Basic Facts about Brownian Motion, Stochastic Integration and Stochastic Differential Equations

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Abstract : The aim of these lectures is to familiarise the reader with these basic facts, stated here for Brownian motion and semimartingales taking values in flat space $\mathbb{R}^n$, so that the same reader may become ready for an exposure of the variants of these facts for processes valued in manifolds.

Contents

1 Lecture A

1.1 Martingales and Stopping times

It is an astonishing fact that many complex notions, results, discussions involving (continuous-time, say) stochastic processes, may be tackled in terms of the simple notion of a $((\mathcal{F}_t), P)$ martingale. A $((\mathcal{F}_t), P)$ adapted, and integrable process $(M_t, t \geq 0)$ is a $((\mathcal{F}_t), P)$ martingale if :

$$\forall\, s \leq t\,, \qquad E[M_t|\mathcal{F}_s] = M_s$$

This notion, which is a mathematical manner of presenting a fair game, is intimately related to $(\mathcal{F}_t)$ stopping times, i.e :

random variables $T : (\Omega, \mathcal{F}) \to [0, \infty]$ such that : $\forall\, t\,, \quad (T \leq t) \in \mathcal{F}_t$

Theorem 1.1. ***(Doob's optional stopping theorem)***
1) Let (M_t) be $(\mathcal{F}_t)$ adapted and integrable. Then, it is a $((\mathcal{F}_t), P)$ martingale iff : $\forall$ $(\mathcal{F}_t)$ stopping time T, which only takes a finite number of values, $E[M_T] = E[M_0]$.
2) Assume that $(M_t), t \geq 0)$ is a uniformly integrable martingale. Then, for every pair (S, T) of $(\mathcal{F}_t)$ stopping times such that : $S \leq T$, one has :

$$E[M_T|\mathcal{F}_S] = M_S$$

- In a good Markovian setting, if $f : E \to \mathbb{R}$ belongs to the domain $\mathcal{D}(\mathcal{L})$ of the infinitesimal generator $\mathcal{L}$ of the E-valued Markovian process (X_t), then one has :

  $$P_t f(x) = f(x) + \int_0^t P_s(\mathcal{L}f)(x)ds\,,$$

  where (P_t) denotes the semigroup of (X_t).
  This identity is in fact equivalent to :

  $$C_t^f \stackrel{def}{=} f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s)ds$$

  is a $(P_x, (\mathcal{F}_t))$ martingale.

- Kunita (1969) took advantage of this equivalence to define the extended generator $\mathcal{L}_e$ as an operator defined on bounded functions f such that there exists $\tilde{f}$ with :

  $$f(X_t) - f(x) - \int_0^t \tilde{f}(X_s)ds$$

  is a $(P_x, (\mathcal{F}_t))$ martingale; Kunita's definition of $\mathcal{L}_e$ is then :

  $$\mathcal{L}_e(f) = \tilde{f}$$

1.2 A fundamental example of a continuous martingale : Brownian motion

Recall the central limit theorem

$$\frac{1}{\sqrt{n}}(X_1 + X_2 + ... + X_n) \underset{n\to\infty}{\overset{(law)}{\longrightarrow}} \mathcal{N}(0, \sigma^2)\,,$$

where $X_1 + X_2 + ... + X_n$ are iid, centered variables with second moment $\sigma^2 = E(X_1^2) < \infty$, and $\mathcal{N}(0, \sigma^2)$ denotes a centered Gaussian variable with variance σ^2.

This central result admits a process extension (: Donsker's theorem) :

$$\left(\frac{1}{\sqrt{n}}\mathcal{S}_{[nt]}, t \geq 0\right) \underset{n\to\infty}{\overset{(law)}{\longrightarrow}} \mathcal{N}(\sigma B_t, t \geq 0) \tag{1.1}$$

where $\mathcal{S}_k = X_1 + ... + X_k$, and $[x]$ =integer part of x.

It is easily shown from (1.1) that the process $(B_t, t \geq 0)$ on the RHS is Gaussian, centered, with independent increments; moreover, (B_t) and $(B_t^2 - t)$ are also easily shown to be martingales (with respect to $(\mathcal{B}_t = \sigma\{B_s, s \leq t\}$, $t \geq 0))$.

By Kolmogorov's continuity criterion, since (equally easily!) :

$$E[(B_t - B_s)^4] = 3\sigma^4(t-s)^2\,,$$

the process $(B_t, t \geq 0)$ admits a continuous version.

