Enlarging the 2D - Brownian filtration with a subordinated perpetuity.

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1)

1. Introduction.

(1.1) Let $(B_t, t \geq 0)$ and $(\gamma_t, t \geq 0)$ be two independent 1-dimensional Brownian motions; for $\mu < 0$, and $\nu \in \mathbb{R}$, we consider the associated Brownian motions with respective drifts μ and ν:

$$B_t^{(\mu)} = B_t + \mu t \quad , \quad \text{and} \quad \gamma_t^{(\nu)} = \gamma_t + \nu t \quad , \quad t \geq 0,$$

and we define the process: $$X_t^{(\mu,\nu)} = \int_0^t d\gamma_s^{(\nu)} \exp(B_s^{(\mu)}) \ , \ t \geq 0 .$$

It has recently been remarked ([1], [6]) that the law of the so-called subordinated perpetuity:

$$X^{(\mu,\nu)} \overset{def}{=} X_\infty^{(\mu,\nu)} \quad (\text{which is well defined, since } \mu < 0)$$

is the generic type IV Pearson distribution (Pearson [4], Johnson-Kotz ([2], Chapter 12), Wong [5], ...); precisely, one has the following

Theorem 1 ([1], [6]): *The law of $X^{(\mu,\nu)}$ admits the density:* (1)

$$f_{\mu,\nu}(x) = \frac{C_{\mu,\nu}}{(1+x^2)^{\frac{1}{2}-\mu}} \exp\left(2\nu \arctan(x)\right)$$

For simplicity, we shall now write only f for $f_{\mu,\nu}$; the following quantities will play an important role in the sequel:

Il faut donner la Cte; en particulier, $C_{\mu,0} = ?$

(Voir de l'autre papier).

$$\left\{ \frac{1}{\int_0^t ds \exp\left(\alpha B_s^{(\mu)}\right)} , t > 0 \right\} \overset{(\text{law})}{=} \left\{ \frac{1}{\int_0^t ds \exp\left(\alpha B_s^{(-\mu)}\right)} + \frac{1}{\int_0^\infty ds \exp\left(\alpha \tilde{B}_s^{(\mu)}\right)} , t > 0 \right\}$$

where, on the RHS, $B^{(-\mu)}$ denotes a Brownian motion with positive drift $(-\mu)$, and $\tilde{B}^{(\mu)}$ a copy of $B^{(\mu)}$, assumed to be independent of $B^{(-\mu)}$.

(1.2) Our motivation to develop this case study of enlargements come from two origins, at least:

a) there is presently a lot of interest in Mathematical Finance to study how inside trading may modify the pricing framework in a given market; mathematically, this may be translated in terms of an enlargement of filtration;

b) we hope the present study may help us to develop further extensions of Pitman's theorems just as [3] led us to [3′]

(to be developed more explicitly)

2. Proof of Theorem 2.

(2.1) The presentation of the initial enlargement formula given in Chapter 12, p. 33-34 of [7] applies (with only one change, made necessary by the fact that our filtration $\{\mathcal{F}_t\}$ is generated by a 2D-Brownian motion instead of a one-dimensional one.

We may summarize this presentation as follows:

denoting $\phi_x(t) = \phi(t,x)$ the density of the conditional distribution of X given $\mathcal{F}_t$, and writing:

$$\phi_x(t) = \phi_x(0) \exp\left\{ \int_0^t \left(\rho_1(s,x)\, dB_s + \rho_2(s,x)\, d\gamma_s \right) - \frac{1}{2} \int_0^t ds \left[\rho_1^2 + \rho_2^2 \right](s,x) \right\}$$

a generic $\{\mathcal{F}_t\}$ martingale:

$$M_t = \int_0^t \left(m_1(s)\, dB_s + m_2(s)\, d\gamma_s \right)$$

$$\left\{ \frac{1}{\int_0^t ds \exp\left(\alpha B_s^{(\mu)}\right)} , t > 0 \right\} \overset{(law)}{=} \left\{ \frac{1}{\int_0^t ds \exp\left(\alpha B_s^{(-\mu)}\right)} + \frac{1}{\int_0^\infty ds \exp\left(\alpha \tilde{B}_s^{(\mu)}\right)} , t > 0 \right\}$$

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a generic $\{\mathcal{F}_t\}$ martingale:

$$M_t = \int_0^t \left(m_1(s)\, dB_s + m_2(s)\, d\gamma_s \right)$$

is decomposed in $\{\hat{\mathcal{F}}_t\}$ as:

$$M_t = \hat{M}_t + \int_0^t ds \left\{ m_1(s)\, \rho_1(s, X) + m_2(s)\, \rho_2(s, X) \right\}$$

Thus, in order to prove Theorem 2, it only remains to find ρ_1 and ρ_2.

