July, 31^{st}, 1991.

On the distribution of the local times on the Brownian bridge

I intend to show that the law of $(\ell_\mu^y \,;\, y \geq 0)$ may be characterized by an identity in law which generalizes (26), Proposition 3 of July, 30^{th}.
(I shall keep the same notations, and extend them).

Indeed, if we consider $(B_t - \mu t,\, t \geq 0)$, the Brownian motion with drift $(-\mu)$ $(\mu > 0)$, and if we denote by $(\mathcal{L}^x_{(\mu)},\, x \geq 0)$ the family of its local times at positive levels, then we have the following extensions of (23), (25), (26), (26′), (28):

$$(23)_\mu \qquad \sup_{t\geq 0}(B_t - t\mu) \sim \frac{e}{2\mu} \;;$$

Proposition 3_μ: We have the identity in law:

$$(26)_\mu \qquad (\mathcal{L}^y_{(\mu)} \,;\, y \geq 0) \sim \left(\frac{|N|}{\mu}\, \ell_\mu^{y\mu/|N|} \;;\, y \geq 0\right)$$

Corollary: For any given $x > 0$, the identity in law $(25)_\mu$ holds, where:

$$(25)_\mu \qquad \mathcal{L}^x_{(\mu)} \sim \left(\frac{e}{\mu} - 2x\right)^+.$$

Proof of the Corollary: Using (1.a) and $(26)_\mu$ in conjunction, we have:

$$\mathcal{L}^x_{(\mu)} \sim \frac{|N|}{\mu}\, \ell_\mu^{x\mu/|N|} \sim \frac{|N|}{\mu}\left(R - \frac{2x\mu}{|N|}\right)^+$$

2)

and, therefore: $$\mathcal{L}^x_{(\mu)} \sim \left(\frac{|N/R|}{\mu} - 2x\right)^+ \sim \left(\frac{\mathbf{e}}{\mu} - 2x\right)^+,$$

which is $(25)_\mu$ $\square$

We now recall the Ray-Knight thm concerning $(\mathcal{L}^y_{(\mu)}, y \geq 0)$.

<u>Proposition 4</u> : i) <u>We have</u> : $$\mathcal{L}^0_{(\mu)} \sim \frac{\mathbf{e}}{\mu} \; ;$$

ii) <u>Given</u> $\mathcal{L}^0_{(\mu)} = a$, <u>the law</u> of $(\mathcal{L}^y_{(\mu)} ; y \geq 0)$ is $Q_a^{(-\mu)}$, <u>the law</u> of <u>the square</u> of <u>the Ornstein-Uhlenbeck process with dimension</u> 0, <u>and parameter</u> $(-\mu)$, <u>that is</u>, <u>if we denote</u>: $X_x \equiv \mathcal{L}^x_{(\mu)}$, we have:

$$X_x = a + 2\int_0^x \sqrt{X_y}\, d\beta_y - 2\mu \int_0^x dy\, X_y, \tag{30}$$

where $(\beta_y, y \geq 0)$ is a one-dimensional Brownian motion.

<u>Sketch of the proof</u>: Define $x_t = B_t - \mu t$, and use Tanaka's formula for $(x_t - x)^+$; then:

$$(x_t - x)^+ = (-x)^+ + \int_0^t 1_{(x_s > x)}\, dx_s + \frac{1}{2} \mathcal{L}^x_{(\mu),t} \, .$$

Let $t \to \infty$, we obtain: $$\mathcal{L}^x_{(\mu)} = -2\int_0^\infty dB_s\, 1_{(x_s > x)} + 2\mu \int_0^\infty ds\, 1_{(x_s > x)},$$

so that: $$\mathcal{L}^x_{(\mu)} = \mathcal{L}^0_{(\mu)} + 2\int_0^\infty \cancel{\mathfrak{z}}\, dB_s\, 1_{(0 < x_s < x)} - 2\mu \int_0^\infty ds\, 1_{(0 < x_s < x)}$$

and, given: $\mathcal{L}^0_{(\mu)} = a$, we obtain easily

$$X_x = a + 2\int_0^x d\beta_y \sqrt{X_y} - 2\mu \int_0^x dy\, X_y . \qquad \square$$

For computations in the sequel, it will be interesting to relate $Q_a^{(-\mu)}$ and $Q_a^{(0)}$ with the help of Girsanov's thm (in fact, for a lot of computations, we shall be able to use our formulae in: A decomposition of Bessel Bridges!).

