} \, d\pi^j(s, z, u) \]

\[ d\bar{X}_t = -\alpha \bar{X}_t \, dt + d\bar{M}_t \]

\[ - \bar{X}_{t-} \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(\bar{X}_{t-})\} \, d\pi^i(t, z, u) \]
Limit system: heuristic (2)

\[ M^N_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R} \times \mathbb{R}} u^1 \{ z \leq f(X^N_{s,j}) \} d\pi^j(s, z, u) \]
Limit system : heuristic (2)

\[ M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u_1 \{ z \leq f(X_{s-N}^j) \} d\pi^j(s, z, u) \]

\( \bar{M} \) is an integral wrt a BM \( W \)
Limit system : heuristic (2)

\[ M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u1 \{ z \leq f(X_{s-N}^j) \} \, d\pi^j(s, z, u) \]

\( \tilde{M} \) is an integral w.r.t. a BM \( W \)

\[ \langle \tilde{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_{s-N}^j) \, ds \]
Limit system: heuristic (2)

\[ M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u1 \left\{ z \leq f(X^{N,j}_s) \right\} d\pi^j(s, z, u) \]

\( \tilde{M} \) is an integral wrt a BM \( W \)

\[ \langle \tilde{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X^{N,j}_s) ds \]

Then \( \tilde{M} \) should satisfy

\[ \tilde{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}^j_s) dW_s} = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}^N_s(f) dW_s} \]

with \( \bar{\mu}^N := \frac{1}{N} \sum_{j=1}^{N} \delta \bar{X}_j \)
Limit system: heuristic (3)

\[ \tilde{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} \, dW_s \quad \text{where} \quad \mu = \lim_{N} \mu^N \]
Limit system: heuristic (3)

\[ \tilde{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s \text{ where } \mu = \lim_{N} \tilde{\mu}_N \]

\[ d\tilde{X}^i_t = -\alpha \tilde{X}^i_t dt + \sigma \sqrt{\mu_t(f)} dW_t \]

\[ -\tilde{X}^i_{t_-} \int_{\mathbb{R}_+ \times \mathbb{R}} 1\{z \leq f(\tilde{X}^i_{t_-})\} d\pi^i(t, z, u) \]
Limit system: heuristic (3)

\[ \tilde{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s \quad \text{where} \quad \mu = \lim_{N} \bar{\mu}^N \]

\[ d\tilde{X}_t^i = -\alpha \tilde{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \]
\[ - \tilde{X}_t^- \int \mathbb{R}^+ \times \mathbb{R} 1_{\{z \leq f(\tilde{X}_t^-)\}} d\pi^i(t, z, u) \]

\( \mu \) is the limit of empirical measures of \((\tilde{X}^i)_{i \geq 1}\) exchangeable by Proposition (7.20) of [Aldous (1983)] \( \mu \) is the directing measure of \((\tilde{X}^i)_{i \geq 1}\) (conditionally on \( \mu \), \( \tilde{X}^i \) i.i.d. \( \sim \) \( \mu \) )
Limit system: heuristic (3)

\[ \tilde{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s \text{ where } \mu = \lim_N \mu^N \]

\[ d\tilde{X}_t^i = -\alpha \tilde{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \]
\[ \quad - \tilde{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbbm{1}_{\{z \leq f(\tilde{X}_{t-}^i)\}} d\pi^i(t, z, u) \]

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Conditionally on \( W \), the \( \tilde{X}^i \) \((i \geq 1)\) are i.i.d.
by Lemma (2.12) of [Aldous (1983)] \( \mu = \mathcal{L}(\tilde{X}^1|W) = \mathcal{L}(\tilde{X}^i|W) \)
Limit system: heuristic (3)

\[ \bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s \text{ where } \mu = \lim_N \bar{\mu}^N \]

\[ d\bar{X}^i_t = -\alpha \bar{X}^i_t dt + \sigma \sqrt{\mu_t(f)} dW_t \]
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\( \mu \) is the limit of empirical measures of \((\bar{X}^i)_{i \geq 1}\) exchangeable by Proposition (7.20) of [Aldous (1983)] \( \mu \) is the directing measure of \((\bar{X}^i)_{i \geq 1}\) (conditionally on \(\mu, \bar{X}^i\) i.i.d. \(\sim \mu\))

Conditionally on \(W\), the \(\bar{X}^i\) \((i \geq 1)\) are i.i.d. by Lemma (2.12) of [Aldous (1983)] \(\mu = \mathcal{L}(\bar{X}^1|W) = \mathcal{L}(\bar{X}^i|W)\)
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\[ \tilde{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s \text{ where } \mu = \lim_{N} \tilde{\mu}^N \]

\[ d\tilde{X}_t^i = -\alpha \tilde{X}_t^i dt + \sigma \sqrt{\mathbb{E} [f(\tilde{X}_t^i) | W]} dW_t \]

\[ -\tilde{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\tilde{X}_{t-}^i)\}} d\pi^i(t, z, u) \]

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Conditionally on \( W \), the \( \tilde{X}_i^i \) \( (i \geq 1) \) are i.i.d.
by Lemma (2.12) of [Aldous (1983)] \( \mu = \mathcal{L}(\tilde{X}_1^1 | W) = \mathcal{L}(\tilde{X}_i^i | W) \)
Well-posedness of the limit equation (1)

\[ d\tilde{X}_t^i = -\alpha \tilde{X}_t^i dt + \sigma \sqrt{\mathbb{E} \left[ f(\tilde{X}_t^i) | \mathcal{W} \right]} dW_t \]

\[ - \tilde{X}_t^i - \int_{\mathbb{R}_+ \times \mathbb{R}} 1\{z \leq f(\tilde{X}_t^-)\} d\pi^i(t, z, u) \]