We may now define a standard real-valued Brownian motion $(B_t, t \geq 0)$ as being a centered, continuous, Gaussian process with independent increments, such that $B_0 = 0$ and $E(B_t^2) = t$. This process plays a role of prototype in many questions, i.e :

- it is the prototype of a continuous (local) martingale; see **1.6**, below;
- it is the prototype of a continuous Lévy process, that is an homogeneous process with independent increments, and continuous paths, since all such processes may be written as : $\sigma B_t + \mu t$, $t \geq 0$, for two constants σ and μ.

1.3 The predictable bracket between two martingales

Theorem 1.2.

If (M_t) and (N_t) are two locally square integrable martingales (: this condition is automatically fulfilled if M and N are continuous), then there exists a unique predictable process with bounded variation, written $(< M, N >_t, t \geq 0)$ such that : $(M_t N_t - < M, N >_t, t \geq 0)$ is a local martingale.

Comments

- By uniqueness, the application : $(M,N) \to <M,N>$ is bilinear, symmetric, and positive.
- Furthermore, there is the Kunita-Watanabe inequality :

$$\int_0^t |H_s K_s| |d<M,N>_s| \leq \left(\int_0^t H_s^2 d<M>_s\right)^{\frac{1}{2}} \left(\int_0^t K_s^2 d<N>_s\right)^{\frac{1}{2}}$$

  where, for simplicity, we denote $(<M>_t)$ for $<M,M>_t$.

In the Markovian theory, for *a good Markov process* $(X_t)_{t\geq 0}$ (which means, in particular here, that every locally square integrable martingale $(M_t, t \geq 0)$ - with respect to any of the $(P_x)'s$ - satisfies : $d<M>_t = m_t dt$, for some predictable process $m_t \geq 0$) the brackets : $<C^f, C^g>_t$, $f, g \in \mathcal{D}(\mathcal{L})$, satisfy : $<C^f, C^g>_t = \int_0^t \Gamma(f,g)(X_s) ds$ where $\Gamma : \mathcal{D}(\mathcal{L}) \times \mathcal{D}(\mathcal{L}) \to (.)$ denotes the "opérateur carré du champ", whose existence is due to Kunita, and which was rediscovered by Roth (1976); for a good Markov process (X_t), $\mathcal{D}(\mathcal{L})$ is an algebra, and $\Gamma(f,g) = \mathcal{L}(fg) - f\mathcal{L}(g) - g\mathcal{L}(f)$.

1.4 Stochastic integrals; Itô's formula

Itô's formula is the fundamental theorem of stochastic calculus, just as one speaks of the fundamental theorem of ordinary integral/differential calculus.

Theorem 1.3.
Let $(X^1, ..., X^n)$ be a n-dimensional continuous $(\mathcal{F}_t)$ semimartingale. Then, for every $f \in C^2(\mathbb{R}^n)$, we have :

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s).dX_s + \frac{1}{2}\int_0^t \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(X_s) d<X^i, X^j>_s$$

Comments
It is the second order term which is the novelty, with respect to ordinary calculus. However, the geometric interpretation of this second term shall be discussed in the different courses.

1.5 Lévy's characterization of Brownian motion

Lévy's theorem (Theorem 1.5 below) is extremely powerful as it allows to recognize that a given process is a Brownian motion from just one (or two !) martingale properties.

Theorem 1.4.
The only continuous local martingale $(M_t)_{t\geq 0}$ *such that* $(M_t^2 - t, t \geq 0)$ *is also a local martingale is Brownian motion.*

Proof
Assume that $(M_t, t \geq 0)$ satisfies these properties; then, by Itô's formula, for any simple function $f : \mathbb{R}_+ \to \mathbb{R}$, we get :

$$\exp\left(i\int_0^t f(u)dM_u + \frac{1}{2}\int_0^t f^2(u)du\right), \quad t \geq 0,$$

is a local martingale; but, since it is bounded, it is a true martingale, and we get :

$$E\left[\exp\left(i\int_0^t f(u)dM_u\right)\right] = \exp\left(-\frac{1}{2}\int_0^t f^2(u)du\right)$$

which shows at once that the distribution of the increments $(M_{t_1}, M_{t_2} - M_{t_1}, ..., M_{t_n} - M_{t_{n-1}})$ is that of $(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})$, hence the result.

