[N.Y. AUG. 24th/2004]

ON THE TWO-DIMENSIONAL PROCESSES $\left(N_t, \overline{N}_t \equiv \sup_{s \leq t} N_s, t \geq 0\right)$ FOR N A LOCAL MARTINGALE

NOTATION: WHY N & $\overline{N}$?

To avoid possible confusion with M being either a maximum process, or a martingale, and/or S being either a maximum process, or a price process –

TOPICS:
1. THE 2-DIMENSIONAL MARKOV PROCESS $\left(N_t = B_t, \overline{N}_t = \overline{B}_t ; t \geq 0\right)$ with B a Brownian motion.
2. A RESOLUTION OF SKOROKHOD'S PROBLEM.
3. A MORE GENERAL VIEWPOINT: BALAYAGE.
4. AN ADDED WRINKLE TO DOOB'S INEQUALITIES FOR ≥ 0 SUBMARTINGALES.
5. PENALISATIONS OF BM (B_t) WITH A FUNCTION OF $(\overline{B}_t)$.

1. ON MARKOV PROPERTIES OF $(B_t, \overline{B}_t \,;\, t \geq 0)$

- THE MARKOV PROPERTY FOLLOWS FROM

$$E\left[F(B_t, \overline{B}_t) \,\middle|\, \mathcal{F}_s\right] = E\left[F\left(B_s + \hat{B}_{(t-s)}, \overline{B}_s \vee \left(B_s + \overline{\hat{B}}_{(t-s)}\right)\right) \middle/ \mathcal{F}_s\right]$$

where $\hat{B}_u \equiv B_{s+u} - B_s$, $u \geq 0$,

is independent from $\mathcal{F}_s$. THE LAW OF $(\hat{B}_u, \overline{\hat{B}}_u)$

X is well-known to be ($\longrightarrow$ hence, the semigroup)

$$P(B_u \in dx, \overline{B}_u \in dy) = \left(\frac{2}{\pi u^3}\right)^{1/2} (2y - x) \exp\left(-\frac{(2y-x)^2}{2u}\right) dx\, dy \; 1(y \geq x \vee 0)$$

- WHAT ARE THE HARMONIC FUNCTIONS $H(x,y)$ FOR $(B, \overline{B})$?

ANSWER : $H(x,y) = f(y)(y - x) - F(y)$

for f locally integrable / $F(y) = \int_0^y dz\, f(z)$

i.e: $(H(B_t, \overline{B}_t), t \geq 0)$ is a local martingale.

AND MORE GENERALLY:

$(H(N_t, \overline{N}_t), t \geq 0)$ is a local martingale, for any continuous local mart. $(N_t)_{t \geq 0}$

PROOF: CONSIDER $\Phi(N_t, \overline{N}_t)$, $t \geq 0$,

ASSUMED TO BE A LOCAL MARTINGALE.
APPLY ITÔ (IF POSSIBLE!):

$$\Phi(N_t, \overline{N}_t) = \Phi(N_0, \overline{N}_0) + \int_0^t \Phi'_x (N_s, \overline{N}_s)\, dN_s$$
$$\underbrace{+ \int_0^t \Phi'_y (N_s, \overline{N}_s)\, d\overline{N}_s + \frac{1}{2}\int_0^t \Phi''_{x^2} (N_s, \overline{N}_s)\, d\langle N\rangle_s}_{= 0}.$$

BUT: $d\overline{N}_s$ and $d\langle N\rangle_s$
are mutually singular, since:

$$d\overline{N}_s \equiv 1_{(\overline{N}_s - N_s = 0)}\, d\overline{N}_s \,;\quad d\langle N\rangle_s = 1_{(\overline{N}_s - N_s \neq 0)}\, d\langle N\rangle_s$$

SO:

$$\begin{cases} \Phi'_y (\overline{N}_s, \overline{N}_s) = 0, \ (d\overline{N}_s) \text{ a.s.} \\ \Phi''_{x^2} (N_s, \overline{N}_s) = 0, \ d\langle N\rangle_s \text{ a.s.} \end{cases}$$

(1) $\underline{\Phi'_y (y,y) = 0}$, $\quad \underline{\Phi''_{x^2}(x,y) = 0.}$ (2)

HENCE: (3) $\underline{\Phi(x,y) = \alpha(y)\, x + \beta(y)}$, from (2)

and from (1): $\alpha'(y)\, y + \beta'(y) = 0$

Thus, $\beta(y) = -\int_0^y u\, \alpha'(u)\, du = \int_0^y \alpha(u)\, du - \alpha(y)\, y$

AND: $\Phi(x,y) = \int_0^y \alpha(u)\, du - \alpha(y)(y - x).$

CONVERSELY: It suffices to have f locally integrable, by MONOTONE CLASS THM:

For f regular (: C^1):

(4) $$f(\overline{N}_t)(\overline{N}_t - N_t) - F(\overline{N}_t) = -\int_0^t f(\overline{N}_s)\,dN_s$$

THEN, THE FORMULA EXTENDS BY MCT.

