(2) (Pour Jim : Attention, le 26 Juillet au soir, cette Note a été complétée avec le bas de la page 4, et la page 5) 1)

On ranked lengths of excursions, II

Berkeley, July 26th, 1994.

The starting point of this Note is the following:

the distribution of the sequence $(V_1(1), \ldots, V_n(1), \ldots)$ is complicated, whereas that of:

$$\left(H^{(1)}, \ldots, H^{(n)} \equiv \inf\{t : V_n(t) \geq 1\}, \ldots\right)$$

is much easier to understand; precisely:

(1) the variables $(H^{(1)}, H^{(2)} - H^{(1)}, \ldots, H^{(n+1)} - H^{(n)}, \ldots)$ are independent, and the variables $(H^{(n+1)} - H^{(n)})$, for each $n \geq 1$, are identically distributed.

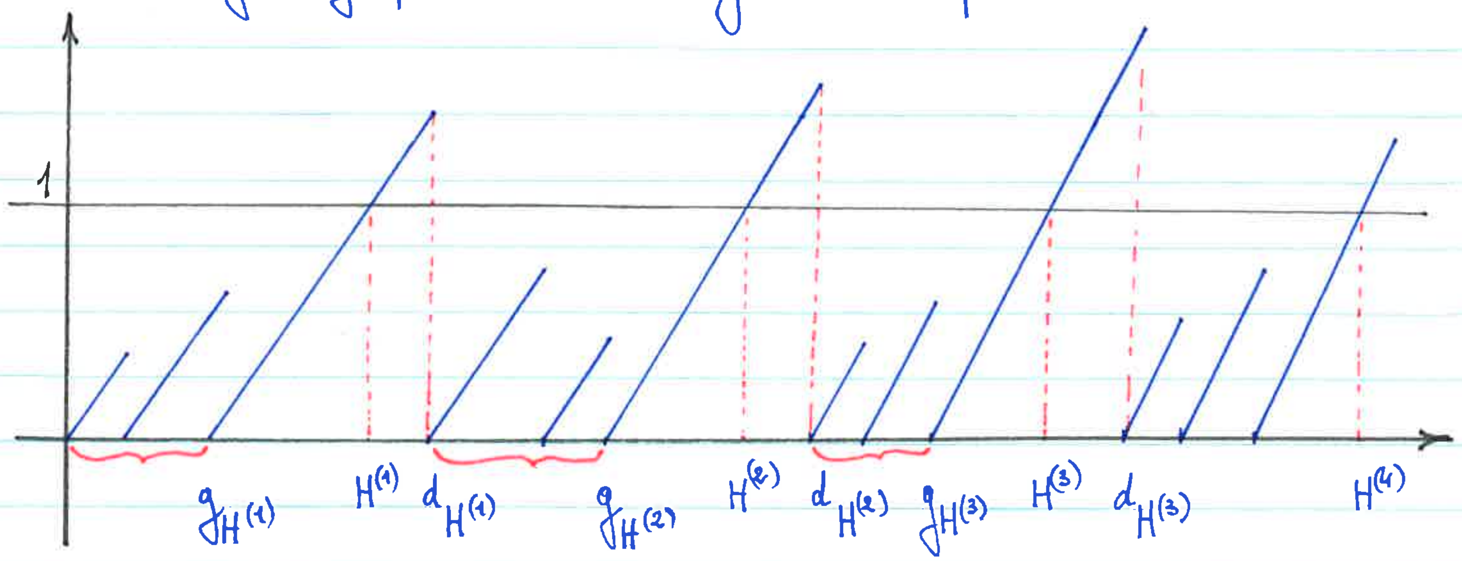
Their common distribution is obtained by writing:

$$H^{(2)} - H^{(1)} = H^{(1)} + (d_{H^{(1)}} - H^{(1)}) + (H^{(2)} - d_{H^{(1)}}),$$

which is the sum of 3 independent variables, such that:

$$H^{(1)} \overset{(law)}{=} H^{(2)} - d_{H^{(1)}}, \text{ and: } P(d_{H^{(1)}} - H^{(1)} \in du) = \frac{du}{2(1+u)^{3/2}}.$$

The following picture will be useful in the sequel



In this Note, we shall exploit the simple ~~the~~ description (above) of the law of the sequence $(H^{(n)}, n \geq 1)$ to deduce a description of the sequence $(V_n(1), n \geq 1)$, which although more complicated than (1) is, in the end, quite explicit.