Problems:

- conditional expectation in the Brownian term (McKean-Vlasov frame)
- unbounded jumps (non-Lipschitz compensator \( x \mapsto -xf(x) \))
- jump term and Brownian term
Well-posedness of the limit equation (1)

\[
d\tilde{X}_t^i = -\alpha \tilde{X}_t^i dt + \sigma \sqrt{\mathbb{E} \left[ f(\tilde{X}_t^i) | W \right]} dW_t \\
- \tilde{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1\{z \leq f(\tilde{X}_{t-}^i)\} d\pi^i(t, z, u)
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Problems:
- conditional expectation in the Brownian term (McKean-Vlasov frame)
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Solution: consider \( a : \mathbb{R} \rightarrow \mathbb{R}_+ \) increasing, bounded, lower-bounded, \( C^2 \) such that

\[
|a''(x) - a''(y)| + |a'(x) - a'(y)| \\
+ |xa'(x) - ya'(y)| + |f(x) - f(y)| \leq C|a(x) - a(y)|
\]
Well-posedness of the limit equation (2)

\[
a(\bar{X}_t^i) = a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s \\
+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds \\
+ \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} [a(0) - a(\bar{X}_{s-}^i)] 1\{z \leq f(\bar{X}_{s-}^i)\} d\pi^i(s, z, u)
\]
Well-posedness of the limit equation (2)

\[
a(\tilde{X}_t^i) = a(\tilde{X}_0^i) - \alpha \int_0^t \tilde{X}_s^i a'(\tilde{X}_s^i) ds + \sigma \int_0^t a'(\tilde{X}_s^i) \sqrt{\mathbb{E}[f(\tilde{X}_s^i)|W]} dW_s
\]

\[
+ \frac{\sigma^2}{2} \int_0^t a''(\tilde{X}_s^i) \mathbb{E}[f(\tilde{X}_s^i)|W] ds
\]

\[
+ \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} [a(0) - a(\tilde{X}_{s-}^i)] 1 \{z \leq f(\tilde{X}_{s-}^i)\} d\pi^i(s, z, u)
\]

To prove trajectory uniqueness:

- \(u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\tilde{X}_s^i)|]\) (problem with Brownian term)
- \(u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\tilde{X}_s^i)|^2]\) (problem with jump term)
Well-posedness of the limit equation (2)

\[ a(\bar{X}_t^i) = a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s \]

\[ + \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds \]

\[ + \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} [a(0) - a(\bar{X}_{s-}^i) \mathbb{1}_{\{z \leq f(\bar{X}_{s-}^i)\}} d\pi^i(s, z, u) \]

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- \( u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\bar{X}_s^i)|] \) (problem with Brownian term)
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Idea of [Graham (1992)] : \( u(t) = \mathbb{E} \left[ \sup_{0 \leq s \leq t} |a(\hat{X}_s^i) - a(\bar{X}_s^i)| \right] \)
Well-posedness of the limit equation (2)

\[ a(\bar{X}_{t}^i) = a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s \]
\[ + \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds \]
\[ + \int_{[0,t] \times \mathbb{R}^+ \times \mathbb{R}} [a(0) - a(\bar{X}_{s^-}^i)] 1_{\{z \leq f(\bar{X}_{s^-}^i)\}} d\pi^i(s, z, u) \]

To prove trajectorial uniqueness:

- \( u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|] \) (problem with Brownian term)
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Idea of [Graham (1992)]: \( u(t) = \mathbb{E} \left[ \sup_{0 \leq s \leq t} |a(\hat{X}_s^i) - a(\check{X}_s^i)| \right] \)

\( \forall t \geq 0, u(t) \leq C(t + \sqrt{t})u(t) \)
Well-posedness of the limit equation (2)

\[
a(\tilde{X}_t^i) = a(\tilde{X}_0^i) - \alpha \int_0^t \tilde{X}_s^i a'(\tilde{X}_s^i) \, ds + \sigma \int_0^t a'(\tilde{X}_s^i) \sqrt{\mathbb{E}[f(\tilde{X}_s^i)\mid W]} \, dW_s \\
+ \frac{\sigma^2}{2} \int_0^t a''(\tilde{X}_s^i) \mathbb{E}[f(\tilde{X}_s^i)\mid W] \, ds \\
+ \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} \left[a(0) - a(\tilde{X}_s^-^i)\right]1_{\{z \leq f(\tilde{X}_s^-^i)\}} \, d\pi^i(s, z, u)
\]

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- \( u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\tilde{X}_s^i)|] \) (problem with Brownian term)
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Well-posedness of the limit equation (2)

\[
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\]

To prove trajectorial uniqueness:

- \( u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|] \) (problem with Brownian term)
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Idea of [Graham (1992)] : \( u(t) = \mathbb{E} \left[ \sup_{0 \leq s \leq t} |a(\hat{X}_s^i) - a(\check{X}_s^i)| \right] \)

\( \forall t \geq 0, u(t) \leq C(t + \sqrt{t}) u(t) \implies \exists t_0 > 0, u(t_0) = 0 \)

Iteratively \( \forall n \in \mathbb{N}, u(nt_0) = 0, \) whence \( \forall t > 0, u(t) = 0 \)
Discussion about the function $f$

Any $f \in C_b^1(\mathbb{R}, \mathbb{R}_+)$ satisfying $f'(x) \leq C(1 + |x|)^{-(1+\epsilon)}$ ($\epsilon > 0$)
Discussion about the function $f$

