Aug. 19th, 1993

More transforms of a spider-martingale.

1. On the joint law of times spent in the rays by the spider up to T_*.

The origin of the developments presented below is my desire to extend formula $(7)^k$ in: "Some formulae for the Brownian spider, I", Aug. 14th, which is:

$$(7)^k \qquad E\left[\exp\left(-\frac{\lambda^2}{2}T_*\right)\right] = \frac{\sum_i \left(1/\sinh(\lambda a_i)\right)}{\sum_i \coth(\lambda a_i)}$$

to obtain the joint Laplace transform of the

$$T_*^{(i)} \overset{def}{=} \int_0^{T_*} ds\, 1_{(B_s \in I_i)} \quad , \quad i = 1, 2, \cdots, k.$$

Here is this extension:

$$(7')^k \qquad E\left[\exp\left(-\frac{1}{2}\sum_i \lambda_i^2 T_*^{(i)}\right)\right] = \frac{\sum_i \left(\lambda_i/\sinh(\lambda_i a_i)\right)}{\sum_i \lambda_i \coth(\lambda_i a_i)}$$

which may be refined into:

$$(7')^k_i \qquad E\left[\exp\left(-\frac{1}{2}\sum_j \lambda_j^2 T_*^{(j)}\right) 1_{(B_{T_*} = x_i)}\right] = \frac{\lambda_i / \sinh(\lambda_i a_i)}{\sum_j \lambda_j \coth(\lambda_j a_j)}$$

Note: For simplicity, I will write $T_*^{\langle\lambda\rangle}$ for $\left(\sum_j \lambda_j^2 T_*^{(j)}\right)$;

I will also use the notation: $T_t^{(j)} = \int_0^t ds\, 1_{(B_s^{(j)} > 0)} \equiv \int_0^t ds\, 1_{(B_s \in I_j)}$

The proof relies on the two following facts:

a) The process $S_t = (S_t^{(1)}, S_t^{(2)}, \dots, S_t^{(k)})$ defined by:

$$S_t^{(i)} = \prod_{j \neq i} \left(\cosh(\lambda_j B_t^{(j)}) \exp\left(-\frac{\lambda_j^2}{2} T_t^{(j)}\right)\right) \left(\frac{\sinh(\lambda_i B_t^{(i)})}{\lambda_i} \exp\left(-\frac{\lambda_i^2}{2} T_t^{(i)}\right)\right)$$

is a spider-martingale;

b) the one-dimensional process:

$$C_t = \prod_{j=1}^{k} \cosh(\lambda_j B_t^{(j)}) \exp\left(-\frac{\lambda_j^2}{2} T_t^{(j)}\right)$$

is an ordinary martingale.

c) Below, we shall see that a) and b) are particular cases of some more general construction of spider-martingales, and/or martingales from a given spider martingale.

So, let us assume a) and b) for the moment, and prove $(7')_i^k$ right away. From the fundamental property of a spider-martingale, we know that:

$$E\left[S_{T_*}^{(i)} \right] = \sigma, \quad \text{a constant which does not depend on } i.$$

This identity may be written as:

$$(1) \qquad \left(\frac{\sinh(\lambda_i a_i)}{\lambda_i} \right) E\left[\exp\left(-\frac{1}{2} T_*^{\langle\lambda\rangle}\right) 1_{(B_{T_*} = x_i)} \right] = \sigma$$

Now, from b), we get: $E\left[C_{T_*} \right] = 1$, that is:

$$(2) \qquad \sum_{i=1}^{k} \cosh(\lambda_i a_i) \; E\left[\exp\left(-\frac{1}{2} T_*^{\langle\lambda\rangle}\right) 1_{(B_{T_*} = x_i)} \right] = 1.$$

We multiply and divide the i^{th} expectation which appears in (2) by : $\frac{\sinh(\lambda_i a_i)}{\lambda_i}$.

Thus, we obtain, by using (1) and (2) jointly :

$$\left[(2) \Longrightarrow\right] \qquad (3) \quad \left\{\sum_{i=1}^{k} \left(\lambda_i \coth(\lambda_i a_i)\right)\right\}\sigma = 1 ,$$

so that we have now obtained the value of σ and, going back to (1), we obtain, from (3), the formula $(7')_i^k$.

<u>Comments :</u>

1. Of course, the formula $(7')_i^k$ may be decomposed at time g_{T_*} ;
2. There is certainly a proof by excursion theory....
3. In the end, we should be able to describe the joint law of

$$\left| \begin{array}{l} A_t^{(1)} = \sup_{s \leq t} B_s^{(1)} \; ; \; A_t^{(2)} = \sup_{s \leq t} B_s^{(2)} \; ; \; \cdots\cdots \; ; \; A_t^{(k)} = \sup_{s \leq t} B_s^{(k)} \; ; \\ T_t^{(1)} = \int_0^t ds \, 1_{(B_s^{(1)} > 0)} \; ; \; T_t^{(2)} = \int_0^t ds \, 1_{(B_s^{(2)} > 0)} \; ; \; \cdots\cdots \; ; \; T_t^{(k)} = \int_0^t ds \, 1_{(B_s^{(k)} > 0)} . \end{array} \right.$$

[To be developed].

2. Constructing martingales and spider-martingales from a given spider-martingale.

(2.1) Creation of martingales from a spider-martingale.

