- Aug. 9th, 2004 -

A note about strictly cont. local martingales.

Consider $(D_t, t \geq 0)$ a ≥ 0 local martingale, and for simplicity, let us assume that: $t \longrightarrow E[D_t]$ is continuous.
Take: $D_0 = 1$
Then, $t \longrightarrow E[D_t]$ is a decreasing, cont. function on $[0,\infty)$, which may be represented as the tail of the distribution of a r.v. $\underset{\geq 0}{T_0}$; ie:

$$(1) \quad E[D_t] = E[1_{(T_0 \geq t)}].$$

From this, we deduce, for $s<t$:

$$(2) \quad E[D_s - D_t] = E[1_{(s \leq T_0 \leq t)}].$$

and I think that some general measure argt ~~alt~~ (ie: ~ Monotone class Thm) allows ~~us~~ to get:

$$(3) \quad E\left[-\int f(u)\, dD_u\right] = E[f(T_0)],$$

$\forall f: (0,\infty] \longrightarrow (0,\infty)$

(All we need is to know a priori that: $f \longrightarrow E\left(-\int f(u)\, dD_u\right)$ is a measure, but I think this is OK ...).

Example: $D_t = x/R_t$ under $P_x^{(3)}$, the law of BES(3), starting from x; then, we know ~~happen~~ the abs. cont. relationship:

$$(4) \quad P^{(3)}_{x \,|\mathcal{F}_t} = \left(\frac{X_{t\wedge T_0}}{x}\right) \cdot W_{x \,|\mathcal{F}_t},$$

where, on the RHS, W_x denotes Wiener measure, with

$$W_x(X_0 = x) = 1.$$

Thus, from (4), we deduce:

$$(5) \qquad \left(\frac{x}{R_t}\right) \cdot P_x^{(3)}\Big|_{\mathcal{F}_t} = 1_{(t<T_0)} \cdot W_x\Big|_{\mathcal{F}_t}.$$

from which we deduce in particular (1) for $D_t = \left(\frac{x}{R_t}\right)$, but, in fact, much more!

$$(6). \qquad E_x^{(3)}\left[H_t\left(\frac{x}{R_t}\right)\right] = W_x\left[H_t \, 1_{(t<T_0)}\right],$$

$\forall\, H_t \geq 0$, $\mathcal{F}_t$ measurable -

(We may of course, modify this example for other Bessel processes, with the abs. cont. relationship ~~[illegible]~~

$d_- = 2(1-\nu)$; $\quad d_+ = 2(1+\nu)$.

Also, connection with the Doléans – Föllmer measure associated with (D_t). : This is a measure on $\Omega \times \mathbb{R}_+$, $\mathcal{P}$ ← previsible. here: $E_x^{(3)}\left[\int H_s \, dD_s\right]$

$$\| = W_x\left[H_{T_0}\right].$$

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