August, 1st, 1994

A two-parameter hierarchy of probabilities, the action of the shift on the R_k's, and an approximation of local time.

1. Motivations.

It seems interesting to introduce the two-parameter family:

$$P^{\#}_{k,m} = \frac{c_{k,m}}{(\sqrt{\tau_1})^k \left(\sqrt{V_1(\tau_1)}\right)^m} \cdot P^{\#}$$

In my document (7), I have looked uniquely at $P^{\#}_{k,0}$, and found an "explicit" density for $P^{\#}_{k,0}|\mathcal{R}_p$ with respect to $P^{\#}_{|\mathcal{R}_p}$.

Here, I shall look at the same quantities for $P^{\#}_{0,m}$, and apply this to obtain a martingale proof of the following result:

$$(1) \qquad \lim_{n\to\infty} \left(\frac{\pi}{2} n^2 V_n(t)\right) = L_t^2, \text{ uniformly on compacts.}$$

Once this is done, we will look again at "explicit" computations for $P^{\#}_{k,m}|\mathcal{R}_p$ with respect to $P^{\#}_{|\mathcal{R}_p}$.

2. Some absolute continuity relationships.

We first remark that, for every $m>0$, the sequence:

$$(2) \qquad \left\{ \Delta^{(m)}_n \overset{\text{def}}{=} \frac{(m+n)!}{m!\, n!} (R_1 \cdots R_n)^{\frac{m}{2}} \equiv \frac{(m+n)!}{m!\, n!} \left(\frac{V_{n+1}(1)}{V_1(1)}\right)^{m/2}, n\geq 1 \right\}$$

is a $(P, (\mathcal{R}_n))$ martingale (Proof: Immediate from the knowledge of the distribution of the R_n's).

As a Corollary of (5), we deduce the convergence result (1) immediately, ~~but~~ by taking e.g : $m=2$ (we also use Dini's thm. to obtain the uniform convergence on compacts !!).

Let us look now closer at (5), and write this identity in the following form:

$$(5') \qquad E\left[\left(\frac{L_1}{\sqrt{V_1(1)}}\right)^m \Big| \mathcal{R}_p\right] = \frac{(m+p)!}{p!}\left(\frac{\pi}{2}\,\frac{V_{p+1}(1)}{V_1(1)}\right)^{m/2}$$

Thus, we have obtained the following

<u>Thm</u>: <u>For every</u> p, <u>the</u> $(p+1)$ <u>random variables</u> $R_1, R_2, \ldots, R_p$ and $\left(\frac{L_1}{\sqrt{V_{p+1}}}\right)$ <u>are independent</u>, <u>and, moreover</u>:

$$\sqrt{\frac{2}{\pi}}\cdot\frac{L_1}{\sqrt{V_{p+1}}} \overset{(\text{law})}{=} Z_{p+1}$$

(: gamma variable with parameter $(p+1)$).

(10')

!

Quelques remarques sur le document (10).

1. La propriété : $\sqrt{\frac{2}{\pi}} \frac{L_1}{\sqrt{V_{p+1}}} \overset{(loi)}{=} Z_{p+1}$ est bien en accord avec les identités en loi de l'algèbre beta-gamma ; en effet, on peut écrire :

$$\sqrt{\frac{2}{\pi}} \frac{L_1}{\sqrt{V_p}} = \left(\frac{\sqrt{V_{p+1}}}{\sqrt{V_p}}\right) \left(\sqrt{\frac{2}{\pi}} \frac{L_1}{\sqrt{V_{p+1}}}\right),$$

et, dans le membre de droite, $\sqrt{R_p} \equiv \frac{\sqrt{V_{p+1}}}{\sqrt{V_p}}$ a pour loi $Z_{p,1}$ et est indépendante de $\sqrt{\frac{2}{\pi}} \frac{L_1}{V_{p+1}}$, qui a pour loi Z_{p+1}.

On retrouve donc bien ainsi l'identité : $Z_p \overset{(loi)}{=} Z_{p,1} Z_{p+1}$.

* i.e. : le second thm. qui figure en (10).