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$f(x) = c + d \arctan(\alpha + \beta x)$ satisfy the hypothesis
Discussion about the function $f$

Any $f \in C^1_b(\mathbb{R}, \mathbb{R}_+)$ satisfying $f'(x) \leq C(1 + |x|)^{(1+\varepsilon)}$ ($\varepsilon > 0$)

$f(x) = c + d \arctan(\alpha + \beta x)$ satisfy the hypothesis

\[ X_{N, i} > x_0 \approx X_{N, i} < x_0 \]
Discussion about the function \( f \)

Any \( f \in C^1_b(\mathbb{R}, \mathbb{R}_+) \) satisfying \( f'(x) \leq C(1 + |x|)^{-(1+\varepsilon)} \) (\( \varepsilon > 0 \))

\[
f(x) = c + d \arctan(\alpha + \beta x)
\]

satisfy the hypothesis

"Neuron \( i \) active / inactive" \( \approx "X^{N,i} > x_0 \) / \( X^{N,i} < x_0" \)
Simulations of $X^{N,1}$

- $N = 10$
- $N = 1000$
Another version of the limit system

The strong limit system:

\[ d\bar{X}_t^i = -\alpha \bar{X}_t^i \, dt + \sigma \sqrt{\mathbb{E} \left[ f(\bar{X}_t^i) \right]} \, dW_t \]

\[ - \bar{X}_t^i - \int_{\mathbb{R}^+ \times \mathbb{R}} 1_{\{z \leq f(\bar{X}_{t-}^i)\}} \, d\pi^i(t, z, u) \]
Another version of the limit system

The strong limit system:

\[
d\tilde{X}_t^i = -\alpha \tilde{X}_t^i dt + \sigma \sqrt{\mathbb{E} \left[ f(\tilde{X}_t^i) \mid W \right]} dW_t
- \tilde{X}_t^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\tilde{X}_{t-}^i)\}} d\pi^i(t, z, u)
\]

The weak limit system:

\[
d\tilde{Y}_t^i = -\alpha \tilde{Y}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t
- \tilde{Y}_t^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\tilde{Y}_{t-}^i)\}} d\pi^i(t, z, u)
\]

where \(\mu_t = \mathcal{L}(\tilde{Y}_t^1 \mid \mu_t)\) is the directing measure of \((\tilde{Y}_t^i)_{i \geq 1}\)
Equivalence between the two systems

An auxiliary system:

\[ d\tilde{X}_t^{N,i} = -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N} \sum_{j=1}^{N} f(\tilde{X}_t^{N,j})} dW_t \]

\[ - \tilde{X}_t^{N,i} \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(\tilde{X}_t^{N,j})\} d\pi^i(t, z, u) \]
Equivalence between the two systems

An auxiliary system:

\[
\begin{align*}
    d\tilde{X}_t^{N,i} &= -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N} \sum_{j=1}^{N} f(\tilde{X}_t^{N,j})} dW_t \\
    &- \tilde{X}_t^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1\{z \leq f(\tilde{X}_t^{N,j})\} d\pi^i(t, z, u)
\end{align*}
\]

Let \( u_N(t) = \mathbb{E} \left[ \sup_{s \leq t} |a(\tilde{Y}^1_s) - a(\tilde{X}^{N,1}_s)| \right] \)
Equivalence between the two systems

An auxiliary system:

\[
\begin{align*}
    d\tilde{X}_{t}^{N,i} &= -\alpha\tilde{X}_{t}^{N,i}dt + \sigma \sqrt{\frac{1}{N}\sum_{j=1}^{N} f(\tilde{X}_{t}^{N,j})}dW_{t} \\
    &\quad - \tilde{X}_{t-}^{N,i} \int_{\mathbb{R}^{+} \times \mathbb{R}} 1\{z \leq f(\tilde{X}_{t-}^{N,j})\} d\pi^{i}(t, z, u)
\end{align*}
\]

Let \( u_{N}(t) = \mathbb{E} \left[ \sup_{s \leq t} |a(\bar{Y}_{s}^{1}) - a(\tilde{X}_{s}^{N,1})| \right] \)

\[
u_{N}(t) \leq C(t + \sqrt{t})u_{N}(t) + \mathcal{C}N^{-1/2} \mu_{s}(f) - N^{-1} \sum_{j=1}^{N} f(\bar{Y}_{s}^{j})
\]
Equivalence between the two systems

An auxiliary system:

\[
d\tilde{X}_t^{N,i} = -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N} \sum_{j=1}^{N} f(\tilde{X}_t^{N,j})} dW_t - \tilde{X}_t^{N,i} - \int_{\mathbb{R}_+ \times \mathbb{R}} 1\{z \leq f(\tilde{X}_t^{N,j})\} d\pi^i(t, z, u)
\]

Let \( u_N(t) = \mathbb{E}\left[\sup_{s \leq t} |a(\tilde{Y}_s^1) - a(\tilde{X}_s^{N,1})|\right] \)

\[
u_N(t) \leq C(t + \sqrt{t}) u_N(t) + C N^{-1/2} \mu_s(f) - N^{-1} \sum_{j=1}^{N} f(\tilde{Y}_s^j)
\]

For \( 0 \leq t \leq T \) (small enough)

\[
u_N(t) \leq C N^{-1/2} \quad \text{as } N \to \infty
\]
Convergence of \((X^N,i)_{1 \leq i \leq N}\)

\[dX^N,i_t = - \alpha X^N,i_t \, dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u1\{z \leq f(X^N,j)\} \, d\pi^j(t, z, u)\]

\[d\bar{X}^i_t = - \alpha \bar{X}^i_t \, dt + \sigma \sqrt{\mu_t(f)} \, dW_t\]

\[d\bar{X}^i_t = - \bar{X}^i_t \, dt + \sqrt{\mu_t(f)} \, dW_t\]

