1)

A note on Williams decomposition of the BES(3) process at its ultimate minimum

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Abstract: We propose a concise proof of Williams decomposition of the BES(3) process, starting from $r>0$, at its ultimate minimum.

The proof is done with the help of the progressive enlargement formula with respect to g, the time of the ultimate minimum.

This discussion is strongly motivated by, and linked with, the reading guide of Nikeghbali–Platen about general last passage times.

1. Introduction and some statements from [NP].

(1.1) In their survey [NP] about last passage times, the authors illustrate some of their formulae with the following example:

let $(R_t, t\geq 0)$ be a BES(3) process on $\mathbb{R}_+$, starting from $r>0$. Denote by $(\mathcal{R}_t)_{t\geq 0}$ its natural filtration, and let $I_t = \inf_{s\leq t} R_s$.

The following results are found in [NP], around Corollary 4.10

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a) I_∞ is distributed as rU, with U uniform on $[0,1]$;

b) The (Azéma-) supermartingale associated with g, the time at which R reaches I_∞ is:

$$Z_t \equiv P(g > t \mid \mathcal{R}_t) = I_t / R_t .$$

c) The Laplace transform of the law of g is:

$$E[\exp(-\lambda g)] = \frac{1}{\sqrt{2\lambda}\, r}\left\{ 1 - \exp\left(-\sqrt{2\lambda}\, r\right)\right\}$$

d) The density of g is: $\quad p(t) = \frac{1}{\sqrt{2\pi t}\, r}\left(1 - e^{-(r^2/2t)}\right)$

(1.2) Recall that, if $(B_t, t \geq 0)$ is a Brownian motion starting from 0, and $a \in \mathbb{R}$, then the law of the first hitting time of a by B, denoted as $T_a^{(B)}$ is:

$$(1) \qquad P\left(T_a^{(B)} \in dt\right) = \frac{dt}{\sqrt{2\pi t^3}} |a| \exp\left(-a^2/2t\right).$$

This well known fact allows to rewrite c) and d) above as follows:

$$(2) \qquad g \overset{(\text{law})}{=} T_{rU}^{(B)} ,$$

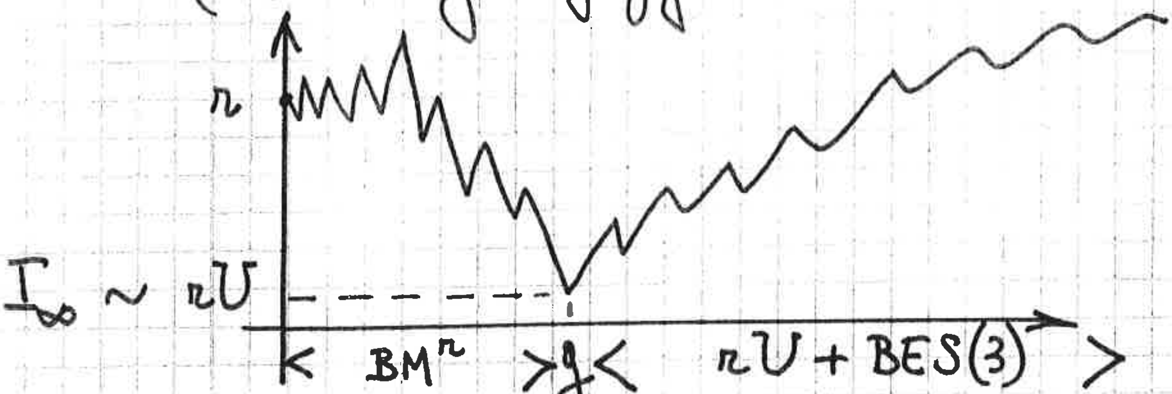
where U, uniform on $[0,1]$, is independent from B.

We let the reader verify (2) from c) and d).

In fact, (2) may be understood via the classical decomposition of R before and after g, due to Williams [W]

2. Williams decomposition of R, before and after g, via progressive enlargement.

(2.1) The following figure describes well the decomposition.



In fact, this figure is nothing else but Fig 5 in Revuz-Yor ([RY], Proposition 3.10 and Theorem 3.11 in Chap 6, Sect. 3) when the BES(3) path is considered starting from level c (= r, here).

(2.2) We now state precisely Williams' decomposition theorem before and after g.

Theorem (Williams [W]) Consider the following 3 independent random objects:

i) $(B'_t, t \geq 0)$, a BM starting from $r > 0$;

ii) U a uniform r.v. on $[0,1]$

iii) $(\tilde{R}_t, t \geq 0)$ a BES(3) process starting from 0.

Then, the process:

$$(3) \qquad R_t = \begin{cases} B'_t, & \text{for } t \leq g = \inf\{u : B'_u = \\ rU + \tilde{R}_{(t-g)}, & t \geq g \end{cases}$$

is a BES(3) process starting from r...

We note that the pre-g-Brownian motion found in (3) explains the result (2'); indeed, if $B'_t = r - B^{(0)}_t$, then: (2') $g = \inf\{u : B^{(0)}_u = r(1-U)\}$ from which (2) follows.

(2.3) We now proceed to the proof of the theorem, via the enlargement formula which describes the additive decomposition of the BES(3) process (R_t) in the filtration $(\mathcal{R}^g_t)$ containing $(\mathcal{R}_t)$, and making g a stopping time —

- We have,

(4) $R_t = r + B_t + \int_0^t \frac{ds}{R_s}$, with (B_t) a Brownian motion with respect to $(\mathcal{R}_t)$. (see, e.g., [J])

- Then, the enlargement formula in $(\mathcal{R}^{(g)}_t)$ yields:

(5) $$r + B_t = B'_t + \int_0^{g\wedge t} \frac{d\langle B, Z\rangle_u}{Z_u} + \int_g^t \frac{d\langle B, 1-Z\rangle_u}{(1-Z_u)}$$

with (B'_t) a Brownian motion with respect to $(\mathcal{R}^{(g)}_t)$.