2. Dans le même esprit, on déduit du même théorème* la propriété suivante : les variables $\left(\sqrt{\frac{2}{\pi}} \frac{L_1}{\sqrt{V_{p+1}}} - \sqrt{\frac{2}{\pi}} \frac{L_1}{\sqrt{V_p}}, p \geq 1\right)$

~~Not~~ sont indépendantes, et équidistribuées, avec pour loi commune la loi exponentielle de paramètre 1.

(C'est toujours une conséquence du théorème, et de l'algèbre beta-gamma)

3. Considérons maintenant la relation (8), p. 6, du document (7), que je réécris sous la forme suivante :

$$E\left[\sqrt{2T}\left(\left(\sqrt{\frac{2}{\pi}} \frac{L_1}{\sqrt{V_{p+1}}}\right)\sqrt{V_{p+1}}\right)^k \Big| \mathcal{R}_p\right]$$

(1)

$$= E\left[\left(Z_{1+p}\right)^k\right] \int_0^\infty dx \left(\frac{x e^{x^2/2}}{\psi(x)}\right)^k \gamma_{a,p}(x)$$

Because of this equality, it is now tempting to think that, in fact:

$$(2?) \qquad \left(\sqrt{\frac{2}{\pi}}\,\frac{L_1}{\sqrt{V_{p+1}}}\right) \text{ is independent of } \sigma\{\mathcal{R}_p, V_{p+1}\} \equiv \sigma\{V_1, V_2, \dots, V_{p+1}\}$$

and that the identity (1) may be rewritten as:

$$(3?) \qquad E\left[\left(\sqrt{2T}\sqrt{V_{p+1}}\right)^k \,\middle|\, \mathcal{R}_p\right] = \int_0^\infty dx \left(\frac{x\,e^{x^2/2}}{\psi(x)}\right)^k \gamma_{a,p}(x).$$

However, (2?) is certainly wrong for the 2 following reasons:

(i) if it were true, then we would deduce from the equality

$$L_1 = \left(\frac{L_1}{\sqrt{V_{p+1}}}\right)\sqrt{V_{p+1}}$$

the identity in law: $(4?) \qquad |N| \overset{(law)}{=} \left(\sqrt{\frac{\pi}{2}}\, Z_{1+p}\right)\sqrt{V_{p+1}}$

It should be fairly easy to show that this does not hold.

(ii) Other contradictions arise from (3?).
Indeed, the function: $\quad u(x) = \dfrac{x\,e^{x^2/2}}{\psi(x)}$ is

$$(4) \qquad \boxed{\frac{x\,e^{x^2/2}}{\psi(x)} \equiv \frac{x}{\tilde{\psi}(x)} = \frac{1}{\sqrt{\frac{\pi}{2}} + \int_x^\infty \frac{du}{u^2}\exp\left(-\frac{u^2}{2}\right)}}$$

[Pour la correction du 2, voir le doc. (6), p. 3

and, as x varies in $]0,\infty[$, the function $u(x)$ varies between 0 and $\sqrt{\frac{2}{\pi}}$.
But, on the left hand side of (3?), the variable $\sqrt{2TV_{p+1}}$ varies on the entire half-line.

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The "full" computations for the 2-parameter "hierarchy" of probabilities. $(10'')$ 1)

I continue the computation of

$$(1) \qquad E^{\#}_{k,m}(F_p) \equiv E^{\#}\left(\frac{c_{k,m}\, F_p}{\left(\sqrt{z_1}\right)^{k}\left(\sqrt{V_1(z_1)}\right)^{m}}\right)$$

which were done for $k=0$ in Doc $(10')$, and for $m=0$ in Doc. (7).

Here, I shall follow closely the computations made in Doc. (7); the notation $(j)_7$ refers to the equation (j) in Doc(7).

We have:

$$E^{\#}_{k,m}(F_p) = c_{k,m}\, E^{\#}\left(\frac{1}{\left(V_1(z_1)\right)^{\frac{k+m}{2}}}\; \frac{F_p(R'_n,\, n\geq 1)}{\left(1+R'_1+R'_1R'_2+\cdots\cdots+\right)^{k/2}}\right)$$

$$\underset{\uparrow}{=} c_{k,m} \lim_{n\to\infty} E\left[F_p\, \frac{\left(n\sqrt{\frac{\pi}{2}}\right)^{k+m}\left(R_1\cdots R_n\right)^{\frac{k+m}{2}}}{\left(1+R_1+R_1R_2+\cdots\cdots\right)^{k/2}}\right]$$

[this generalizes $(1)_7$]

Now, let $\ell = k+m$, and define: $c'_{k,m} = c_{k,m}\left(\sqrt{\frac{\pi}{2}}\right)^{\ell}(\ell!)$.