Goal : \((X^N,i)_{1 \leq i \leq N}\) converges to \((\bar{X}^i)_{i \geq 1}\) in \(D^{\mathbb{N}^*}\)
Convergence of \((X_N^i)_{1 \leq i \leq N}\)

\[
dX_t^{N,i} = -\alpha X_t^{N,i} \, dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u1 \{z \leq f(X_{t-}^{N,j})\} \, d\pi^j(t, z, u) \\
- X_t^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1 \{z \leq f(X_{t-}^{N,i})\} \, d\pi^i(t, z, u)
\]

\[
d\bar{X}^i_t = -\alpha \bar{X}^i_t \, dt + \sigma \sqrt{\mu_t(f)} \, dW_t \\
- \bar{X}_t^{i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1 \{z \leq f(\bar{X}_{t-}^{i})\} \, d\pi^i(t, z, u)
\]

Goal: \((X^N,i)_{1 \leq i \leq N}\) converges to \((\bar{X}^i)_{i \geq 1}\) in \(D^{\mathbb{N}^*}\)

Equivalent condition (Proposition (7.20) of [Aldous (1983)]):
\(\mu^N := \sum_{j=1}^{N} \delta_{X_N^j}\) converges to \(\mu := \mathcal{L}(\bar{X}^1|W)\) in \(\mathcal{P}(D)\)
Outline of the proof

**Step 1.** $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
Equivalent condition : $(X^{N,1})_N$ is tight on $D$
Proof : Aldous’ criterion

**Step 2.** Identifying the limit distribution of $(\mu^N)_N$
Proof : any limit of $\mu^N$ is solution of a martingale problem
Martingale problem: Principle

SDE:

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_+ \times E} \Phi(X_t, u)1\{z \leq f(X_t)\} d\pi(t, z, u) \]
Martingale problem: Principle

SDE:
\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_+ \times E} \Phi(X_t-, u)1_{\{z \leq f(X_t-)\}} d\pi(t, z, u) \]

Martingale problem: for \( g \) smooth

\[ g(Y_t) - g(Y_0) - \int_0^t Lg(Y_s)ds \] is a local martingale,

\[ Lg(x) = b(x)g'(x) + \frac{1}{2} \sigma(x)^2 g''(x) + f(x) \int_E (g(x + \Phi(x, u)) - g(x)) d\nu(u) \]
Martingale problem: Principle

**SDE:**

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_+ \times E} \Phi(X_{t-}, u) 1_{\{z \leq f(X_{t-})\}} d\pi(t, z, u) \]

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SDE ⇒ martingale problem: Ito’s formula

\[ g(X_t) - g(X_0) - \int_0^t Lg(X_s)ds = \int_0^t \sigma(X_s)g'(X_s)dW_s \]

\[ + \int_0^t \int_0^\infty \int_E (g(X_{s-} + \Phi(X_{s-}, u)) - g(X_{s-})) 1_{\{z \leq f(X_{s-})\}} d\tilde{\pi}(s, z, u) \]
Martingale problem : Principle

SDE:
\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_+ \times E} \Phi(X_{t-}, u)1_{\{z \leq f(X_{t-})\}} d\pi(t, z, u) \]

Martingale problem : for \( g \) smooth
\[ g(Y_t) - g(Y_0) - \int_0^t Lg(Y_s) ds \]
\[ \text{is a local martingale,} \]
\[ Lg(x) = b(x)g'(x) + \frac{1}{2} \sigma(x)^2 g''(x) + f(x) \int_E (g(x + \Phi(x, u)) - g(x)) d\nu(u) \]

SDE \Rightarrow \text{martingale problem : Ito’s formula}
\[ g(X_t) - g(X_0) - \int_0^t Lg(X_s) ds = \int_0^t \sigma(X_s)g'(X_s)dW_s \]
\[ + \int_0^t \int_0^\infty \int_E (g(X_{s-} + \Phi(X_{s-}, u)) - g(X_{s-})) 1_{\{z \leq f(X_{s-})\}} d\tilde{\pi}(s, z, u) \]

Martingale problem \Rightarrow \text{SDE : representation theorems}
Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ ($Q = \mathcal{L}(\mu)$)
Martingale problem

Given \( Q \in \mathcal{P}(\mathcal{P}(D)) \) (\( Q = \mathcal{L}(\mu) \))

Canonical space \( \Omega := \mathcal{P}(D) \times D^2 \) with \( \omega = (\mu, (Y^1, Y^2)) : \)

Meaning : \((Y^1, Y^2)\) mixture of iid directed by \( \mu \)
Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ ($Q = \mathcal{L}(\mu)$)

Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$ :

Meaning: $(Y^1, Y^2)$ mixture of iid directed by $\mu$

$$P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m)m \otimes m(B)dQ(m)$$
Martingale problem

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Canonical space \( \Omega := \mathcal{P}(D) \times D^2 \) with \( \omega = (\mu, (Y^1, Y^2)) \):

Meaning: \( (Y^1, Y^2) \) mixture of iid directed by \( \mu \)

\[
P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m)m \otimes m(B)dQ(m)
\]

\( Q \) is solution of \( (\mathcal{M}) \) if for all \( g \in C^2_b (\mathbb{R}^2) \),

\[
g(Y^1_t, Y^2_t) - g(Y^1_0, Y^2_0) - \int_0^t Lg(\mu_s, Y^1_s, Y^2_s)ds \text{ is a martingale}
\]
Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D)) \ (Q = \mathcal{L}(\mu))$

Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$:

Meaning: $(Y^1, Y^2)$ mixture of iid directed by $\mu$

$$P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m)m \otimes m(B) dQ(m)$$

$Q$ is solution of $(\mathcal{M})$ if for all $g \in C^2_b(\mathbb{R}^2)$,

$$g(Y^1_t, Y^2_t) - g(Y^1_0, Y^2_0) - \int_0^t Lg(\mu_s, Y^1_s, Y^2_s) ds$$ is a martingale

$$Lg(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^2 \partial_{i,j}^2 g(x) + f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x))$$
Uniqueness for the martingale problem

Let $Q$ be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where $\mu$ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i \geq 1}$
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$$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P$$ (from the martingale problem)
Uniqueness for the martingale problem

Let $Q$ be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where $\mu$ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i \geq 1}$

$$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P \text{ (from the martingale problem)}$$

Representation theorems imply (admitted)

$$\forall i \in \{1, 2\}, \ d\bar{Y}^i_t = -\alpha \bar{Y}^i_t dt + \sqrt{\mu_t(f)} dW_t$$

$$- \bar{Y}^i_{t-} \int_{\mathbb{R}^+} 1_{\{z \leq f(\bar{Y}^i_{t-})\}} d\pi^i(t, z)$$
Uniqueness for the martingale problem

Let $Q$ be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where $\mu$ is the directing measure of some exchangeable system $(\bar{Y}_i)_{i \geq 1}$

$$\mathcal{L}(\mu, \bar{Y}_1, \bar{Y}_2) = P \text{ (from the martingale problem)}$$

Representation theorems imply (admitted)

$$\forall i \in \mathbb{N}^*, \ d\bar{Y}^i_t = -\alpha \bar{Y}^i_t dt + \sqrt{\mu_t(f)} dW_t$$

$$- \bar{Y}^i_{t-} \int_{\mathbb{R}^+} 1_{\{z \leq f(\bar{Y}_{t-}^i)\}} d\pi^i(t, z)$$
Uniqueness for the martingale problem

Let $Q$ be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where $\mu$ is the directing measure of some exchangeable system $(\tilde{Y}^i)_{i \geq 1}$

$\mathcal{L}(\mu, \tilde{Y}^1, \tilde{Y}^2) = P$ (from the martingale problem)

Representation theorems imply (admitted)

$$\forall i \in \mathbb{N}^*, d\tilde{Y}_t^i = -\alpha \tilde{Y}_t^i dt + \sqrt{\mu_t(f)} dW_t$$

$$- \tilde{Y}_t^i - \int_{\mathbb{R}^+} 1\{z \leq f(\tilde{Y}_{t-}^i)\} d\pi^i(t, z)$$

Then the law of $\mu = \mathcal{L}(\tilde{Y}^1|\mathcal{W})$ is uniquely determined
Convergence of $\mu^N$ to the solution of $(\mathcal{M})$

Let $\mu$ be the limit of (a subsequence of) $\mu^N$

$\mathcal{L}(\mu)$ is solution of $(\mathcal{M})$ if

$$
\mathbb{E} [F(\mu)] = 0
$$

for any $F$ of the form

$$
F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\left[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t L\phi(m_r, \gamma_r)dr\right]
$$
Convergence of $\mu^N$ to the solution of ($M$)

Let $\mu$ be the limit of (a subsequence of) $\mu^N$

$L(\mu)$ is solution of ($M$) if

$$\mathbb{E}[F(\mu)] = 0$$

for any $F$ of the form

$$F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})\ldots\phi_k(\gamma_{s_k})\left[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t L\phi(m_r, \gamma_r)dr\right]$$
Convergence of $\mu^N$ to the solution of $(M)$

Let $\mu$ be the limit of (a subsequence of) $\mu^N$

$\mathcal{L}(\mu)$ is solution of $(M)$ if

$$E[F(\mu)] = 0$$

for any $F$ of the form

$$F(m) := \int_{D^2} m \otimes m(d\gamma) \phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k}) \left[ \phi(\gamma_t) - \phi(\gamma_s) \right]$$

$$+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr$$

$$- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r))dr - \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr$$

$$- \frac{\sigma^2}{2} \int_s^t m_r(f) \sum_{i_1, i_2=1}^{2} \partial_{i_1, i_2}^2 \phi(\gamma_r)dr$$
The expression of $F(\mu^N)$

\[
F(\mu^N) := \int_{D^2} \mu^N \otimes \mu^N(d\gamma) \phi_1(\gamma_{s_1}) \cdots \phi_k(\gamma_{s_k}) \left[ \phi(\gamma_t) - \phi(\gamma_s) \right]
\]
\[
+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r) dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r) dr
\]
\[
- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r)) dr
\]
\[
- \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr
\]
\[
- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2=1}^2 \partial_{i_1, i_2}^2 \phi(\gamma_r) dr
\]
The expression of $F(\mu^N)$

\[
F(\mu^N) := \\
\int_{D^2} \mu^N \otimes \mu^N(d\gamma) \phi_1(\gamma_{s_1}) \cdots \phi_k(\gamma_{s_k}) \left[ \phi(\gamma_t) - \phi(\gamma_s) \right] \\
+ \alpha \int_s^t \gamma_1^1 \partial_1 \phi(\gamma_r) dr + \alpha \int_s^t \gamma_2^1 \partial_2 \phi(\gamma_r) dr \\
- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r)) dr \\
- \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr \\
- \frac{\sigma^2}{2} \int_s^t \mu^N_r(f) \sum_{i_1, i_2 = 1}^2 \partial_{i_1, i_2}^2 \phi(\gamma_r) dr
\]
The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^N, i, X_{s_1}^N, j) \cdots \phi_k(X_{s_k}^N, i, X_{s_k}^N, j) \left[ \phi(X_t^N, i, X_t^N, j) - \phi(X_s^N, i, X_s^N, j) \right]$$