Let $M_t = (M_t^{(1)}, \dots, M_t^{(k)})$ be a k-spider martingale.
I would like to find some reasonable conditions which ensure that, if:

$$f : \begin{array}{l} (\mathbb{R}_+)^k \times (\mathbb{R}_+)^k \longrightarrow \mathbb{R} \\ ((x_1, \dots, x_k); (t_1, \dots, t_k)) \longrightarrow f((x_1, \dots, x_k); (t_1, \dots, t_k)) \end{array}$$

then:

$$F_t = f((M_t^{(1)}, \dots, M_t^{(k)}); \langle M^{(1)} \rangle_t, \dots, \langle M^{(k)} \rangle_t)$$

is an ordinary martingale.
[<u>Note</u>: For clarity, I will write $f([x_1, x_2, \dots, x_k]; [t_1, t_2, \dots, t_k])$ to separate better space and time.]

From Itô's formula, we get:

$$(4) \quad dF_t = \sum_i f'_{x_i}\, dM_t^{(i)} + \sum_i \left(f'_{t_i} + \frac{1}{2} f''_{x_i^2}\right) d\langle M^{(i)} \rangle_t$$

(there are no rectangular brackets, since: $d\langle M^{(i)}, M^{(j)} \rangle_t = 0$, for $i \neq j$)

$$(4') \quad = d(\text{mart.}) + (dA_t) \sum_i f'_{x_i}([0, 0, \dots 0]; [\langle M^{(j)} \rangle_t ; j \leq k])$$
$$+ \sum_i d\langle M^{(i)} \rangle_t \left(f'_{t_i} + \frac{1}{2} f''_{x_i}\right)\left([0, 0, \dots, M_t^{(i)}, 0, \dots 0]; [\langle M^{(j)} \rangle_t, j \leq k]\right)$$

Hence, we have

Proposition 1: $F_t = f\left([M_t^{(1)}, \dots, M_t^{(k)}]; [\langle M^{(j)}\rangle_t ; j \leq k]\right)$

is a martingale as soon as the two following conditions are satisfied:

$$(5) \qquad \sum_i f'_{x_i}\left([0,0,\dots,0]; [t_1, -, t_k]\right) = 0$$

$$(6) \qquad \text{for every } i, \quad \left(f'_{t_i} + \frac{1}{2} f''_{x_i^2}\right)\left([0,0,-,x_i,0,-0]; [t_1,-,t_k]\right) = 0$$

In particular, it is sufficient that:

for every i, and for every $(t_1, -, t_{i-1}, \cdot, t_{i+1}, -, t_k)$, the function of two variables (x_i, t_i):

$$(7) \qquad \varphi_i^{(t_1,-,t_{i-1},\cdot,t_{i+1},-,t_k)}(x_i,t_i) \stackrel{\text{def}}{=} f\left([0,-,0,x_i,0,-0]; [t_1,-,t_k]\right)$$

is a space-time harmonic function such that: $\frac{\partial}{\partial x_i}\varphi_i^{(-,\cdot,-)}(0,t_i) = 0.$

Example 1: Take $f\left([x_1,-,x_k]; [t_1,-,t_k]\right) = \prod_{i=1}^k g_i(x_i,t_i)$,

where the g_i's are space-time harmonic functions.

Then, $(F_t, t \geq 0)$ is a martingale iff:

$$(8) \qquad \sum_{i=1}^k \left(\prod_{j\neq i} g_j(0,t_j)\right)\left(\frac{\partial}{\partial x_i} g_i\right)(0,t_i) = 0$$

In particular, it is a martingale as soon as:

$$(9) \qquad \text{for every } i, \quad \left(\frac{\partial}{\partial x_i} g_i\right)(0,t_i) = 0.$$

(2.2) Creation of spider-martingales from a given spider-martingale.

Here, I will start with k functions $f^{(1)}, \dots, f^{(k)}$ as in (2.1), but I assume, furthermore, that:
these functions take values in $\mathbb{R}_+$, and:

(10) $$f^{(i)}\left([x_1, x_2, \dots, x_k]; [t_1, \text{—}, t_k]\right) = 0 \quad \text{iff} \quad x_i = 0.$$

Then, I would like to find some reasonable conditions on:

$$\left\{\mathbb{F}_t = \left(F_t^{(1)}, \dots, F_t^{(k)}\right), \ t \geq 0\right\} \text{ to be a spider-martingale.}$$

Inspecting Itô's formula (4') for $dF_t^{(j)}$, for all j's simultaneously, we obtain the following:

Proposition 2: $(\mathbb{F}_t, t \geq 0)$ is a spider-martingale as soon as the two following conditions are satisfied:

(11) the quantity $\sum_i \frac{\partial}{\partial x_i} f^{(j)}\left([0, 0, \text{—}, 0]; [t_1, \text{—}, t_k]\right)$

does not depend on j;

(12) for every j, and every i, $0 \equiv \left(\frac{\partial}{\partial t_i} + \frac{1}{2}\frac{\partial^2}{\partial x_i^2}\right) f^{(j)}\left([0, 0, -, x_i, 0, -, 0]; [t_1, -, t_k]\right)$

Example 2: Take:

$$f^{(j)}\left([x_1, \text{—}, x_k]; [t_1, \text{—}, t_k]\right) = \prod_{i=1}^{k} g_i^{(j)}(x_i, t_i).$$

where the $g_i^{(j)}$'s are space-time harmonic functions.

[To be developed].

3. [To be developed] <u>Degrees of liberty for the existence of a k-spider martingale, when p $(<k)$ of its legs are given.</u>

4. [To be developed] <u>Hermite polynomials, symmetric functions of k variables, and spider-martingales.</u>

We can apply the results in paragraph 2 to get more spider-martingales associated with a given one, using, instead of (cosh) and (sinh), the Hermite polynomials of (resp.) even and odd orders.

5. [To be developed] <u>Spider-martingales and planar martingales.</u>

There seem to be certain similarities between spider-martingales and conformal martingales (i.e: time changes of planar BM) which I would like to explore.