1)

On the distribution of the integral of the exponential of Brownian motion, and the Hartman – Watson distribution.

In this paper, we study in particular the density of the Hartman - Watson distribution, which is defined as the probability η_r $(r>0)$ such that:

$$(0) \qquad \frac{I_{|\nu|}(r)}{I_0(r)} = \int_0^\infty e^{-\frac{\nu^2}{2}t}\, d\eta_r(t).$$

This probability admits a density which can be expressed in terms of the function:

$$(1) \qquad \Psi_r(t) = e^{\pi^2/2t} \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-(y^2/2t)} e^{-r(\operatorname{ch} y)} (\operatorname{sh} y) \sin\left(\frac{\pi y}{t}\right) dy$$

Note that one has:

$$(2) \qquad \Psi_r(t) = e^{\pi^2/2t} E\left[e^{-r\,\operatorname{ch}(B_t)} \operatorname{sh}(B_t) \sin\left(\frac{\pi}{t} B_t\right)\right],$$

where B_t denotes BM at time t.

~~To~~ It is of some interest to understand ~~the~~ well the graph / variation of $(\Psi_r(t), t\geq 0)$.

• From (1), we obtain that, as $t \longrightarrow \infty$,

$$\psi_r(t) \simeq \frac{1}{\sqrt{2\pi t}} \left(\frac{\pi}{t}\right) \int_0^\infty dy\, y\,(\operatorname{sh} y)\, e^{-r(\operatorname{ch} y)}$$

(3)
$$= \sqrt{\frac{\pi}{2}} \left(\frac{1}{t^{3/2}}\right) \left(\left(\frac{1}{r}\right) e^{-r} \right)_{\text{(plus compliqué!)}}$$

As a consequence, if H_r is a random variable which is distributed as η_r, then:

$$E\left[(H_r)^\alpha \right] < \infty \quad \text{iff} \quad \alpha < 1/2.$$

• We are also quite interested in the asymptotic study of $\psi_r(t)$ as $t \longrightarrow 0$, due to the factor: $\left(e^{\pi^2/2t}\right)$, which in some sense, must "cancel" with the integral part in (1).

Instead of studying directly the asymptotics of this integral, as $t \longrightarrow 0$, we try to use some Tauberian theorem, based on the asymptotics of $I_\nu(r)$, $\nu \longrightarrow \infty$.

With the help of the series development of $I_\nu(r)$, we get immediately:

(4)
$$\boxed{I_\nu(r) \sim \frac{(r/2)^\nu}{\Gamma(\nu+1)}. \qquad \nu \longrightarrow \infty.}$$

and, we may then use Stirling's formula:

(5)
$$\Gamma(\nu+1) \underset{\nu \to \infty}{\sim} \left(\nu^\nu\right)\left(\exp(-\nu)\right)\sqrt{2\pi\nu}.$$

so that we obtain: 3)

$$(4') : \qquad I_\nu(x) \underset{(\nu\to\infty)}{\sim} \left(\frac{x}{2}\right)^\nu \left(\nu^{-\nu}\right) \exp(+\nu) \frac{1}{\sqrt{2\pi\nu}}.$$

It then seems that we have to use <u>some exponential Tauberian theorem</u> which may be found in : Bingham - Goldie - Teugels - , p.254,

$$\Rightarrow \quad \frac{\log\left(I_\nu(n)\right)}{g(\nu)} - \frac{\log\left(f(\nu)\right)}{g(\nu)} \xrightarrow[(\nu\to\infty)]{} 0$$

$$\nu\left(\log\frac{n}{2}\right) - \underbrace{\nu\left(\log\nu\right)} + \nu - \frac{1}{2}\log\left(2\pi\nu\right).$$

$$g(\nu) = \nu\left(\log\nu\right).$$

$$-\left(\frac{\log I_\nu(n)}{\nu\left(\log\nu\right)}\right) \longrightarrow 1.$$

$$\Downarrow$$