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1.6 The Dambis / Dubins-Schwarz (1965) and Knight (1970) representation theorems

1.6.1 The Dambis / Dubins-Schwarz theorem gives an important representation of a generic continuous local $(\mathcal{F}_t)$ martingale $(M_t)_{t\geq 0}$, as a time change of a Bronwian motion; precisely :

$$M_t = \beta_{<M>_t}, t \geq 0$$

where $(\beta_u, u \geq 0)$ is a $(\mathcal{F}_{\tau_u})$ Brownian motion $[\tau_u = \inf\{t :< M >_t > u]$. The details are given in the paragraph 1.6.2.

1.6.2 To any continuous local martingale $(M_t)_{t\geq 0}$, one may associate its quadratic variation process $(< M >_t, t \geq 0)$ which may be defined by

$$< M >_t = P - \lim_{n\to\infty} \sum_{t_1^{(n)}<...<t_p^{(n)}\leq t} (M_{t_{i+1}^{(n)}} - M_{t_i^{(n)}})^2 \text{ where } \sup_i (t_{i+1}^{(n)} - t_i^{(n)}) \underset{n\to\infty}{\longrightarrow} 0$$

It is not difficult to show that

$$M_t^2 - < M >_t, \; t \geq 0, \tag{1.2}$$

is a local martingale, and since $(< M >_t)$ is increasing and continuous, the property (1.2) characterizes $(< M >_t)$. Note that this discussion is closely related to Theorem 1.2, where in greater generality, the existence and uniqueness of $(< M >_t)$ is asserted.

Theorem 1.5. ***(Dambis / Dubins-Schwarz (1965))***
If $(M_t, t \geq 0)$ is a continuous local $(\mathcal{F}_t)$ martingale such that $< M >_\infty = \infty$, then the process :

$$\beta_u = M_{\tau_u}\,, \qquad \tau_u = \inf\{t :< M >_t > u\}$$

is a $(\mathcal{F}_{\tau_u})$ Brownian motion, and M may be represented as :

$$M_t = \beta_{<M>_t}\,, \qquad t \geq 0\,.$$

Proof of the theorem (Sketch) :
By time-changing, one finds that both (β_u) and $(\beta_u^2 - u)$ are two continuous $(\mathcal{F}_{\tau_u})$ martingales (to be carefully justified). It then suffices to use Lévy's theorem to conclude; then, undo the time change...

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1.6.3 This theorem extends partially to any k-tuple $(M^1, ..., M^k)$ of continuous $(\mathcal{F}_t)$ local martingales, such that : $< M^i, M^j >_t = 0$, for $i \neq j$.
Such a k-tuple may be represented as :

$$M_t^i = \beta^i_{<M^i>_t}\,,\ t \geq 0\,,\ i = 1, 2, ..., k,$$

with k independent Brownian motions $(\beta_i)_{1 \leq i \leq k}$. This theorem is due to F. Knight (1970); a difference with the Dubins-Schwarz theorem is that, here, we have lost "the notion of time"...

1.6.4 However, there is a particular case where time is not lost, i.e : when the $(< M^i >_t)$ are equal. Call (A_t) this common process, and :

$$\tau_u = \inf\{t : A_t > u\}\,,\ u \geq 0\,.$$

Then, $(\beta_u^i)_{u \geq 0; i=1,2,...,k}$ is a $\mathbb{R}^k$ - Brownian motion - with respect to $(\mathcal{F}_{\tau_u})$. This very interesting case occurs in particular for $k = 2$, with conformal martingales $(Z_t \equiv X_t + iY_t, t \geq 0)$ denoted here in a complex manner. Then, a conformal martingale $(Z_t, t \geq 0)$ may be defined as a $\mathbb{C}$-valued continuous local martingale such that $(Z_t^2, t \geq 0)$ is also a $\mathbb{C}$-martingale, this is equivalent to the "Cauchy-Riemann equations" : $< X >_t = < Y >_t$; $< X, Y >_t = 0$. The

reason for this name is that, if $(Z_t, t \geq 0)$ denotes now a $\mathbb{C}$-valued Brownian motion, and $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, then $(f(Z_t), t \geq 0)$ is a conformal martingale.
The $\mathbb{C}$-extension of the Dubins-Schwarz-Knight theorem may then be written as :

$$\boxed{f(Z_t) = \widehat{Z}_{\int_0^t |f'(Z_u)|^2 du} \quad , \; t \geq 0 \, ,} \tag{1.3}$$

where f' is the $\mathbb{C}$-derivative of f, and $\widehat{Z}$ denotes another $\mathbb{C}$-valued Brownian motion. This is an extremely powerful result due to P. Lévy (1943), which expresses the *conformal invariance of* $\mathbb{C}$*-valued* Brownian motion.
It is easily shown, as a consequence of (1.3), using the exponential function that, if $Z_0 = a$, then $(Z_t, t \geq 0)$ shall never visit $b \neq a$ (of course, almost surely). As a consequence, (1.3) may be extended to any meromorphic from $\mathbb{C}$ to itself, when $P(Z_0 \in \mathcal{S}) = 0$ with $\mathcal{S}$ the set of singular points of f.