Reading Doc. (7), bottom of p.3, we get:

$$E^{\#}_{k,m}(F_p) = c'_{k,m}\, E\left[F\left(R_{\ell+1}, R_{\ell+2}, \ldots, R_{\ell+p}\right)\frac{1}{\left(\Sigma_{\ell+1}\right)^{k/2}}\right],$$

and, from the top of p.4, Doc. (7):

$$E\left[\frac{1}{(\Sigma_{\ell+1})^{k/2}}\,\Big|\,\mathcal{R}_{\ell+p}\right]$$

$$=\frac{1}{\Gamma\left(\frac{k}{2}\right)}\int_0^\infty dt\; t^{\left(\frac{k}{2}-1\right)}\, e^{-t\left(1+R_{\ell+1}+(R_{\ell+1}R_{\ell+2})+\cdots+(R_{\ell+1}\cdots R_{\ell+p})\right)}\,\varphi_{\ell+p+1}\left(t R_{\ell+1}\cdots R_{\ell+p}\right)$$

$$=\frac{1}{\Gamma\left(\frac{k}{2}\right)\left(R_{\ell+1}\cdots R_{\ell+p}\right)^{k/2}}\int_0^\infty dt\; t^{\frac{k}{2}-1}\, e^{-t\,\theta_p(R_{\ell+1},\cdots,R_{\ell+p})}\,\varphi_{\ell+p+1}(t).$$

We now replace the vector $(R_{\ell+1},\text{---},R_{\ell+p})$ by $(R_1,\text{---},R_p)$ with the help of $(2)_7$. Thus, we obtain:

$$(2)\qquad E^{\#}_{k,m}(F_p)=c'_{k,m}\,E\left[\frac{(\ell+p)!}{\ell!\,p!}\frac{1}{\Gamma\left(\frac{k}{2}\right)}F_p\;(R_1\cdots R_p)^{m/2}\int_0^\infty dt\; t^{\frac{k}{2}-1}e^{-t\,\theta_p(R_1,-,R_p)}\,\varphi_{\ell+p+1}(t)\right]$$

In particular, we have obtained the following generalization of $(3')_7$:

$$(3)\qquad P^{\#}_{k,m}\Big|_{\mathcal{R}_p}=d_{k,m}\left(\theta_p(R_1,-,R_p)\,;(R_1\cdots R_p)\right)\cdot P\Big|_{\mathcal{R}_p}$$

where :

$$d_{k,m}(a,b)=c'_{k,m}\,\frac{(\ell+p)!\;b^{m/2}}{\ell!\;p!\;\Gamma\left(\frac{k}{2}\right)}\int_0^\infty dt\; t^{\frac{k}{2}-1}\,e^{-ta}\,\varphi_{\ell+p+1}(t)$$

$\ell=k+m$, and $c'_{k,m}=c_{k,m}\left(\sqrt{\frac{\pi}{2}}\right)^{\ell}(\ell!)$.

Some consequences of formula (3):

By comparing (1) and (3), and using again the admissibility of τ_1, we learn the following:

(in the sequel, we note : $a=\theta_p(R_1,-,R_p)$, and $b=R_1\cdots R_p\equiv\frac{V_{p+1}}{V_1}$)

$$E\left[c_{k,m}\frac{(L_1)^\ell}{(\sqrt{V_1})^m}\Big|\mathcal{R}_p\right]=c'_{k,m}\frac{(\ell+p)!\,b^{m/2}}{\ell!\,p!\,\Gamma\left(\frac{k}{2}\right)}\int_0^\infty dt\, t^{\frac{k}{2}-1}e^{-ta}\varphi_{\ell+p+1}(t),$$

hence:

$$E\left[\frac{(L_1)^\ell}{(\sqrt{V_1})^m}\Big|\mathcal{R}_p\right]=\left(\sqrt{\frac{\pi}{2}}\right)^\ell\frac{(\ell+p)!\,b^{m/2}}{p!\,\Gamma\left(\frac{\ell-m}{2}\right)}\int_0^\infty dt\, t^{\frac{\ell-m}{2}-1}e^{-ta}\varphi_{\ell+p+1}(t)$$