- Moreover, we deduce from b) that:

(6) $$\frac{d\langle B, Z\rangle_u}{Z_u} = -\frac{du}{R_u}, \quad u \leq g$$

(7) $$\frac{d\langle B, 1-Z\rangle_u}{1-Z_u} = \frac{I_\infty \, du}{R_u(R_u - I_\infty)}, \quad u > g$$

These two identities imply easily:

- using (4) and (6), the form of the pre-g-process;
- using (4) and (7), the form of the post-g-process.

• For the proof of (3) to be complete, it remains to prove that B' is independent of $I_\infty \overset{(law)}{=} rU$, or, more exactly that, given $I_\infty = a$, the pre-g-process is $(B'_u, u \leq T'_a)$ with obvious notation. This property is asserted in the following

Proposition: Let $(F_u, u \geq 0)$ denote a ≥ 0, predictable process on path-space; let P_r denote the law of R, starting from r, and let P'_r denote the law of Brownian motion B' starting from r. Then, one has, for $a < r$:

$$E_r\left[F_g \mid I_\infty = a\right] \underset{①}{=} E_r\left[F_{T_a} \mid T_a < \infty\right] \underset{②}{\equiv} E_r\left[F_{T_a} \mid I_\infty < a\right] \tag{8}$$

$$\underset{③}{=} E'_r\left[F(B'_u; u \leq T'_a)\right]$$

Proof of Proposition:

(1) The equality between the RHS of ① and ③ follows from the classical Doob-absolute continuity relationship:

$$P_r\big|_{\mathcal{F}_t} = \left(\frac{X_{t\wedge T_0}}{r}\right) \cdot P'_r\big|_{\mathcal{F}_t}$$

((X_t) denotes the coordinate process on path-space) on canonical path space $C([0,\infty), \mathbb{R})$. This equality may be extended when replacing t by a stopping time. In particular we have:

$$P_r\big|_{\mathcal{F}_{T_a}} \cap (T_a < \infty) = \left(\frac{a}{r}\right) \cdot P'_r\big|_{\mathcal{F}_{T_a}}$$

for $0 < a < r$, which yields the desired result.

(ii) The equality ② follows simply from: $(T_a < \infty) = (I_\infty < a)$, P_r a.s.

(iii) It remains to prove ①

• We first start with the identity: (here, for simplicity, we note P instead of P_r)

$$(9) \quad E\left[1_{(g \le t)}\, \varphi(I_\infty)\right] = E\left[\int_0^t \varphi(I_s)\, d(1-Z_s)\right]$$

for every $\varphi: [0,\infty) \to \mathbb{R}_+$, Borel.

Indeed, the LHS of (9) equals:

$$E\left[1_{(g \le t)}\, \varphi(I_t)\right] = E\left[(1-Z_t)\,\varphi(I_t)\right]$$

$$= E\left[\int_0^t \varphi'(I_s)\, dI_s\, (1-Z_s)\right] + E\left[\int_0^t \varphi(I_s)\, d(1-Z_s)\right]$$

assuming here that φ is C^1.

We note that the expectation involving φ' is equal to 0, since $1-Z_s = 0$, dI_s a.e. Thus, a monotone class argument yields that (9) holds for every $\varphi \ge 0$, Borel.

• Next, from the additive decomposition of $(1-Z_s)$, we obtain:

$$(10) \quad \begin{cases} E\left[1_{(g \le t)}\, \varphi(I_\infty)\right] = E\left[\int_0^t \varphi(I_s)\left(-\dfrac{dI_s}{I_s}\right)\right] \\ \qquad = E\left[\int_{I_t}^r \varphi(a)\, \dfrac{da}{a}\right] \\ = E\left[\int_0^r \varphi(a)\, \dfrac{da}{a}\, 1_{(T_a \le t)}\right], \text{ since } (I_t \le a) = (t \ge T_a) \end{cases}$$

• Now, the identity (10) (between the extreme terms)

†) still holds when we replace t by a generic stopping time. Hence, another use of the monotone class theorem yields:

$$(11)\quad E\left[F_g\ \varphi(I_\infty)\right] = E\left[\int_0^r \varphi(a)\frac{da}{a}\ F_{T_a}\ 1_{(T_a<\infty)}\right]$$

Finally, recalling that $I_\infty \overset{(law)}{=} rU$ under P_r allows to deduce the identity ① from (11).

3. A concluding remark.

The statement of the theorem invites for a proof mixing initial enlargement with I_∞ and progressive enlargement with g.

In fact we have not proceeded exactly like this; the Proposition plays the role of the initial enlargement method and relies on a classical Girsanov relationship between P_r and P'_r.

In conclusion, we like to present the above as an example (certainly among many of the same kind) of a mélange of enlargements techniques and Girsanov theorem —

References:

[J] Th. Jeulin: Semi-martingales et grossissements d'une filtration. LNM Springer 833 (1980).

[NP] A. Nikeghbali, E. Platen: A reading guide for last passage times with financial applications in view. To appear in Finance and Stochastics (2012)

[RY] D. Revuz, M. Yor: Continuous Martingales and Brownian motion. 3rd edition. Springer (1999)

[W] D. Williams: Path decomposition and continuity