We first present a few relations between the two sequences.

<u>1st relation</u>: For every $n \geq 1$, $\frac{1}{V_n(1)} \overset{(law)}{=} H^{(n)}$

This follows immediately from the scaling property.

<u>2nd relation</u>: For every $n \geq 1$, $H^{(n)}$ is admissible, i.e:

$$(V_k(1), k \geq 1) \overset{(law)}{=} \left(\frac{V_k(H^{(n)})}{H^{(n)}}, k \geq 1 \right)$$

We shall also need the following

<u>Lemma 1</u>: $V_2(H^{(1)}) \equiv V_1(g_{H^{(1)}})$ <u>is distributed as</u> U^2 (U denotes <u>a uniform variable on</u> $[0,1]$), <u>and is independent of</u> $\left(\beta_u^{(1)} \equiv \frac{1}{\sqrt{g_{H^{(1)}}}} B_{u g_{H^{(1)}}}, u \leq 1 \right)$

<u>Consequently</u>, $V_2(H^{(1)})$ <u>is independent of</u>

$$\left(\frac{V_n(g_{H^{(1)}})}{g_{H^{(1)}}} \equiv \frac{V_{n+1}(H^{(1)})}{g_{H^{(1)}}}, n \geq 1 \right)$$

<u>Corollary 2</u>: $\frac{V_2(1)}{V_1(1)}$ <u>is distributed as</u> U^2, <u>and is independent of</u> $\left(\frac{V_{n+1}(1)}{V_2(1)}, n \geq 1 \right)$

Proof: We use the fact that $H^{(1)}$ is admissible, so that:

$$\left(\frac{V_2(1)}{V_1(1)} ; \frac{V_{n+1}(1)}{V_2(1)}, n \geq 1\right) \overset{(law)}{=} \left(V_2(H^{(1)}) ; \frac{V_{n+1}(H^{(1)})}{V_2(H^{(1)})}, n \geq 1\right),$$

and we write:

$$\frac{V_{n+1}(H^{(1)})}{V_2(H^{(1)})} = \left(\frac{V_{n+1}(H^{(1)})}{g_{H^{(1)}}}\right) \Big/ \left(\frac{V_2(H^{(1)})}{g_{H^{(1)}}}\right).$$

$\square$

Now, we continue to look at:

$\dfrac{V_3(1)}{V_2(1)}$, and the "remaining" ratios: $\dfrac{V_{n+2}(1)}{V_3(1)}$, and so on ....

and it is not difficult to show the following

Proposition 3: $\dfrac{V_3(1)}{V_2(1)}$ is distributed as $\max(U^2, V^2) \overset{(law)}{=} U$,

and is independent of the sequence $\left(\dfrac{V_{n+2}(1)}{V_3(1)}, n \geq 1\right)$

More generally, we have:

Proposition 4: $\dfrac{V_{k+1}(1)}{V_k(1)}$ is distributed as $\max(U_1^2, U_2^2, \dots, U_k^2) \overset{(law)}{=} Z_{\frac{k}{2},1}$,

and is independent of $\left(\dfrac{V_{n+k}(1)}{V_{1+k}(1)}, n \geq 1\right)$

Now, we introduce the notation:

$$\boxed{R_1 = \frac{V_2(1)}{V_1(1)}, \quad R_2 = \frac{V_3(1)}{V_2(1)}, \quad \text{———} \quad R_n = \frac{V_{n+1}(1)}{V_n(1)}, \quad \text{———}}$$

This is a sequence of independent variables, with $R_n \overset{(law)}{=} Z_{\frac{n}{2},1}$.

