(9)

August 1^{st}, 1994.

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On ranked lengths of excursions: a summary of some recent results

P. Lévy (1939) obtained the striking result:

$$(1) \qquad A_1^+ \overset{(law)}{\underset{(a)}{=}} \frac{1}{\tau_1} A_{\tau_1}^+ \qquad \text{is arc sine distributed.}$$

Looking for a better understanding of P. Lévy's identity (a), PY[92] obtained the following:

$$(2) \qquad (V_n(1), n \geq 1) \overset{(law)}{=} \left(\frac{1}{\tau_1} V_n(\tau_1), n \geq 1 \right)$$

and

$$(3) \qquad P\left(1-g_1 = V_n(1) \,\middle|\, (V_k(1), k \geq 1)\right) = V_n(1).$$

A number of results about the sequence $(V_k(1), k \geq 1)$ and related quantities are presented in PPY (92) and Perman (93).
Despite this, a number of simple questions about the distribution of the sequence $(V_k(1), k \geq 1)$ remain difficult to solve explicitly, e.g. give an explicit formula for

$$(4) \qquad E\left[V_n(1) \right] = P\left(1-g_1 = V_n(1)\right) \qquad , \text{etc....}$$

(this equality follows from (3)).

In the present paper, we give a number of new results about the sequence $(V_k(1), k \geq 1)$; most of these results are obtained by combining (2) and (3), together with ~~some~~ extensions of (2) to some other random times T (than constants, and (τ_t)) which we shall call admissible if they satisfy:

$$(2') \qquad (V_n(1), n \geq 1) \overset{(law)}{=} \left(\frac{V_n(T)}{T}, n \geq 1 \right)$$

To ~~mem~~memorize easier, we introduce the following notations:

$$\theta_n(R_1, \ldots, R_n) \equiv \frac{1}{R_n} + \frac{1}{R_n R_{n-1}} + \ldots + \frac{1}{(R_n R_{n-1} \ldots R_1)}$$

and $\Sigma_n = 1 + R_n + R_n R_{n+1} + \ldots$,

so that we may write $(5)_{n+1}$ as:

$$(5')_{n+1} \qquad \rho_{n+1} = 1 + \theta_n(R_1, \ldots R_n) + R_{n+1}\Sigma_{n+2}.$$

This gives a representation of ρ_{n+1} in terms of 3 independent random variables:

$$\theta_n(R_1, \ldots R_n),\ R_{n+1},\ \text{and } \Sigma_{n+2},$$

and, to understand better this decomposition, we remark that, from the scaling property, we have:

$$(6) \qquad \rho_{n+1} \overset{(law)}{=} H^{(n+1)}$$

(we write simply $H^{(n+1)}$ for $H_1^{(n+1)}$)

The admissibility of $H^{(n+1)}$ will help us to understand better the identity $(5')_{n+1}$ which we now turn into an identity in law:

$$(5'')_{n+1} \qquad H^{(n+1)} = 1 + \theta_n(R_1, \ldots, R_n) + R_{n+1}\Sigma_{n+2}$$

But, for the moment, we now describe the simple structure of the sequence $(H^{(n)}, n \geq 1)$.

Theorem 3: 1. The variables $(H^{(1)}, H^{(2)} - H^{(1)}, \ldots, H^{(n+1)} - H^{(n)}, \ldots)$ are independent, and the variables $H^{(n+1)} - H^{(n)}$, for each $n \geq 1$, are identically distributed.