2. A RESOLUTION OF SKOROKHOD'S PROBLEM.

THE PROBLEM (: EMBEDDING PROBLEM) IS:

|| let $\mu(dx)$ be a probability on $\mathbb{R}$; $\int \mu(dx)\,|x| < \infty$

$$\int \mu(dx)\,x = 0.$$

|| FIND T a stopping time of BM (B_t) such that: law of $(B_T) = \mu$

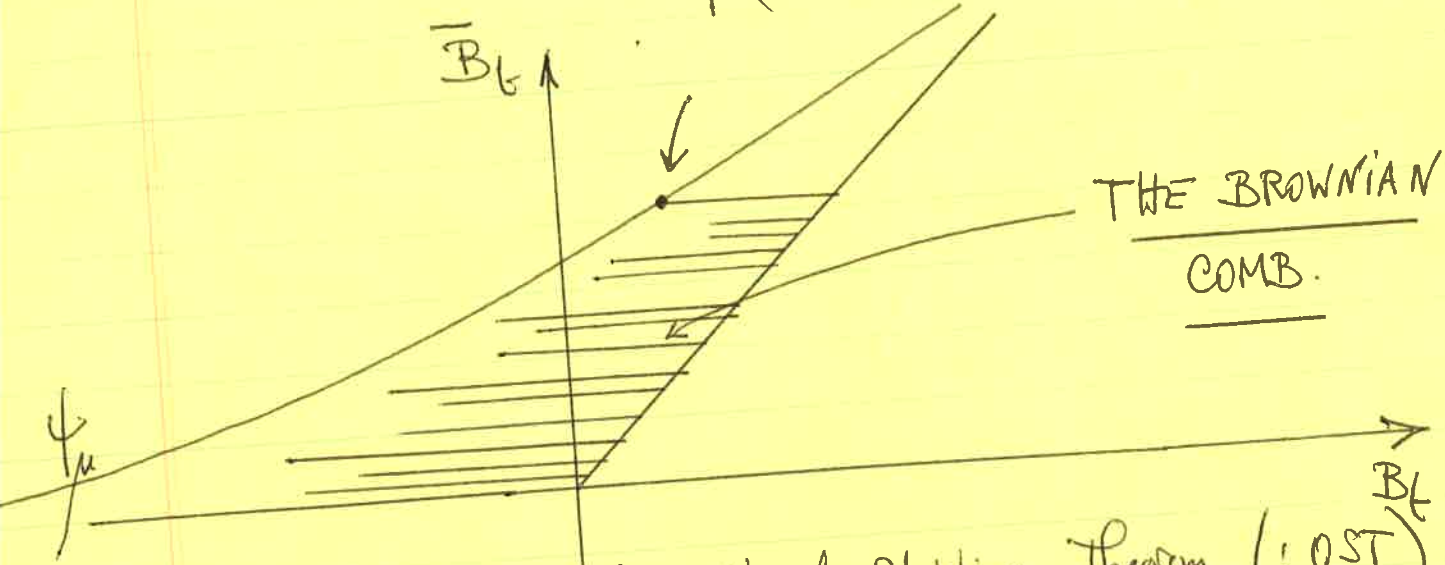
and T "small enough", ie:

- if $\int \mu(dx)\,x^2 < \infty$, then: $E(T) < \infty$
- in general, $(B_{t\wedge T}, t \geq 0)$ UNIFORMLY INTEGRABLE

A SOLTN (: AZÉMA-Y):

$$T_\mu = \inf\{t : \overline{B}_t \geq \psi_\mu(B_t)\}$$

where $\psi_\mu(x) = \frac{1}{\mu([x,\infty))} \int_{[x,\infty)} t \, d\mu(t)$



SOLUTION: Use the Optional Stopping Theorem (: OST) for previous martingales. So:

$\forall f$ bounded Borel $\quad \cancel{E[f(S_T)(}$

(5) $\quad E\left[f(\overline{B}_T)(\overline{B}_T - B_T) - F(\overline{B}_T)\right] = 0.$

Define $\varphi(x) = E(B_T \mid \overline{B}_T = x)$.