References :

- B. Davis :
  Brownian Motion and Analytic Functions.
  Ann. Prob. (1979).

- R. Durrett :
  Brownian Motion and Martingales in Analysis.
  Wadsworth (1984).

1.7 Stratonovich versus Itô's integrals

Let H and X be two continuous semimartingales in the same filtration. The Stratonovich integral of H with respect to X may be defined, and denoted as :

$$\int_0^t H_s \circ dX_s = \int_0^t H_s dX_s + \frac{1}{2} < H, X >_t \tag{1.4}$$

where on the RHS, $\left(\int_0^t H_s dX_s , \; t \geq 0 \right)$ denotes the Itô integral.
We now discuss both the interest, and the intrinsic character of this definition :

a) Assume that (X_t) is a continuous semimartingale, and $f : \mathbb{R} \to \mathbb{R}$ is a C^3 function.
   Then, Itô's formula :

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d < X >_s$$

may be written in the simpler form :

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s$$

as a Stratonovich integral.
At this point, the reader may think of this as an adhoc definition, but in fact there is a nice geometric justification, which we now discuss :

b) The sequence of integrals :

$$\sum_{\tau_n} \int_0^1 f(X_{t_i} + h(X_{t_i+1} - X_{t_i}))dh(X_{t_i+1} - X_{t_i})$$

converges as $n \to \infty$, towards :

$$\int_0^t f(X_u) \circ dX_u$$

as soon as f is in C^1; one needs to be more careful with integrals such as :

$$\sum_{\tau_n} \int_0^1 f(X_{t_i+s(t_{i+1}-t_i)})d\mu(s)(X_{t_i+1} - X_{t_i}),$$

as $n \to \infty$.

The previous discussion extends easily to $(X_t, t \geq 0)$ an $\mathbb{R}^d$-valued continuous semimartingale, and yields the following extension of Itô's formula.

Proposition 1.6.
Let $\pi = \sum_{i=1}^d f_i(x_1, ..., x_d)dx_i$ be a closed differential form of C^1 class on an open set U of $\mathbb{R}^d$, and let (X_t) be a continuous semimartingale taking values in U. Then :

$$\int_{X_{(0,t)}(\omega)} \pi = \int_0^t \sum_{i=1}^d f_i(X_s)dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial f}{\partial x_j}(X_s)d < X^i, X^j >_s \qquad (1.5)$$

where $\int_{X_{(0,t)}(\omega)} \pi$ denotes the integral of π, along the continuous path $(X_s(\omega), s \geq 0)$, for $0 \leq s \leq t$. Recall that, if $\gamma : [0,1] \to U$ is a continuous path, then $\int_\gamma \pi$ is defined as $\hat{\pi}(\gamma(1)) - \hat{\pi}(\gamma(0))$, where $\hat{\pi}$ denotes a primitive of π along a chain of balls covering the graph of γ.

Sketch of proof.
Formula (1.5) follows from the approximation :

$$\int_{X_{(0,t)}(\omega)} \pi = a.s. \lim_{n\to\infty} \sum_{\tau_n} \int_0^1 \sum_{j=1}^d f_j(X_{t_i} + h(X_{t_{i+1}} - X_{t_i}))dh(X^j_{t_{i+1}} - X^j_{t_i})$$

where (τ_n) is a sequence of subdivisions of $[0,t]$, whose meshes tend to 0.
In particular, if $(Z_u, u \leq T_U)$ denotes complex Brownian motion considered before its exit time from the open domain U, and if $f : U \to \mathbb{C}$ is an holomorphic function, then :

$$\int_{Z_{(0,t)}(\omega)} f(z)dz = \int_0^t f(Z_s)dZ_s\,, \qquad t < T_U,\ a.s.$$

Example 1.7.1. $U = \mathbb{C} \setminus \{0\};\ f(z) = \frac{1}{z}$
This allows to give a representation of the continuous determination of the argument of $(Z_u, u \geq 0)$ *around* 0 *as :*

$$\theta_t - \theta_0 = Im\left(\int_0^t \frac{dZ_s}{Z_s}\right) = \int_0^t \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2}$$

References :

- M. Yor :
  Sur quelques approximations d'intégrales stochastiques.
  Sém. Prob. XI., Springer (1977).

