1).

On some space-time probability measures beneath Black-Scholes type formulae.

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1. The Black-Scholes paradigm.

(1.1) This paper originates from the following result:

consider the put quantity:

$$(1) \qquad C^{-}(K,t) \overset{\text{def}}{=} E\left[(K-\mathcal{E}_t)^{+}\right],$$

where $K \geq 0$, and $\mathcal{E}_t = \exp\left(B_t - \frac{t}{2}\right)$, $t \geq 0$, with $(B_t)_{t \geq 0}$ standard Brownian motion.

Theorem 1: There exists a probability, which we shall denote by γ, on $[0,1] \times [0,\infty)$ such that:

$$(2) \qquad C^{-}(K,t) = \gamma\left([0,K] \times [0,t]\right)$$

We give three descriptions of this probability:

Description # 1: (3) $\gamma(dK, ds) = \frac{1}{2} E\left[d_{K,s}\left(\mathcal{L}^K_s\right)\right]$

where $(\mathcal{L}^K_s)$ is the family of local times of $(\mathcal{E}_s, s \geq 0)$; a little more formally:

$$\frac{1}{2} E\left[\int_0^1 \int_0^\infty q(K,s)\, d_{K,s}\left(\mathcal{L}^K_s\right)\right]$$

(Then, we should refer to [3], and discuss).

extends as a probability, from simple functions q to ≥ 0 Borel ones

Description # 2: The probability γ on $[0,1]\times[0,\infty)$ is the law of the pair:

$$(4)\quad \left(\max\left\{ \exp(-2\mathbb{e})\,;\ \exp\left(-\sqrt{(2\mathbb{e}')(4B_1^2)}\right)\right\},\ 4B_1^2 \right)$$

where B_1, $\mathbb{e}$, $\mathbb{e}'$ are independent, with $\mathbb{e}$, $\mathbb{e}'$ 2 standard exponential variables –

Description # 3: The probability γ on $[0,1]\times[0,\infty)$ is the law of the pair:

$$(5)\quad \left(U,\ Y_{\left(\frac{1}{2},\ \ell(U)\right)} \right), \qquad \ell(x) = \log(1/x)$$

where U is uniform on $[0,1]$, independent of the "process" $(Y_{a,b}\,;\ a,b>0)$ which is now defined.

Lemma: Denote by $\mathcal{N}$ the distribution function of the standard Gaussian variable, i.e.:

$$(6)\quad \mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy\, \exp(-y^2/2).$$

Then, to any $a, b > 0$, one can associate a r.v. $Y_{a,b}$ taking values in $[0,\infty)$, such that:

$$(7)\quad P(Y_{a,b} \le t) = \mathcal{N}\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right), \quad t \ge 0.$$

The density of $Y_{a,b}$ is given by:

$$(8)\quad \frac{d}{dt}\left(P(Y_{a,b} \le t)\right) = \frac{1}{\sqrt{2\pi}}\, e^{ab} \exp\left(-\frac{1}{2}\left(a^2 t + \frac{b^2}{t}\right)\right)\left\{\frac{a}{2\sqrt{t}} + \frac{b}{2t^{3/2}}\right\}$$

If we denote: $T_b^{(a)} = \inf\{t : B_t + at = b\}$, resp: $G_b^{(a)} = \sup\{t : B_t + at = b\}$, then:

$$(9)\quad P(Y_{a,b} \in dt) = \frac{1}{2}\left\{ P(T_b^{(a)} \in dt) + P(G_b^{(a)} \in dt) \right\}$$

Remark: It may be worth mentioning that this Lemma admits a wide extension, since to any distribution function F on $\mathbb{R}$, and any $a, b > 0$, we can associate a new distribution function $F_{a,b}$ on $\mathbb{R}_+$ via:

$$F_{a,b}(t) = F\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right), \quad t \geq 0$$

However, the particular case $F \equiv \mathcal{N}$ is extremely well "fitted" within our discussion.

Upon comparing the descriptions #2 and #3, it is quite natural to look for some understanding of the implied identities in law between the first, resp: second, components of (4) and (5); precisely, we wish to check directly that:

$$(14) \quad \max\left\{ \exp(-2\mathbb{e}) ; \exp\left(-\sqrt{(2\mathbb{e}')\,4B_1^2}\right)\right\} \overset{(law)}{=} U$$

as well as:

$$(15) \quad Y_{\left(\frac{1}{2}, \ell(U)\right)} \overset{(law)}{=} (4B_1^2) \left(\overset{(law)}{=} g_0 = \sup\left\{t : B_t - \frac{t}{2} = 0\right\}\right)$$

[In fact, I should have written $G_0^{(-1/2)}$ following notation in the Lemma!!]