(2.2) the process $\{\phi_x(t);\, t \geq 0,\, x\}$ of conditional densities given $\{\mathcal{F}_t\}$

is now found to be (with the help of Theorem 1):

$$\boxed{\phi_x(t) = \frac{1}{e_t^{(\mu)}}\, f\left(\frac{x - X_t}{e_t^{(\mu)}}\right)}, \quad \text{where: } e_t^{(\mu)} = \exp(B_t^{(\mu)}).$$

Note, in particular, that this provides an interesting family of $\{\mathcal{F}_t\}$ martingales.

Next, ρ_1 and ρ_2 are obtained after writing $\{\phi_x(t),\, t \geq 0\}$ in exponential form; one finds:

$$\rho_1(s, x) = -\psi\left(\frac{x - X_s}{e_s^{(\mu)}}\right), \quad \text{and} \quad \rho_2(s, x) = -\varphi\left(\frac{x - X_s}{e_s^{(\mu)}}\right),$$

which completes the proof of Theorem 2.

3. A discussion of the enlargement formula (3).

To start with, we would like to compare formula (3) with the (enlargement) formula for the filtration of $(B_t, t \geq 0)$, enlarged with:

$$A^{(\mu)} \equiv A_\infty^{(\mu)} = \int_0^\infty ds\, \exp(2B_s^{(\mu)}),$$

which is presented in [3], where it reads as follows:

$$(5) \qquad B_t^{(\mu)} = B_t^{*(-\mu)} - \int_0^t ds\, \frac{\exp(2B_s^{(\mu)})}{(A^{(\mu)} - A_s^{(\mu)})}.$$

with the notation $\{B_t^{*(-\mu)}\}$ for an $\mathcal{F}_t^* \equiv \mathcal{F}_t \vee \sigma\{A^{(\mu)}\}$ Brownian motion.

(at this point, we warn the reader that in [3], the negative drift is denoted by $(-\mu)$, so that μ in [3] is changed in $(-\mu)$ here).
We now make several comparisons between formulae (3) and (5).

(3.1) Let us consider the case $\nu = 0$ in Theorem 2, so that the function ψ becomes:

$$\psi_0(z) \equiv (2\mu) + \frac{1-2\mu}{1+z^2}$$

Let us further remark that formula (5) is also the enlargement formula in $\mathcal{G}^*_t = \mathcal{F}_t \vee \sigma(A^{(\mu)}) \vee \sigma(\hat{\gamma}_u, u \geq 0)$, where $(\hat{\gamma}_u, u \geq 0)$ denotes the Dubins-Schwarz Brownian motion associated with the martingale:

$$X^{(\mu)}_t = \int_0^t d\gamma_s \exp(B^{(\mu)}_s) = \hat{\gamma}_{A^{(\mu)}_t}.$$

Since $\hat{\mathcal{F}}_t \equiv \mathcal{F}_t \vee \sigma(X^{(\mu)}_\infty) \subset \mathcal{G}^*_t$, for every t, in order that formulae (5) and (3) be coherent, the following conditional expectation relation must hold:

$$E\left[\frac{\exp(2B^{(\mu)}_s)}{A^{(\mu)} - A^{(\mu)}_s} \,\Big|\, \hat{\mathcal{F}}_s \right] = \frac{(1-2\mu)}{1 + \left(\frac{X - X_s}{e^{(\mu)}_s}\right)^2},$$

or equivalently, since $(e^{(\mu)}_s)^2 \equiv \exp(2B^{(\mu)}_s)$ is $\hat{\mathcal{F}}_s$ measurable

Est ce que in dans le cas $\nu \neq 0$, on ne pourrait pas aussi continuer les comparaisons?!

$$\boxed{E\left[\frac{1}{A^{(\mu)} - A^{(\mu)}_s} \,\Big|\, \hat{\mathcal{F}}_s \right] = \frac{(1-2\mu)}{(e^{(\mu)}_s)^2 + (X - X_s)^2}}$$

Pour cela, il faudrait probablement faire le grossissement

To be checked precisely ....

This relationship should follow from the known facts: conditionally on $\mathcal{F}_s$, the pair: $(X - X_s, A - A_s)$ is ~~Conjoint~~ distributed as $\left(e^{(\mu)}_s \sqrt{H}\, N, (e^{(\mu)}_s)^2 H\right)$, where $H \overset{(law)}{=} \frac{1}{2\gamma_\mu}$, and N is $\mathcal{N}(0,1)$ independent from H. dont la loi nous est tj

~~PAT ATS~~ avec $A^{(\mu)}$ $A^{(\mu,1)}$

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