- Ph. Protter :
  Stochastic Integration and Differential Equations.
  Second edition, Springer (2004).

2 Lecture B

2.1 Stochastic Differential Equations (: SDEs)

Immediately after he constructed stochastic integrals with respect to Brownian motion, Itô used his definition to consider SDE's; i.e solutions of :

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \tag{2.1}$$

where : $\sigma : \mathbb{R}^n \to \mathcal{M}_{n\times n}\,,\ b : \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 2.1.
Assume that σ *and* b *are Lipschitz. Then, the equation (2.1) admits only one solution, which may be obtained with the help of Picard's iteration procedure.*

2.2 Some examples : linear SDE's, Bessel processes,...

Example 2.2.1. *(Linear SDE's)*
For simplicity, we take here $n = 1$, and we may consider :

$$X_t = x + B_t + \lambda \int_0^t ds\, X_s$$

the SDE satisfied by the Ornstein-Uhlenbeck process with parameter λ. It solves explicitly as :

$$X_t = e^{\lambda t}(x + \int_0^t e^{\lambda s} dB_s)\,,$$

with the help of var. of constant.
In fact, one may solve quite generally the following linear SDE, where both H and Z are continuous semimartingales :

$$X_t = H_t + \int_0^t X_s\, dZ_s\,.$$

Then, one obtains :

$$X_t = \mathcal{E}(Z)_t \left(H_0 + \int_0^t \mathcal{E}(Z)_s^{-1}(dH_s - d < H, Z >_s) \right)$$

where

$$\mathcal{E}(Z)_t = \exp\{Z_t - Z_0 - \frac{1}{2} < Z >_t\}$$

Example 2.2.2. ***(Bessel processes)***
Consider first $R_t = |B_t|$, the Euclidean norm of n-dimensional Brownian motion ($n \geq 1$). Then, it is not difficult to justify, via Itô's formula, that (R_t) satisfies :

$$R_t = R_0 + \beta_t + \frac{n-1}{2} \int_0^t \frac{ds}{R_s} \qquad ((\beta_t) : 1 - dim.B.M.) \qquad (2.2)$$

Of course, the singular drift creates some difficulty in the discussion of this SDE, but if one considers : $Z_t = R_t^2$, then (2.2) transforms in :

$$Z_t = Z_0 + 2 \int_0^t \sqrt{Z_s} d\beta_s + n.t\,; \quad Z_t \geq 0\,,$$

an equation which is shown to enjoy path uniqueness.

2.3 Their Markov property

Denote by $(P_x)_{x\in\mathbb{R}^n}$ the family of the laws - considered on $\Omega_{can} \equiv \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ - of the solutions of equation (2.1). These are well-defined (e.g : with the help of Picard's iteration procedure). These laws are Markovian, i.e :

$$E_x[F \circ \theta_t | \mathcal{F}_t] = E_{X_t}(F)$$

The infinitesimal generator, of (X_t), under these laws, may easily be given in terms of σ and b.

2.4 Another approach : the martingale problem

This approach may be explained as follows :

(i) We saw, in Theorem 1.5, the simple characterization of Brownian motion in terms of just two martingale properties;

(ii) it is a natural question to ask whether there may exist a similar characterization for "diffusions", i.e : Markov processes (taking values in $\mathbb{R}^n$, say), with continuous paths.
In fact, such a characterization exists, and one may go back and forth between the formulation of a martingale problem, and a presentation as a solution of Itô's SDE (2.1); more precisely :

- if (X_t) solves the SDE (2.1), then, for every $f \in \mathcal{C}^2\mathbb{R}$,

$$f(X_t) - f(X_0) - \int_0^t ds \mathcal{L} f(X_s)$$

is a local martingale, where :

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2(x)\frac{d^2 f}{dx^2} + b(x)\frac{df}{dx} ;$$

Any probability P on Ω_{can} which satisfies this property, with respect to (X_t) the canonical process, and its canonical filtration will be said to be a solution to the martingale problem $\mathcal{M}_{\sigma^2,b}$.

- Conversely, if P is a solution to the martingale problem $\mathcal{M}_{\sigma^2,b}$, then, by taking $f(x) = x$, and $f(x) = x^2$, and applying the Itô-Lévy characterization of Brownian motion, one can show that under P, (X_t) is the solution of an Itô SDE with diffusion coefficient σ, and drift coefficient b.