$$+ \alpha \int_{s}^{t} X_r^N, i \partial_1 \phi(X_r^N, i, X_r^N, j) \, dr + \alpha \int_{s}^{t} X_r^N, j \partial_2 \phi(X_r^N, i, X_r^N, j) \, dr$$

$$- \int_{s}^{t} f(X_r^N, i)(\phi(0, X_r^N, j) - \phi(X_r^N, i, X_r^N, j)) \, dr$$

$$- \int_{s}^{t} f(X_r^N, j)(\phi(X_r^N, i, 0) - \phi(X_r^N, i, X_r^N, j)) \, dr$$

$$- \sigma^2 \frac{2}{2} \int_{s}^{t} \mu_r^N(f) \sum_{i_1, i_2=1}^{2} \partial^2_{i_1, i_2} \phi(X_r^N, i, X_r^N, j) \, dr$$
The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^N, i, X_{s_1}^N, j) \cdots \phi_k(X_{s_k}^N, i, X_{s_k}^N, j) \left[ \phi(X_t^N, i, X_t^N, j) - \phi(X_s^N, i, X_s^N, j) \right]$$

$$+ \alpha \int_s^t X_r^N, i \partial_1 \phi(X_r^N, i, X_r^N, j) \, dr + \alpha \int_s^t X_r^N, j \partial_2 \phi(X_r^N, i, X_r^N, j) \, dr$$

$$- \int_s^t f(X_r^N, i)(\phi(0, X_r^N, j) - \phi(X_r^N, i, X_r^N, j)) \, dr$$

$$- \int_s^t f(X_r^N, j)(\phi(X_r^N, i, 0) - \phi(X_r^N, i, X_r^N, j)) \, dr$$

$$- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2=1}^{2} \partial^2_{i_1, i_2} \phi(X_r^N, i, X_r^N, j) \, dr \right]$$

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The expression of $F(\mu^N)$

\[
F(\mu^N) := \\
\frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j}) \cdots \phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j}) \left[ \phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) \right] \\
+ \alpha \int_s^t X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) \, dr + \alpha \int_s^t X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) \, dr \\
- \int_s^t f(X_r^{N,i})(\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j})) \, dr \\
- \int_s^t f(X_r^{N,j})(\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) \, dr \\
- \frac{\sigma^2}{2} \int_s^t \frac{1}{N} \sum_{k=1}^{N} f(X_r^{N,k}) \sum_{i_1,i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^{N,i}, X_r^{N,j}) \, dr
\]
The expression of $F(\mu^N)$

$$
F(\mu^N) := \frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^N,i, X_{s_1}^N,j) ... \phi_k(X_{s_k}^N,i, X_{s_k}^N,j) \left[ \phi(X_t^N,i, X_t^N,j) - \phi(X_s^N,i, X_s^N,j) \right] \\
+ \alpha \int_s^t X_r^N,i \partial_1 \phi(X_r^N,i, X_r^N,j) \, dr + \alpha \int_s^t X_r^N,j \partial_2 \phi(X_r^N,i, X_r^N,j) \, dr \\
- \int_s^t f(X_r^N,i)(\phi(0, X_r^N,j) - \phi(X_r^N,i, X_r^N,j)) \, dr \\
- \int_s^t f(X_r^N,j)(\phi(X_r^N,i, 0) - \phi(X_r^N,i, X_r^N,j)) \, dr \\
- \frac{\sigma^2}{2} \int_s^t \frac{1}{N} \sum_{k=1}^{N} f(X_r^N,k) \sum_{i_1,i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^N,i, X_r^N,j) \, dr
$$
The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j}) \ldots \phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j}) \left[ \phi(X_{t}^{N,i}, X_{t}^{N,j}) - \phi(X_{s}^{N,i}, X_{s}^{N,j}) \right]$$

$$+ \alpha \int_{s}^{t} X_{r}^{N,i} \partial_1 \phi(X_{r}^{N,i}, X_{r}^{N,j}) \, dr + \alpha \int_{s}^{t} X_{r}^{N,j} \partial_2 \phi(X_{r}^{N,i}, X_{r}^{N,j}) \, dr$$

$$- \int_{s}^{t} f(X_{r}^{N,i})(\phi(0, X_{r}^{N,j}) - \phi(X_{r}^{N,i}, X_{r}^{N,j})) \, dr$$

$$- \int_{s}^{t} f(X_{r}^{N,j})(\phi(X_{r}^{N,i}, 0) - \phi(X_{r}^{N,i}, X_{r}^{N,j})) \, dr$$

$$- \int_{s}^{t} \int_{\mathbb{R}} \frac{u^2}{2} \frac{1}{N} \sum_{k=1}^{N} f(X_{r}^{N,k}) \sum_{i_1, i_2=1}^{2} \partial^2_{i_1, i_2} \phi(X_{r}^{N,i}, X_{r}^{N,j}) d\nu(u) \, dr$$
The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j}) \cdots \phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j}) \left[ \phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) \right]$$

$$+ \alpha \int_{s}^{t} X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) \, dr + \alpha \int_{s}^{t} X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) \, dr$$