— Checking (14) is easy, since:

a) $\mathbb{e} \overset{(law)}{=} \sqrt{2\mathbb{e}' B_1^2}$; this is a probabilistic translation of the duplication formula for the gamma function; see, e.g., Chaumont-Yor ([ ]; Exercise (??)).

b) $\exp(-2\mathbb{e}) \overset{(law)}{=} U^2$

Thus, (14) writes: $\max\left(U^2 ; (U')^2\right) \overset{(law)}{=} U$,

which is obvious, by looking at the distribution functions!!

$$\left(\text{i.e.: } (\sqrt{K}) \times (\sqrt{K}) = K\right)$$

– Checking the validity of (15) is a little less immediate! From the description (9) of the law of $Y_{a,b}$, we see that (15) amounts to:

$$P\left(G_0^{(-1/2)} \in dt\right) = \frac{1}{2}\int_0^1 du \left\{ P\left(T_{\ell(u)}^{(1/2)} \in dt\right) + P\left(G_{\ell(u)}^{(1/2)} \in dt\right) \right\}$$

(16) $$= \frac{1}{2}\int_0^\infty dx\, e^{-x} \left\{ P\left(T_x^{(1/2)} \in dt\right) + P\left(G_x^{(1/2)} \in dt\right) \right\}$$

We shall use the well-known formulae:

$$P\left(T_a^{(\nu)} \in dt\right) = \frac{dt\, a}{\sqrt{2\pi t^3}}\, e^{-\frac{(a-\nu t)^2}{2t}} \qquad (a>0).$$

(17) $$P\left(G_a^{(\nu)} \in dt\right) = |\nu| \frac{dt}{\sqrt{2\pi t}} \exp\left(-\frac{(a-\nu t)^2}{2t}\right).$$

Due to (17), (16) is equivalent to:

$$\left(\frac{1}{2}\right)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{t}{8}\right) = \frac{1}{2}\left\{\int_0^\infty dx\, e^{-x}\left\{\cancel{\left(\frac{1}{2}\right)} \frac{x}{\sqrt{2\pi t^3}}\, e^{-\frac{\left(x-\frac{t}{2}\right)^2}{2t}} + \left(\frac{1}{2}\right)\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-t/2)^2}{2t}}\right\}\right\}$$

~~attention~~

[(18)?]

[(18)?] simplifies into:

[(19)?] $$1 = \cancel{\tfrac{1}{2}} \int_0^\infty dx\, e^{-x}\left\{\left(\frac{x}{t}\right) e^{-\frac{(x^2-tx)}{(2t)}} + \frac{1}{2} e^{-\left(\frac{x^2-tx}{2t}\right)}\right\}$$

The RHS equals:

$$\cancel{\tfrac{1}{2}} \int_0^\infty dx\, e^{-(x/2)}\left\{\frac{x}{t}\, e^{-\left(\frac{x^2}{2t}\right)} + \frac{1}{2} e^{-(x^2/2t)}\right\}$$

Now, integration by parts yields:

$$\int_0^\infty dx \left(\frac{x}{t}\, e^{-x^2/2t}\right) e^{-x/2} = 1 - \int_0^\infty dx \left(\frac{1}{2}\right) e^{-x/2} e^{-x^2/2t}$$

Consequently, we have indeed checked (19).

(1.2) An extension of Theorem 1.

In the next statement, we shall replace the standard Brownian martingale $(\mathcal{E}_t, t \geq 0)$ by the semimartingale:

$$\mathcal{E}_t^{\sigma,-\nu} = \exp(\sigma B_t - \nu t).$$

Then, we can show:

Theorem 2: There exists a probability on $[0,1] \times [0,\infty)$, which we shall denote by $\gamma_{\sigma,\nu}$ such that:

(20)
$$C^-_{\sigma,\nu}(K,t) \overset{df}{=} E\left[\left(K - \mathcal{E}_t^{\sigma,-\nu}\right)^+\right] = \gamma_{\sigma,\nu}\left([0,K] \times [0,t]\right)$$

Moreover, $\gamma_{\sigma,\nu}$ is the law of

(21)
$$\left(U, Y_{\left(\frac{\nu}{\sigma}, \frac{\ell(U)}{\sigma}\right)}\right) \qquad \left(\ell(u) = \ln\left(\frac{1}{u}\right)\right).$$

where U is uniform on $[0,1]$, independent of the process $(Y_{a,b}\,; a,b > 0)$ introduced in the preceding Lemma.