- To summarize : A priori, the SDE presentation of a diffusion may not be given, but one may always (at least : often!) present the diffusion in this form...
  In any case, if b and/or σ are not regular, the "real work" still needs to be done, i.e is there uniqueness in law for $\mathcal{M}_{\sigma^2,b}$? The Itô presentation may help as shown by Stroock-Varadhan : Multidimensional Diffusions, Springer (1979). P. Priouret : Ecole d'Eté de St-Flour (1974) gives a detailed discussion of these points.

2.5 Weak and Strong solutions

If (X_t) solves (2.1), then it is called a strong solution if (X_t) is adapted to the natural filtration of B. If not, (X_t) is called a weak solution.

Theorem 2.2. *(Watanabe-Yamada [-])*
Pathwise uniqueness implies that the solution is strong.

Another uniqueness is that of uniqueness in law, i.e : on a given probability space, there may exist several pathwise solutions to equation (2.1), but their distributions are equal. The classical case which is discussed in every text-book is that of : $\begin{cases} \sigma(x) = sgn(x) \\ b(x) = 0 . \end{cases}$
Indeed, in that case (with $x = 0$), if (X_t) is a solution, then so is $(-X_t)$. But, both of them are Brownian motions, thanks to Lévy's theorem 1.5.

3 Lecture C

3.1 The Martingale representation theorem for Brownian motion

is the following

Theorem 3.1.
If $(M_t, t \geq 0)$ is a local martingale with respect to the filtration $(\mathcal{B}_t, t \geq 0)$ generated by an n-dimensional Brownian motion $(B_t, t \geq 0)$, it may be represented as :

$$M_t = c + \int_0^t m_s.dB_s\,, \tag{3.1}$$

where $(m_s)_{s\geq 0}$ is a $(\mathcal{B}_s)$ predictable process, taking values in R^n, and $x.y$ denotes the Euclidean scalar product between x and y. For (3.1) to make sense, it is necessary that m satisfies : $\int_0^t |m_s|^2 ds < \infty$ a.s, for every t.

Proof

(i) [Warm-up] Again, similarly to the proof of Theorem 1.5, we first prove the result for exponential martingales of the form :

$$\mathcal{E}_t^f := \exp\left(\int_0^t f(s).dB_s - \frac{1}{2}\int_0^t f^2(s)ds\right)$$

for $f \in L^2_{\ell oc}(\mathbb{R}_+; \mathbb{R}^n)$.
By Itô's formula, we get :

$$\mathcal{E}_t^f = 1 + \int_0^t \mathcal{E}_s^f f(s).dB_s ;$$

thus, in case

$$M_t = \mathcal{E}_t^f \,, \text{ we get : } m_t = \mathcal{E}_t^f f(t)$$

(ii) We now show that, if $X \in L^2(\mathcal{B}_\infty; \mathbb{R})$, then there exists a predictable process (x_s), taking values in $\mathbb{R}^n$, such that :

$$X = E(X) + \int_0^\infty x_s.dB_s \,,$$

with :

$$E\left[\int_0^\infty |x_s|^2 ds\right] < \infty$$

(this ensures that : $E[X^2] = (E[X])^2 + E\left[\int_0^\infty |x_s|^2 ds\right]$). Consequently, in order to prove the theorem, it suffices to show that a total family of variables in $L^2(\mathcal{B}_\infty)$ may be represented in the preceding form.
Such a total family may be obtained from :

$$\mathcal{E}_\infty^f \equiv \exp\left(\int_0^\infty f(s).dB_s - \frac{1}{2}\int_0^\infty |f(s)|^2 ds\right) ,$$

where we now assume $f \in L^2(\mathbb{R}_+, \mathbb{R}^n)$.
The stochastic integral representation of this variable is obtained from i), since : $\mathcal{E}_t^f = E[\mathcal{E}_\infty^f | \mathcal{B}_t]$. The fact that $\Lambda \equiv \{\mathcal{E}_\infty^f; f \in L^2(\mathbb{R}_+, \mathbb{R}^n)\}$ is total in $L^2(\mathcal{B}_\infty)$ follows from :
if $Y \in L^2(\mathcal{B}_\infty)$ satisfies : $E[Y\mathcal{E}] = 0$, $\forall \mathcal{E} \in \Lambda$, then :

$$E\left[Y \exp\left(\int_0^\infty f(s).dB_s\right)\right] = 0 \,, \; \forall f \in L^2(\mathbb{R}_+)$$

but this implies that $Y = 0$, hence Λ is total.