From there, it is relatively easy to "describe" the distribution of the sequence $(V_n(1), n \geq 1)$.

Theorem 5: The sequence

$$S = \left(\frac{V_2(1)}{V_1(1)}, \frac{V_3(1)}{V_1(1)}, \text{———}, \frac{V_n(1)}{V_1(1)}, \text{—} \right)$$

is distributed as: $\left(R_1, R_1R_2, R_1R_2R_3, \text{————} \right)$

Moreover, $V_1(1)$ is measurable with respect to S, since:

$$\frac{1 - V_1(1)}{V_1(1)} = \sum_{k=1}^{\infty} S_k, \quad \text{and we have:}$$

$$\frac{1}{V_1(1)} = 1 + R_1 + R_1R_2 + R_1R_2R_3 + \cdots\cdots$$

Finally, from Theorem 5, it is easy to describe the sequence of reciprocals:

$$\left(\frac{1}{V_1(1)}, \frac{1}{V_2(1)}, \text{————}, \frac{1}{V_n(1)}, \text{———} \right)$$

We have the following

Theorem 6: The sequence $\left(\frac{1}{V_1(1)}, \frac{1}{V_2(1)}, \text{———}, \frac{1}{V_n(1)}, \text{——} \right)$

is distributed as $\left(\rho_1, \rho_2, \text{————}, \rho_n, \text{———} \right)$ (: ρ for reciprocal).

where:

$$\rho_1 = 1 + R_1 + R_1R_2 + R_1R_2R_3 + \text{———} + R_1R_2 \cdots R_n + \text{————}$$

$$\rho_2 = 1 + \frac{1}{R_1} + R_2 + R_2R_3 + \text{———} + R_2R_3 \cdots R_n + \text{————}$$

$$\rho_3 = 1 + \frac{1}{R_2} + \frac{1}{R_2R_1} + R_3 + R_3R_4 + R_3R_4R_5 + \text{———} + \cdots\cdots$$

and, in general:

$$\boxed{\rho_{n+1} = 1 + \left(\frac{1}{R_n} + \frac{1}{R_n R_{n-1}} + \frac{1}{R_n R_{n-1} R_{n-2}} + \text{———} + \frac{1}{R_n R_{n-1} \cdots R_1}\right) + \left(R_{n+1} + R_{n+1}R_{n+2} + R_{n+1}R_{n+2}R_{n+3} + \text{———}\right)}$$

Proof: As for Theorem 5, we introduce the sequence:

$$S^{(2)} = \left(\frac{V_3(1)}{V_2(1)}, \frac{V_4(1)}{V_2(1)}, \frac{V_5(1)}{V_2(1)}, \text{———}\right)$$

$$\overset{(law)}{=} \left(R_2, R_2R_3, R_2R_3R_4, \text{———}\right),$$

and we obtain:

$$\sum_{k=1}^{\infty} S_k^{(2)} = \frac{1-(V_1(1)+V_2(1))}{V_2(1)} = \frac{1}{V_2(1)} - 1 - \frac{V_1(1)}{V_2(1)}$$

From this, we deduce the identity in law of $\frac{1}{V_2(1)}$ and ρ_2.

The next identities in law will hold jointly with this identity.
We introduce:

$$S^{(3)} = \left(\frac{V_4(1)}{V_3(1)}, \frac{V_5(1)}{V_3(1)}, \frac{V_6(1)}{V_3(1)}, \text{———}\right)$$

$$\overset{(law)}{=} \left(R_3, R_3R_4, R_3R_4R_5, \text{———}\right)$$

Then, we obtain:

$$\sum_{k=1}^{\infty} S_k^{(3)} = \frac{1-(V_1(1)+V_2(1)+V_3(1))}{V_3(1)}$$

$$= \frac{1}{V_3(1)} - 1 - \left(\frac{V_1(1)}{V_3(1)} + \frac{V_2(1)}{V_3(1)}\right),$$

from which we deduce the identity in law between $\frac{1}{V_3(1)}$ and ρ_3.
It is now clear that the formula is true for all n $\square$