Writing $V_1=(V_{p+1})\left(\frac{1}{b}\right)$, we obtain:

$$E\left[\frac{(L_1)^\ell}{(\sqrt{V_{p+1}})^m}\Big|\mathcal{R}_p\right]=\left(\sqrt{\frac{\pi}{2}}\right)^\ell\frac{(\ell+p)!}{p!}\frac{1}{\Gamma\left(\frac{\ell-m}{2}\right)}\int_0^\infty dt\, t^{\frac{\ell-m}{2}-1}e^{-ta}\varphi_{\ell+p+1}(t),$$

which, using: $\ell=k+m$ again, may be written as:

$$(4)\quad E\left[\left(\sqrt{\frac{2}{\pi}}\frac{L_1}{\sqrt{V_{p+1}}}\right)^\ell\left(\sqrt{V_{p+1}}\right)^k\Big|\mathcal{R}_p\right]=\frac{(\ell+p)!}{p!}\frac{1}{\Gamma\left(\frac{k}{2}\right)}\int_0^\infty dt\, t^{\frac{k}{2}-1}e^{-ta}\varphi_{\ell+p+1}(t)$$

We now transform (4) by making the change of variables $t=\frac{x^2}{2}$, and we use:

$$\varphi_n\left(\frac{x^2}{2}\right)=\frac{e^{\frac{x^2}{2}(n-1)}}{(\psi(x))^n}.$$

Thus, we obtain, by writing: $X_{p+1}\equiv\sqrt{\frac{2}{\pi}}\frac{L_1}{\sqrt{V_{p+1}}}$:

$$(4')\quad E\left[(X_{p+1})^\ell\left(\sqrt{V_{p+1}}\right)^k\Big|\mathcal{R}_p\right]=\frac{(\ell+p)!}{p!\,\Gamma\left(\frac{k}{2}\right)}\int_0^\infty dx\,\frac{x^{k-1}}{2^{\frac{k}{2}-1}}\,e^{-\frac{x^2a}{2}}\,\frac{e^{\frac{x^2}{2}(\ell+p)}}{(\psi(x))^{\ell+p+1}}.$$

Now, recall the two following facts:

a) $X_{p+1}\overset{(\text{law})}{=}Z_{p+1}$, and X_{p+1} is independent of $\mathcal{R}_p$;

b) $E\left[(Z_{p+1})^\ell\right]=\frac{(\ell+p)!}{p!}$.

Using a) and b), we can write the following variants of (4'):

$$(5)\quad E\left[f(X_{p+1}) \left(\sqrt{V_{p+1}}\right)^k \middle| \mathcal{R}_p \right] = \int_0^\infty dx \, \frac{x^{k-1}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} \frac{e^{-\frac{x^2}{2}(a-p)}}{(\psi(x))^{p+1}} E\left[f\left(Z_{p+1} \frac{e^{x^2/2}}{\psi(x)}\right)\right]$$

Denote by $g_{p+1}(z)$ the density of Z_{p+1}. Now, using a) above, we easily obtain:

$$g_{p+1}(z)\, E\left[\left(\sqrt{V_{p+1}}\right)^k \middle| \mathcal{R}_p, X_{p+1}=z\right]$$

$$= \int_0^\infty dx \left(\frac{x^{k-1}}{2^{\frac{k}{2}-1}}\right) \frac{1}{\Gamma\left(\frac{k}{2}\right)} \frac{e^{-\frac{x^2}{2}(a-p)}}{(\psi(x))^{p+1}} \psi(x) e^{-\frac{x^2}{2}} g_{p+1}\left(\psi(x) e^{-\frac{x^2}{2}} z\right)$$