<u>Proposition 5:</u> <u>We have</u>, <u>for</u> $a > 0$, <u>and any</u> $x \geq 0$:

$$(31)_x \quad Q_a^{(-\mu)}\big|\mathcal{F}_x \equiv \exp\left[-\frac{\mu}{2}(X_x - a) - \frac{\mu^2}{2}\int_0^x dy\, X_y\right] \cdot Q_a^{(0)}\big|\mathcal{F}_x ,$$

<u>and</u>, <u>with</u> $x = \infty$,

$$(31)_\infty \quad Q_a^{(-\mu)}\big|\mathcal{F}_\infty \equiv \exp\left(\frac{\mu a}{2} - \frac{\mu^2}{2}\int_0^\infty dy\, X_y\right) \cdot Q_a^{(0)}\big|\mathcal{F}_\infty .$$

<u>Proof:</u> $(31)_x$ is an immediate application of Girsanov's thm, and $(31)_\infty$ is obtained from $(31)_x$, by letting $x \to \infty$, which presents no (uniform) integrability problem since the exponential martingale which is featured in $(31)_x$ is uniformly bounded.

<u>Remark:</u> As a check, we remark that, from $(31)_\infty$, we obtain:

$$(32) \qquad Q_a^{(0)}\left(\exp - \frac{\mu^2}{2}\int_0^\infty dy\, X_y\right) = \exp\left(-\frac{\mu a}{2}\right)$$

which is certainly correct, since, using another R-K thm, the left-hand side of (32) is : $\quad E\left[\exp\left(-\frac{\mu^2}{2}\tau_a^+\right)\right] = \exp\left(-\frac{\mu a}{2}\right),$

where : $\tau_a^+ = \int_0^\infty dx\, L^x_{\tau_a} \equiv \int_0^{\tau_a} ds\, 1_{(B_s>0)}$, and $\tau_a \equiv \inf\{t : L^0_t > a\}$

$\left((L^x_t)_{t\geq 0, x\in\mathbb{R}}\right.$ is here the family of Brownian local times $\left.\right)$

<u>Application</u>: From Proposition 3_μ and Proposition 5, we may now deduce a formula for the "Laplace transform" of the law of the Brownian bridges; indeed, let $m(dy)$ be a ≥ 0 σ-finite measure on $\mathbb{R}_+$; then, we deduce from $(26)_\mu$ that :

$$E\left[\exp\left(-\int m(dy)\,\ell^{y}_{(\mu)}\right)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty dx\,\exp\left(-\frac{x^2}{2}\right)E\left[\exp\left(-\int m(dy)\frac{x}{\mu}\,\ell_p^{y\mu/x}\right)\right].$$

(33)

$$\left(\text{changing variables with } x=\mu z\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty dz\,\mu\,\exp\left(-\frac{\mu^2 z^2}{2}\right)E\left[\exp\left(-\int m(dy)\,z\,\ell_p^{y/z}\right)\right].$$

On the other hand, the left-hand side of (33) is, thanks to Proposition 4, equal to :

$$\mu\int_0^\infty dx\, e^{-\mu x}\, Q_x^{(-\mu)}\left(\exp-\int m(dy)X_y\right),$$ so that we now have the following:

<u>Proposition 6</u> : <u>For any</u> $\mu>0$,

(34) $$\boxed{\sqrt{\frac{2}{\pi}}\int_0^\infty dz\,\exp\left(-\frac{\mu^2 z^2}{2}\right)E\left[\exp\left(-\int m(dy)\,z\,\ell_\mu^{y/z}\right)\right] = \int_0^\infty dx\, e^{-\mu x}\, Q_x^{(-\mu)}\left[\exp-\int m(dy)X_y\right]}$$

I believe that a lot of explicit computations can be obtained from ~~there~~ formula (34), since Laplace transforms for $Q_x^{(-\mu)}$ are readily obtained (see : <u>A decomposition of Bessel bridges</u>), and in particular because the family $(Q_x^{(-\mu)} ; x \geq 0)$ is additive in x :

$$Q_{x+x'}^{(-\mu)} = Q_x^{(-\mu)} \oplus Q_{x'}^{(-\mu)}$$

with our old notations.

(<u>Notabene</u> : We must be careful about the notation $Q_x^{(-\mu)}$, because our old notation Q_x^δ concerns $BESQ_x^\delta$!).