Proof: From the definition (20), we may write:

$$E\left[\left(K - \mathcal{E}_t^{\sigma,-\nu}\right)^+\right] = E\left[\int_0^K dx\, 1_{(\sigma B_t - \nu t < (\ln(x)))}\right]$$

$$= \int_0^K dx\, P\left(B_1 < \frac{\nu\sqrt{t}}{\sigma} - \frac{\ln(x)}{\sigma\sqrt{t}}\right)$$

$$= \int_0^K dx\, P\left(Y_{\frac{\nu}{\sigma}, \frac{\ell(x)}{\sigma}} \leq t\right),$$

which implies the description (21).

(Appendix): 4 pages.

Note that (21) corresponds to Description (# 3) in the particular case $\sigma = 1$, $\nu = 1/2$.

We would like to see whether there is a description #2; in particular, what is the law of $Y_{\left(\frac{\nu}{\sigma}, \frac{\ell(U)}{\sigma}\right)}$?

This law is :

$$\theta_{\nu,\sigma}(dt) = \frac{1}{2}\int_0^1 du \left\{ P\left(T^{(\nu/\sigma)}_{\frac{\ell(u)}{\sigma}} \in dt\right) + P\left(G^{(\nu/\sigma)}_{\frac{\ell(u)}{\sigma}} \in dt\right)\right\}.$$

$$= \frac{1}{2}\int_0^\infty dx\, e^{-x} \left\{ P\left(T^{(\nu/\sigma)}_{(x/\sigma)}\, dt\right) + P\left(G^{(\nu/\sigma)}_{(x/\sigma)} \in dt\right)\right\}$$

$$\left(= \frac{\sigma}{2}\int_0^\infty dy\, e^{-y\sigma} \left\{ P\left(T^{(\nu/\sigma)}_{y} \in dt\right) + P\left(G^{(\nu/\sigma)}_{y} \in dt\right)\right\}.\right)$$

$$= \frac{dt}{2}\int_0^\infty dx\, e^{-x} \left\{ \frac{(x/\sigma)}{\sqrt{2\pi t^3}}\, e^{-\frac{(x-\nu t)^2}{2\sigma^2 t}} + \frac{\nu}{\sigma}\frac{1}{\sqrt{2\pi t}}\, e^{-\left(\frac{(x-\nu t)^2}{2\sigma^2 t}\right)}\right\}$$

$$= \left(\frac{dt}{2}\right)\left(\frac{1}{\sqrt{2\pi t}\,(\sigma)} \exp\left(-\frac{\nu^2 t}{2\sigma^2}\right)\right)\int_0^\infty dx\, e^{-x}\left\{ \frac{x}{t}\, e^{-\frac{x^2}{2\sigma^2 t} + \frac{x\nu}{\sigma^2}} + \nu\, e^{-\frac{x^2}{2\sigma^2 t} + \frac{x\nu}{\sigma^2}}\right\}$$

$$= \left(\frac{dt}{2}\right)\frac{\sigma}{\sqrt{2\pi t}} \exp\left(-\frac{\nu^2 t}{2\sigma^2}\right)\int_0^\infty dx\, e^{-x}\left\{ \frac{x}{t\sigma^2}\, e^{-\frac{x^2}{2\sigma^2 t} + \frac{x\nu}{\sigma^2}} + \frac{\nu}{\sigma^2}\, e^{-\frac{x^2}{2\sigma^2 t} + \frac{x\nu}{\sigma^2}}\right\}$$

We now introduce the parameter : $\mu = \left(1 - \frac{\nu}{\sigma^2}\right)$, and we compute the integral:

$$I = \int_0^\infty dx\, e^{-\mu x}\left\{ \frac{x}{t\sigma^2}\, e^{-\frac{x^2}{2\sigma^2 t}} + \frac{\nu}{\sigma^2}\, e^{-\frac{x^2}{2\sigma^2 t}}\right\}.$$

$$\equiv I_1 + \left(\frac{\nu}{\sigma^2}\right) I_2\,,$$

where : $$I_1 = \int_0^\infty dx\, e^{-\mu x}\left(\frac{x}{t\sigma^2}\right) e^{-x^2/2\sigma^2 t}$$

$$= -\int_0^\infty dx\, \underbrace{e^{-\mu x}}_{v(x)}\Big[\underbrace{-\frac{x}{t\sigma^2}\, e^{-x^2/2\sigma^2 t}}_{u'(x)}\Big] \quad ; \; u(x) = e^{-x^2/2\sigma^2 t}$$

$$= -\left\{ \big(u(x)v(x)\big)_{x=0}^{x=\infty} - \int_0^\infty dx\, v'(x)u(x) \right\}.$$