■

Comments on Theorem 3.1.
(c.1) Wiener's chaotic decomposition of $L^2(\mathcal{B}_\infty)$, as :

$$L^2(\mathcal{B}_\infty) = \bigoplus_{k=0}^{\infty} \mathcal{C}_k \ ,$$

where

$$\mathcal{C}_k = \left\{ \int_0^\infty dB_{t_1}. \int_0^{t_1} dB_{t_2}.... \int_0^{t_{k-1}} dB_{t_k} f_k(t_1, ..., t_k); f_k \in L^2(\Delta_k; \mathbb{R}^n) \right\}$$

with

$$\Delta_k \equiv \{(t_1, ..., t_k); t_1 > t_2 > ... > t_k\}$$

may also be obtained easily for the variables $\{\mathcal{E}^f_\infty\}$, and then extended to all variables $X \in L^2(\mathcal{B}_\infty)$.
The chaos decomposition of $\mathcal{E}^f_\infty$ is :

$$\mathcal{E}^f_\infty = 1 + \sum_{k=1}^{\infty} \int_0^\infty dB_{t_1}. \int_0^{t_1} dB_{t_2}.... \int_0^{t_{k-1}} dB_{t_k} f(t_1) f(t_2) ... f(t_k)$$

(c.2) Another proof, due to C. Dellacherie, of the martingale representation theorem for Brownian motion consists in remarking that, as a consequence of Theorem 5, W, the distribution of Brownian motion, alias : Wiener's measure is an extremal point in the convex set $\mathcal{M}$ of all probabilities on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ which make the canonical process $(X_t)_{t \geq 0}$ a martingale. Then, one can show that : $\text{ext}(\mathcal{M})$ consists precisely of the probabilities P with respect to which $(X_t)_{t \geq 0}$ enjoys the martingale representation property.

3.2 Girsanov's theorem

This theorem underwent a series of "mutations", which seem to have stabilized in the following form.

Theorem 3.2.
Let P and Q two locally equivalent probability measures on $\{\Omega, \mathcal{F}, (\mathcal{F}_t)\}$; denote :

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = D_t = \exp(L_t - \frac{1}{2} < L >_t)$$

Then, if (M_t) is a $(P, \mathcal{F}_t)$ local martingale, it is a $(Q, \mathcal{F}_t)$ semi-martingale, which may be written as :

$$M_t = \widetilde{M}_t + \int_0^t \frac{d < M, D >_s}{D_s} \equiv \widetilde{M}_t + < M, L >_t$$

where $(\widetilde{M}_t)_{t\geq 0}$ *is* $(Q, (\mathcal{F}_t))$ *local martingale.*

Corollary 3.2.1.
If $(M_t)_{t\geq 0}$ *is a* $((\mathcal{F}_t), P)$ *Brownian motion, then* $(\widetilde{M}_t)_{t\geq 0}$ *is a* $((\mathcal{F}_t), Q)$ *Brownian motion.*

We now give a number of examples of applications.

Example 3.2.1.
For $\mu \in \mathbb{R}$, *denoted by* $W^{(\mu)}$ *the law of* $B_t^{(\mu)} \equiv B_t + \mu t$, *Brownian motion with drift* μ, *on* Ω_{can}, *the canonical space* $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, *where* $X_t(\omega) = \omega(t)$, *and* $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$.
Then, there is the Cameron-Martin relationship :

$$W^{(\mu)}_{|\mathcal{F}_t} = \exp\left(\mu X_t - \frac{\mu^2 t}{2}\right) . W_{|\mathcal{F}_t} \tag{3.2}$$

This may be seen immediately from Theorem 2.2, which implies that, under $W^{(\mu)}$, *one gets :*

$$X_t = \widetilde{X}_t + \mu t\,,$$

with $(\widetilde{X}_t)$ *a* $((\mathcal{F}_t), W^{(\mu)})$ *Brownian motion.*

Example 3.2.2.
There is an absolute continuity relationship between the laws $(P_r^{(\mu)})$ *of Bessel processes with "index"* μ, *starting from* $r > 0$, *as* μ *varies in* $\mathbb{R}_+$; *in fact, a very interesting result due to J. Lamperti (1972) asserts that :*

$$\exp(B_t^{(\mu)}) = R^{(\mu)}_{\int_0^t ds \exp(2B_s^{(\mu)})},$$

where $(R_u^{(\mu)}, u \geq 0)$ *denotes a Bessel process with index* μ, *or "dimension"* $\delta = 2(1+\mu)$.
Then, the following relationship is easily deduced from (3.2) by using time-changes :

$$P^{(\mu)}_{r|\mathcal{R}_u} = \left(\frac{R_u}{r}\right)^{\mu} \exp\left(-\frac{\mu^2}{2}\int_0^u \frac{ds}{R_s^2}\right) . P^{(0)}_{r|\mathcal{R}_u}$$