Then, (5) yields:

(6) $\quad E\left[f(\overline{B}_T)(\overline{B}_T - \varphi(\overline{B}_T)) - F(\overline{B}_T)\right] = 0$

which yields:

$$P(\overline{B}_T \geq y) = \exp\left(-\int_0^y \frac{ds}{(s - \varphi(s))}\right).$$

Now, if $B_T = \varphi(\overline{B}_T)$, then the law of B_T is obtained !! : $\varphi \longrightarrow \mu_\varphi$

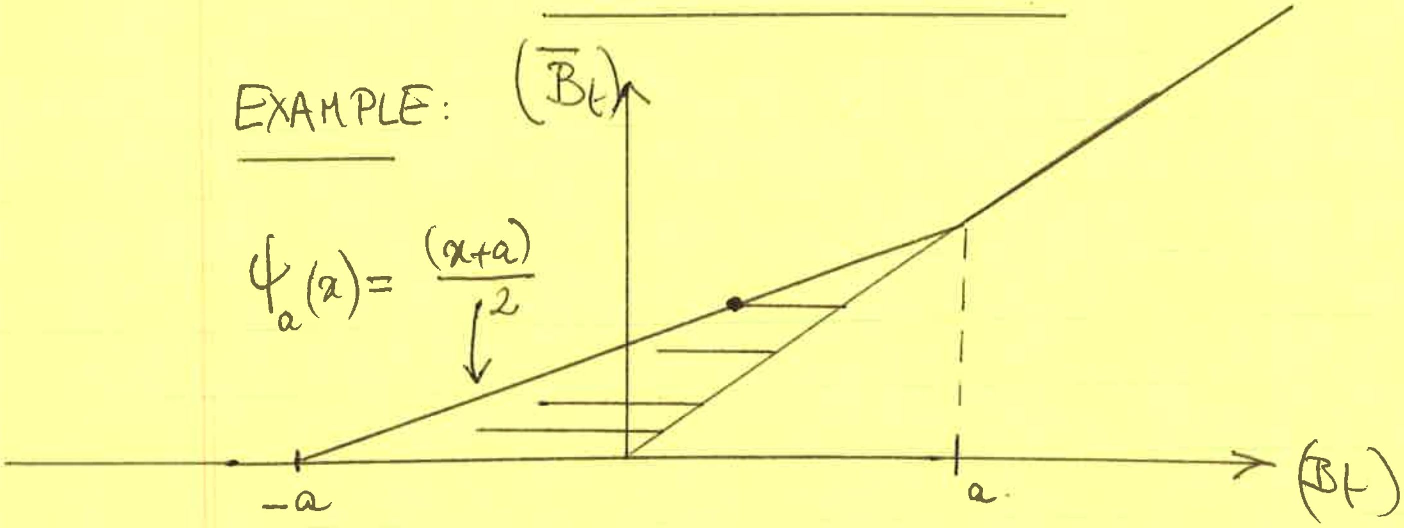
It turns out that the equation:

$$\mu_\varphi = \mu_0 \quad (\text{given})$$

yields:

$$\underline{\varphi = (\psi_{\mu_0})^{-1}}$$

EXAMPLE:



$$T_a \equiv \inf\{t : 2\overline{B}_t - B_t = a\}$$

Then, B_{T_a} is uniform on $[-a, a]$ & indpdt from T_a.

PITMAN'S THEOREM:

$$(R_t = 2\overline{B}_t - B_t,\ t \geq 0) \text{ is } BES(3);$$

$\forall\, T$ stopping time w.r.t. $\mathcal{R}_t \equiv \sigma\{R_s, s \leq t\}$,

$$E\left[g(B_T) \mid \mathcal{R}_T\right] = \frac{1}{2R_T}\int_{-R_T}^{R_T} dx\, g(x)$$

Here: $R_T \equiv a$.

3. A MORE GENERAL VIEWPOINT: BALAYAGE

FOR $(\Sigma_t, t \geq 0)$ a continuous semimartingale,

define $g_t = \sup\{s \leq t : \Sigma_s = 0\}$, last zero $\leq t$

$d_t = \inf\{s \geq t : \Sigma_s = 0\}$, 1st zero, $> t$

Then, if (k_u) is a locally bounded predictable process,

BALAYAGE : $$\boxed{k_{g_t}\Sigma_t = \int_0^t k_{g_s}\, d\Sigma_s} \qquad (7).$$

INTUITIVE MEANING:

A "GLOBAL MULTIPLICATION" OF Σ OVER ITS EXCURSIONS IS A STOCH. INT. IN $(d\Sigma_s)$.

EXAMPLES:

1) $\Sigma_t \equiv N_t$ ($= B_t$, if you wish);

$k_u \equiv f(L_u)$ $\Longrightarrow$ $k_{g_t} = f(L_t)$

local time of N at 0

$$f(L_t)N_t = \int_0^t f(L_s)\,dN_s.$$

2) $\Sigma_t = \bar{N}_t - N_t$;

$k_u \equiv f(\bar{N}_u) \Longrightarrow f(\bar{N}_t)(\bar{N}_t - N_t) = \int_0^t f(\bar{N}_s)\, d(\bar{N}_s - N_s)$

Hence: $f(\bar{N}_t)(\bar{N}_t - N_t) - F(\bar{N}_t)$ is a local martingale.