Thus: $I_1 = 1 + \int_0^\infty dx\, v'(x) u(x)$

$$= 1 \cancel{+} - \mu \int_0^\infty dx\, e^{-\mu x}\, e^{-x^2/2\sigma^2 t}$$

Finally:

$$I_1 + \frac{\nu}{\sigma^2} I_2 = 1 - \mu I_2 + \frac{\nu}{\sigma^2} I_2 \qquad \left(\text{recall: } \mu = 1 - \frac{\nu}{\sigma^2}\right)$$

$$= 1 + \left(2\frac{\nu}{\sigma^2} - 1\right) I_2 .$$

(We note that this simplifies to 1 when: $\frac{2\nu}{\sigma^2} = 1$, in agreement with our previous computation!! $\nu = 1/2$, $\sigma = 1$.).

In any case, we have obtained that the second marginal of $\gamma_{\sigma,\nu}$ is

$$(22)\quad \theta_{\sigma,\nu}(\cancel{B}\,dt) = \left(\frac{dt}{2}\right) \frac{\sigma}{\sqrt{2\pi t}}\, e^{\left(-\frac{\nu^2 t}{2\sigma^2}\right)} \left\{ 1 + \left(2\frac{\nu}{\sigma^2} - 1\right) \int_0^\infty dx\, e^{-\mu x - \frac{x^2}{2\sigma^2 t}} \right\}.$$

(1.3) A signed measure on $\mathbb{R}_+ \times \mathbb{R}_+$. ← Interpretation ??

In this paragraph, we show that the formula:

$$(2)\qquad C^-(K,t) \equiv E\left[(K - \mathcal{E}_t)^+\right] = \gamma\left([0,K] \times [0,t]\right)$$

is true ~~on $\mathbb{R}$~~ for all values $K, t \geq 0$, with γ a signed measure on $\mathbb{R}_+ \times \mathbb{R}_+$, such that: $\gamma\big|_{[0,1]\times[0,\infty)}$ is a probability.

In fact, since $E\left[(K - \mathcal{E}_t)^+\right] = \int_0^K dx\, P(\mathcal{E}_t \leq x)$

$$(23)\qquad = \int_0^K dx\, \mathcal{N}\left(\frac{\sqrt{t}}{2} + \frac{\ln(x)}{\sqrt{t}}\right)$$

We get that: $\gamma(dK, dt) = g(K,t)\, dK\, dt$, with:

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$$g(K,t) = \frac{\partial^2}{\partial K \partial t} \left(E\left[(K - \mathcal{E}_t)^+ \right] \right).$$

which, from (23), yields:

$$g(K,t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right)^2 \right) \cdot \left\{ -\frac{(\ln K)}{2\,t^{3/2}} + \frac{1}{4\,t^{1/2}} \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\ln\frac{1}{K} - \frac{t}{2} \right)^2 \right) \right\} \left\{ \left(\frac{1}{2}\right) + \frac{\ln\left(\frac{1}{K}\right)}{t} \right\}.$$

Thus, we have:

$$(24) \quad \boxed{\gamma(dK, dt) = \begin{cases} [K<1] & \frac{dK}{2}\left\{ P\left(T^{(1/2)}_{(\ln \frac{1}{K})} \in dt \right) + P\left(G^{(1/2)}_{\ln(\frac{1}{K})} \in dt \right) \right\} \\ [K>1] & \frac{dK}{2}\left\{ P\left(G^{(1/2)}_{(\ln \frac{1}{K})} \in dt \right) - P\left(T^{(1/2)}_{\ln(\frac{1}{K})} \in dt \right) \right\} \end{cases}}$$

This is a remarkable formula, which is quite close to our previous discussion in "Unifying"; in fact, we should really compare this with theorem 0 in our [2].

~~(all 8 pages ??)~~

References:

[1] D Madan, B Roynette, M Yor: Option prices as probabilities. Finance Research Letters, 5, 79-87 (2008).

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[3] N. Eisenbaum: Local Time-Space Calculus for reversible semimartingales. Sem. Prob. XL, LNM 1899