This relationship may be extended conveniently for any pair μ, ν *of indexes... However, if* $\mu > 0$, *and* $\nu < 0$, *there is only an absolute continuity relationship, the most famous of which being :* $\mu = +\frac{1}{2}$, $\nu = -\frac{1}{2}$, *then :*

$$P^{(\frac{1}{2})}_{r|\mathcal{R}_u} = \left(\frac{R_{u\wedge T_0}}{r}\right) . W_{r|\mathcal{R}_u}$$

Example 3.2.3. *BM and O.U processes*
The Ornstein-Uhlenbeck process (abbreviated as O.U) with parameter λ may be defined as the solution to :

$$U_t = u + B_t + \lambda \int_0^t U_s ds$$

Its distribution $\widetilde{P}_u^{(\lambda)}$ satisfies :

$$\begin{aligned} \widetilde{P}^{\lambda}_{u|\mathcal{F}_t} &= \exp\left\{\lambda \int_0^t X_s dX_s - \frac{\lambda^2}{2}\int_0^t X_s^2 ds\right\}.W_{u|\mathcal{F}_t} \\ (\dagger) \quad &= \exp\left(\frac{\lambda}{2}(X_t^2 - u^2 - t) - \frac{\lambda^2}{2}\int_0^t X_s^2 ds\right).W_{u|\mathcal{F}_t} \end{aligned}$$

This formula extends easily to the n-dimensional versions (of BM and O.U processes), and leads to formulae such as :

$$\begin{aligned} & E_W\left(\exp\left\{-\frac{\lambda^2}{2}\int_0^1 ds\, X_s^2\right\}|X_i = a\right) \\ = & \left(\frac{\lambda}{\sinh\lambda}\right)^{\frac{1}{2}} \exp\left(-\frac{a^2}{2}((\coth\lambda) - 1)\right), \end{aligned}$$

which is closely related to Lévy's stochastic area formula (for planar BM).

3.3 The Clark-Ocone formula; examples

The Clark-Ocone formula gives an expression of the Itô integrand φ_s of a differentiable functional $F(B_u, u \leq 1)$ on Wiener space, in the direction of the Cameron-Martin space

$$H = \left\{h : [0,1] \to \mathbb{R};\ h(t) = \int_0^t ds\, h'(s) \ \text{with} \ \int_0^1 ds (h'(s))^2 < \infty\right\}$$

This formula is :

$$F(B) = E[F(B)] + \int_0^1 \varphi_s\, dB_s$$

with :

$$\varphi_s = E[F'(B;]s,1])|\mathcal{B}_s],$$

where $F'(B; dt)$ is the signed measure (assumed to exist) such that :

$$\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}(F(B+\varepsilon h) - F(B)) = \int_0^1 F'(B; dt) h(t) = \int_0^1 ds\, h'(s) F'(B;]s,1]),$$

by integration by parts.

References :

- J. Clark:
  The representation of functionals of Brownian motion by stochastic integrals.
  Ann. Math. Stat 41 (1970).

- D. Ocone :
  Malliavin's Calculus and Stochastic Integral Representations of Functionals of diffusion processes.
  Stochastics (1984), vol. 12, p. 161-185.

Examples; Exercises
In practice, given $F \in L^2(\mathcal{B}_1)$, one may compute in many instances :

$$E[F|\mathcal{B}_t] = E_{B_t}[F(\omega/t/.)] \,,$$

and under some hypothesis on F, one may justify the formula :

$$F = E[F] + \int_0^1 \frac{\partial}{\partial x}_{|x=B_s} (E_x[F(\omega/s/.)])\, dB_s$$

Thus, as an example $F = f(B_1)$, for any bounded, Borel f.

3.4 Bismut-type formulae; a little Malliavin calculus ?

This will be developed orally; among the many references to Malliavin calculus, the book by D. Nualart (Springer; 1995) is probably one of the most accessible.