PROOF OF (7): MCT AGAIN!!

Take $k_u \equiv 1_{[0,T]}(u)$

Then: $k_{g_t} = 1_{(g_t \leq T)} = 1_{(t \leq d_T)}$

and $k_{g_t} \Sigma_t = \Sigma_{t \wedge d_T} = \int_0^{t \wedge d_T} d\Sigma_u$, etc...

4. ON DOOB'S INEQUALITIES FOR ≥ 0 SUBMARTINGALES.

Consider a sequence of random variables (X_k), and $\overline{X}_n = \max_{k \leq n}(X_k)$. Then, a succession of trivial identities:

$$f(\overline{X}_n)(\overline{X}_n - X_n) \underset{(a)}{=} f(\overline{X}_{n-1})(\overline{X}_n - X_n)$$

$$\underset{(b)}{=} \sum_{k=1}^{n} f(\overline{X}_{k-1}) \, \Delta_k (\overline{X}_k - X_k)$$

$$= \sum_{k=1}^{n} f(\overline{X}_{k-1})(\overline{X}_k - \overline{X}_{k-1}) - \sum_{k=1}^{n} f(\overline{X}_{k-1})(X_k - X_{k-1})$$

Hence:

$$\boxed{\begin{aligned} & f(\overline{X}_n)(\overline{X}_n - X_n) - F(\overline{X}_n) \\ & = \sum_{k=1}^{n} \int_{\overline{X}_{k-1}}^{\overline{X}_k} \left(f(\overline{X}_{k-1}) - f(x) \right) dx - \sum_{k=1}^{n} f(\overline{X}_{k-1})(X_k - X_{k-1}) \end{aligned}}$$

APPLICATION : $f \geqslant 0$, and $\uparrow$ (so that F is convex)

and (X_k) is a submartingale,

then :

$$S^f_n \equiv \left\{ f(\bar{X}_n)(\bar{X}_n - X_n) - F(\bar{X}_n),\ n \geqslant 0 \right\}$$

is a supermartingale.

Hence :

$$E\left[f(\bar{X}_n)\bar{X}_n - F(\bar{X}_n) \right] \leqslant E\left[f(\bar{X}_n) X_n \right].$$

Applications :

1) $f(y) = 1_{(y \geqslant \lambda)}$; then the supermart

$$S^f_n \equiv 1_{(\bar{X}_n \geqslant \lambda)} (\lambda - X_n).$$

which yields : Doob's maximal inequality :

$$\lambda P(\bar{X}_n \geqslant \lambda) \leqslant E\left[1_{(\bar{X}_n \geqslant \lambda)} X_n \right].$$

2) $f(y) = p y^{p-1}$; $F(y) = y^p$; $p > 1$.

Then : $S^{(p)}_n \equiv (p-1)(\bar{X}_n)^p - p(\bar{X}_n)^{p-1} X_n$

is a supermartingale,

which yields Doob's L^p inequality :

$$(p-1) E\left[(\bar{X}_n)^p\right] \leqslant p\, E\left[(\bar{X}_n)^{p-1} X_n\right]$$

Hence: (Hölder) $E[(\overline{X}_n)^p] \leq \left(\frac{p}{p-1}\right)^p E[X_n^p]$

5. PENALISATIONS OF BM WITH A FUNCTION OF $(\overline{B}_t)$.

CONSIDER: $$W_t^f \equiv \frac{f(\overline{B}_t)}{E_W(f(\overline{B}_t))} \cdot W$$

for $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a probability density fctn.

Then, $W_t^f \xrightarrow[t\to\infty]{(w)} W_\infty^f$, ie:

$\forall s$ fixed, $\forall \Gamma_s \in \mathcal{F}_s$: $W_t^f(\Gamma_s) \longrightarrow W_\infty^f(\Gamma_s)$,

and: $W_\infty^f(\Gamma_s) = E_W[1_{\Gamma_s} S_s^f]$,

where: $$S_s^f = 1 - F(\overline{B}_s) + f(\overline{B}_s)(\overline{B}_s - B_s)$$
$$= 1 - \int_0^s f(\overline{B}_u)\, dB_u .$$

Under W_∞^f, (B_t) satisfies:

$$B_t = B_t^{(f)} - \int_0^t \frac{f(\overline{B}_u)\, du}{(1-F(\overline{B}_u)) + f(\overline{B}_u)(\overline{B}_u - B_u)}$$

PROOF: GIRSANOV.