Since $g_{p+1}(z) = \frac{1}{p!} z^p e^{-z}$, this simplifies to give:

$$(6)\quad e^{-z} E\left[\left(\sqrt{V_{p+1}}\right)^k \middle| \mathcal{R}_p, X_{p+1}=z\right] = \int_0^\infty dx \left(\frac{x^{k-1}}{2^{\frac{k}{2}-1}}\right) \frac{1}{\Gamma\left(\frac{k}{2}\right)} e^{-\left(\frac{x^2}{2}(1+a) + z\psi(x) e^{-\frac{x^2}{2}}\right)}$$

We now introduce T an exponential variable, which is independent of Brownian motion, with parameter 1. Thus, we deduce from (6) that:

$$(7)\quad e^{-z} E\left[\left(\sqrt{2TV_{p+1}}\right)^k \middle| \mathcal{R}_p, X_{p+1}=z\right] = k\int_0^\infty dx\, x^{k-1} e^{-\frac{x^2}{2}(1+a)} e^{-\left(z\psi(x) e^{-\frac{x^2}{2}}\right)}$$

, so that, using integration by parts:

$$(8)\quad e^{-z} E\left[\left(\sqrt{2TV_{p+1}}\right)^k \middle| \mathcal{R}_p, X_{p+1}=z\right] = \int_0^\infty dx\, x^k \left\{-\frac{d}{dx}\left(e^{-\frac{x^2}{2}(1+a)} e^{-z\psi(x) e^{-x^2/2}}\right)\right\}$$

Therefore, we have obtained:

$$(9)\qquad e^{-z}\, P\left(\sqrt{2TV_{p+1}}\in dx \,\middle|\, \mathcal{R}_p, X_{p+1}=z\right) = dx\left\{-\frac{d}{dx}\left(e^{-\frac{x^2}{2}(1+a)-z\psi(x)e^{-\frac{x^2}{2}}}\right.\right.$$

Now, we have:

$$e^{-z}\, P\left(2TV_{p+1}\geq x^2 \,\middle|\, \mathcal{R}_p, X_{p+1}=z\right) = e^{-\frac{x^2}{2}(1+a)-z\psi(x)e^{-\frac{x^2}{2}}},$$

and, finally:

$$(10)\qquad e^{-z}\, E\left[\exp\left(-\frac{x^2}{2V_{p+1}}\right)\,\middle|\, \mathcal{R}_p, X_{p+1}=z\right] = e^{-\frac{x^2}{2}(1+a)-z\psi(x)e^{-\frac{x^2}{2}}}.$$

a formula which, of course, can be found directly from Mihael's paper... but, at the end of this story, there may be some interest in seeing how the computations for the 2-parameter hierarchy are related to the structure of the sequence:

$$(V_1, V_2, \dots, V_n, \dots\dots),$$

which, in the end, is described by the following

Theorem: Define $R_n \equiv \dfrac{V_{n+1}}{V_n}$ $(n\geq 1)$.

Then, the law of the sequence $(V_n, n\geq 1)$ is caracterized by the 3 following properties: (i) the random variables (R_n) are independent, and for each n, $R_n \overset{(law)}{=} U^{2/n}$

Now, fix $p\geq 1$. Then:

(ii) $X_{p+1} \overset{def}{=} \sqrt{\dfrac{\pi}{2}}\,\dfrac{L_1}{\sqrt{V_{p+1}}}$ is independent of the vector $(R_1, \dots, R_p)$ and is distributed as Z_{p+1}.

(iii) <u>the law</u> of V_{p+1} <u>conditionally on</u> $R_1, \ldots, R_p$, and X_{p+1}, <u>is caractrized by</u> (10).

<u>Remarks:</u> a) The statements of the Theorem should be presented somewhat differently since, e.g: the property (i) by itself suffices to caracterize the law of the sequence $(V_n, n \geq 1)$.

b) We could (should?) modify property (ii) as:

(ii') the increments: $X_1, X_2 - X_1, \ldots, X_{p+1} - X_p, \ldots$ are independent and distributed exponentially, with parameter 1.

c) The identity (10) admits many variants, some of which are the following: to relate efficiently formula (10) with Mihael's formulae, it is interesting to give:

$$E\left[\exp\left(-\frac{x^2}{2V_1}\right) \Big| \mathcal{R}_p, X_1 = \xi\right],$$

which I hope to discuss in the next document....