$$- \int_{s}^{t} f(X_r^{N,i})(\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j})) \, dr$$

$$- \int_{s}^{t} f(X_r^{N,j})(\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) \, dr$$

$$- \int_{s}^{t} \int_{\mathbb{R}} \frac{u^2}{2} \frac{1}{N} \sum_{k=1}^{N} f(X_r^{N,k}) \sum_{i_1, i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^{N,i}, X_r^{N,j}) \, d\nu(u) \, dr$$
The expression of \( F(\mu^N) \)

\[
F(\mu^N) := \frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j}) \cdots \phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j}) \left[ \phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) \right] \\
+ \alpha \int_s^t X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) \, dr + \alpha \int_s^t X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) \, dr \\
- \int_s^t f(X_r^{N,i}) (\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j})) \, dr \\
- \int_s^t f(X_r^{N,j}) (\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) \, dr \\
- \int_s^t \int_{\mathbb{R}} \sum_{k=1}^{N} f(X_r^{N,k}) \frac{u^2}{2N} \sum_{i_1, i_2=1}^{2} \partial^2_{i_1, i_2} \phi(X_r^{N,i}, X_r^{N,j}) \, d\nu(u) \, dr
\]
The expression of $F(\mu^N)$

\[
F(\mu^N) := \frac{1}{N^2} \sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j}) \ldots \phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j}) \left[ \phi(X_{t}^{N,i}, X_{t}^{N,j}) - \phi(X_{s}^{N,i}, X_{s}^{N,j}) \right]
\]

\[
+ \alpha \int_s^t X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) dr + \alpha \int_s^t X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) dr
\]

\[
- \int_s^t f(X_r^{N,i})(\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j})) dr
\]

\[
- \int_s^t f(X_r^{N,j})(\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) dr
\]

\[
- \int_s^t \int_{\mathbb{R}^k}^{N} f(X_r^{N,k}) \frac{u^2}{2N} \sum_{i_1,i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^{N,i}, X_r^{N,j}) d\nu(u) dr
\]
The expression of $\phi(X^N,i, X^N,j)$

By Ito’s formula,

$$
\mathbb{E}\phi(X^N_t,i, X^N_t,j) - \phi(X^N_s,i, X^N_s,j) = \\
\mathbb{E} - \alpha \int_s^t X^N_r,i \partial_1 \phi(X^N_r,i, X^N_r,j) dr - \alpha \int_s^t X^N_r,j \partial_2 \phi(X^N_r,i, X^N_r,j) dr \\
+ \int_s^t \int_\mathbb{R} f(X^N_r,i)(\phi(0, X^N_r,j + \frac{u}{\sqrt{N}}) - \phi(X^N_r,i, X^N_r,j)) d\nu(u) dr \\
+ \int_s^t \int_\mathbb{R} f(X^N_r,j)(\phi(X^N_r,i + \frac{u}{\sqrt{N}}, 0) - \phi(X^N_r,i, X^N_r,j)) d\nu(u) dr \\
+ \int_s^t \int_\mathbb{R} \sum_{k=1}^{N} \sum_{k \neq i,j} f(X^N_r,k)(\phi(X^N_r,i + \frac{u}{\sqrt{N}}, X^N_r,j + \frac{u}{\sqrt{N}}) - \phi(X^N_r,i, X^N_r,j)) d\nu(u) dr
$$
Vanishing of $\mathbb{E} \left[ F(\mu^N) \right]$

The reset jump term

$$\left| \phi(0, X^N_{r,j}) - \phi(0, X^N_{r,j} + \frac{u}{\sqrt{N}}) \right|$$
Vanishing of $\mathbb{E} \left[ F(\mu^N) \right]$

The reset jump term

$$\left| \phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) \right| \leq C \frac{|u|}{\sqrt{N}}$$
Vanishing of $\mathbb{E} \left[ F(\mu^N) \right]$

The reset jump term

$$\left| \phi(0, X_{r,j}^N) - \phi(0, X_{r,j}^N + \frac{u}{\sqrt{N}}) \right| \leq C \frac{|u|}{\sqrt{N}}$$

The small jump term

$$N \left| \phi(X_{r,i}^N + \frac{u}{\sqrt{N}}, X_{r,j}^N + \frac{u}{\sqrt{N}}) - \phi(X_{r,i}^N, X_{r,j}^N) - \frac{u^2}{2N} \sum_{i_1, i_2=1}^{2} \partial^2_{i_1, i_2} \phi(X_{r,i}^N, X_{r,j}^N) \right|$$

$$\leq CN \frac{|u|^3}{N^{1/2}} = CN^{-1/2} \leq \mathbb{E} \left[ F(\mu) \right] \rightarrow_{N \rightarrow \infty} 0$$
Vanishing of $\mathbb{E} \left[ F(\mu^N) \right]$

The **reset jump term**

$$\left| \phi(0, X_r^N,j) - \phi(0, X_r^N,j + \frac{u}{\sqrt{N}}) \right| \leq C \frac{|u|}{\sqrt{N}}$$

The **small jump term**

$$\mathbb{N} \left| \phi(X_r^N,i + \frac{u}{\sqrt{N}}, X_r^N,j + \frac{u}{\sqrt{N}}) - \phi(X_r^N,i, X_r^N,j) \right|$$

$$- \frac{u}{\sqrt{N}} \sum_{i_1=1}^{2} \partial_{i_1} \phi(X_r^N,i, X_r^N,j) - \frac{u^2}{2N} \sum_{i_1,i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^N,i, X_r^N,j)$$
Vanishing of $\mathbb{E} \left[ F(\mu^N) \right]$ 