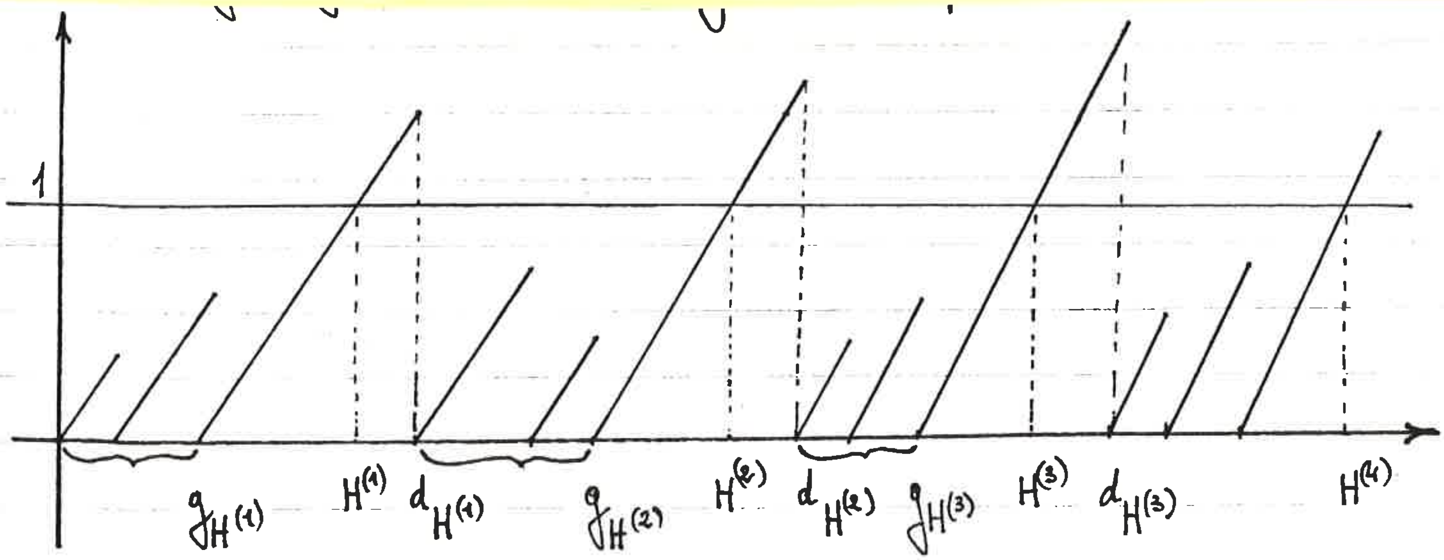
2. Their common distribution is obtained by writing :

$$H^{(2)} - H^{(1)} = H^{(1)} + \left(d_{H^{(1)}} - H^{(1)}\right) + \left(H^{(3)} - d_{H^{(1)}}\right),$$

which is the sum of 3 independent variables, such that :

$$H^{(1)} \overset{(law)}{=} H^{(2)} - d_{H^{(1)}}, \quad \text{and:} \quad P\left(d_{H^{(1)}} - H^{(1)} \in du\right) = \frac{du}{2(1+u)^{3/2}}$$

In the sequel, the following picture will be useful



Some computations for the "hierarchy".

(Unfortunately, lack of time prevented me from writing a logical discussion following the the first 4 pages above, but here are some results).

Define the probabilities $P_k^{\#}$ by:

$$E_k^{\#}\left[F(X_u, u\leq 1)\right] = E\left[\frac{c_k}{(\sqrt{\tau_1})^k} F\left(\frac{B_{u\tau_1}}{\sqrt{\tau_1}}, u\leq 1\right)\right]$$

$$\equiv E^{\#}\left[\frac{c_k}{(\sqrt{\tau_1})^k} F(X_u, u\leq 1)\right]$$

$R_1, R_2, \dots, R_k, \text{—}$ are now defined on the canonical space.
Recall that, as a consequence of (2), the law of $(R_k, k\geq 1)$ is identical under P and under $P^{\#}$. Denote $\mathcal{R}_p = \sigma\{R_1, \text{—}, R_p\}$.

Then, we have the following

<u>theorem 4:</u>

$$P_k^{\#}\big|_{\mathcal{R}_p} = D_p^{(k)}(R_1, \text{—} R_p)\cdot P\big|_{\mathcal{R}_p}$$

<u>where</u> $D_p^{(k)}(R_1, \text{—}, R_p) = d_p^{(k)}\left(\theta_p(R_1, \text{—}, R_p)\right)$,

<u>and the function</u> $d_p^{(k)}(a)$ is given by (: cf, the computations done in Document (7)):

$$d_p^{(k)}(a) = c_{k,p}\int_0^\infty dx \left(\frac{x e^{x^2/2}}{\psi(x)}\right)^k \gamma_{a,p}(x),$$

<u>where</u> $(\gamma_{a,p}(x), x\geq 0)$ <u>is a probability density on</u> $\mathbb{R}_+$, <u>which is defined by</u>:

$$\gamma_{a,p}(x) = -\frac{d}{dx}\left(\frac{e^{-\frac{x^2}{2}(a-p)}}{(\psi(x))^p}\right) \equiv \frac{e^{-\frac{x^2}{2}(a-p)}}{(\psi(x))^{p+1}}\left\{\frac{p}{x}(\psi(x)-1) + ax\psi(x)\right.$$