The reset jump term

$$\left| \phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) \right| \leq C \frac{|u|}{\sqrt{N}}$$

The small jump term

$$N \left| \phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right|$$

$$- \frac{u}{\sqrt{N}} \sum_{i_1=1}^{2} \partial_{i_1} \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N} \sum_{i_1,i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^{N,i}, X_r^{N,j})$$

$$\leq CN \frac{|u|^3}{N\sqrt{N}} = CN^{-1/2} |u|^3$$
Vanishing of $\mathbb{E} \left[ F(\mu^N) \right]$

The reset jump term

$$\left| \phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) \right| \leq C \frac{|u|}{\sqrt{N}}$$

The small jump term

$$N \left| \phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right|$$

$$- \frac{u}{\sqrt{N}} \sum_{i_1=1}^{2} \partial_{i_1} \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N} \sum_{i_1,i_2=1}^{2} \partial_{i_1,i_2}^2 \phi(X_r^{N,i}, X_r^{N,j}) \right|$$

$$\leq CN \frac{|u|^3}{N\sqrt{N}} = CN^{-1/2} |u|^3$$

$$CN^{-1/2} \geq \mathbb{E} \left[ F(\mu^N) \right] \xrightarrow{N \to \infty} \mathbb{E} \left[ F(\mu) \right] = 0$$
Convergence of \((\mu^N)_N\)

\[
dX^N_i(t) = -\alpha X^N_i(t) \, dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\{z \leq f(X^N_j(t))\}} \, d\pi^j(t, z, u)
\]

\[
- X^N_i(t-) \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(X^N_i(t-))\}} \, d\pi^i(t, z, u)
\]

\[
d\bar{X}_t^i = -\alpha \bar{X}_t^i \, dt + \sigma \sqrt{\mu_t(f)} \, dW_t
\]

\[
- \bar{X}_t^i(t-) \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\bar{X}_t^i(t-))\}} \, d\pi^i(t, z, u)
\]
Convergence of \((\mu^N)_N\)

\[
dX_t^{N,i} = -\alpha X_t^{N,i} \, dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}^+ \times \mathbb{R}} u1\{z \leq f(X_t^{N,j})\} \, d\pi^j(t, z, u)
\]

\[
- X_t^{N,i} \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(X_t^{N,i})\} \, d\pi^i(t, z, u)
\]

\[
d\bar{X}_t^i = -\alpha \bar{X}_t^i \, dt + \sigma \sqrt{\mu_t(f)} \, dW_t
\]

\[
- \bar{X}_t^i \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(\bar{X}_t^i)\} \, d\pi^i(t, z, u)
\]

\((\mu^N)_N\) is tight on \(\mathcal{P}(D)\)
Convergence of \((\mu^N)_N\)

\[
dX^N_t, i = -\alpha X^N_t, i \, dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}^+ \times \mathbb{R}} u1\{z \leq f(X^N_t, j)\} \, d\pi^i(t, z, u) \\
- X^N_t, i \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(X^N_t, i)\} \, d\pi^i(t, z, u)
\]

\[
d\bar{X}^i_t = -\alpha \bar{X}^i_t \, dt + \sigma \sqrt{\mu_t(f)} \, dW_t \\
- \bar{X}^i_t \int_{\mathbb{R}^+ \times \mathbb{R}} 1\{z \leq f(\bar{X}^i_t)\} \, d\pi^i(t, z, u)
\]

- \((\mu^N)_N\) is tight on \(\mathcal{P}(D)\)
- let \(\mu\) be the limit of a converging subsequence
Convergence of \((\mu^N)_N\)

\[ dX_{t}^{N,i} = -\alpha X_{t}^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}^+ \times \mathbb{R}} u^1 \{ z \leq f(X_{t}^{N,j}) \} d\pi^j(t, z, u) \]

\[- X_{t-}^{N,i} \int_{\mathbb{R}^+ \times \mathbb{R}} 1 \{ z \leq f(X_{t-}^{N,i}) \} d\pi^i(t, z, u) \]

\[ d\tilde{X}_{t}^{i} = -\alpha \tilde{X}_{t}^{i} dt + \sigma \sqrt{\mu_t(f)} dW_t \]

\[- \tilde{X}_{t-}^{i} \int_{\mathbb{R}^+ \times \mathbb{R}} 1 \{ z \leq f(\tilde{X}_{t-}^{i}) \} d\pi^i(t, z, u) \]

- \((\mu^N)_N\) is tight on \(\mathcal{P}(D)\)
- let \(\mu\) be the limit of a converging subsequence
- \(\mathcal{L}(\mu)\) is the unique solution of \((\mathcal{M})\)
Convergence of \((\mu^N)_N\)

\[
dX^N_i = -\alpha X^N_i \, dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X^N_j)\}} \, d\pi^j(t, z, u)
\]

\[-X^N_i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(X^N_i)\}} \, d\pi^i(t, z, u)\]

\[
d\bar{X}_i = -\alpha \bar{X}_i \, dt + \sigma \sqrt{\mu_t(f)} \, dW_t
\]

\[-\bar{X}_i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(\bar{X}_i)\}} \, d\pi^i(t, z, u)\]

- \((\mu^N)_N\) is tight on \(\mathcal{P}(D)\)
- let \(\mu\) be the limit of a converging subsequence
- \(\mathcal{L}(\mu)\) is the unique solution of \((\mathcal{M})\)
- \(\mu = \mathcal{L}(\bar{X}^1|W)\) is the only limit of \((\mu^N)_N\)
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Thank you for your attention!

Questions?