Sept. 7^{th}, 1993.

Note B : Some Complements on Spider_Martingales.

(6)

Consider a process $\left(M_t = \left(M_t^{(1)}, \dots, M_t^{(k)}\right), t \geq 0\right)$ which satisfies a), b), c), d), as in the Definition of Sect. 4 in [L].

Then, we may complete Prop. 1 with the equivalent property:

(iv) for every k-tuple $(u_1, \dots, u_k)$ such that: $\sum_{i=1}^{k} u_i = 0$, $\left(\sum_{i=1}^{k} u_i M_t^{(i)}, t \geq 0\right)$ is a martingale.

A possible proof is: (iii) $\Longrightarrow$ (iv), since:

$$\sum_i u_i M_t^{(i)} = \left(\sum_i u_i N_t^{(i)}\right) + \left(\sum_i u_i\right) A_t .$$

Conversely, (iv) $\Longrightarrow$ (i), since, from (iv), for every bounded stopping time τ, we have: $\sum_i u_i = 0 \Longrightarrow \sum_i u_i E[M_\tau^{(i)}] = 0$.

This can only happen if the vector $\left(E(M_\tau^{(i)}); i = 1, 2, \dots, k\right)$ is proportional to $(1, 1, \dots, 1)$, i.e: the fundamental property e) is satisfied. □

Remark: If (M_t) is a spider-martingale, we can extend the property (iv) as follows:

if $\left(z_u^{(i)}; i = 1, 2, \dots, k\right)$ are k bounded previsible processes, then:

$$\sum_{i=1}^{k} z_{g_t}^{(i)} M_t^{(i)} \text{ is a martingale as soon as } \sum_i z_u^{(i)} \equiv 0.$$

Proof: From the balayage formula, $\sum_i z_{g_t}^{(i)} M_t^{(i)} = (\text{mart}) + \int_0^t dA_u \left(\sum_i z_u^{(i)}\right)$ □

The property (iv) also invites to define skew-spider-martingales which are generalizations of spider-martingales in the same way skew BM generalizes BM. (To be Completed).

$\left[\text{i.e: } \sum_i u_i p_i = 0 \Longrightarrow \sum_i u_i M_t^{(i)} \text{ is a martingale}\right]$.

END OF MESSAGE

Note B : Some Complements on Spider-martingales [Sept. 8th, 1993].

(Continuation of Sept. 7th).

1)

Let $p = (p_1, \dots, p_k)$ be a k-tuple of reals, with $p_i \neq 0$, for each i, and define a p-spider-martingale to be a process $\{M_t = (M_t^{(1)}, \dots, M_t^{(k)}),\ t \geq 0\}$ such that:

$(iv)_p$ $\boxed{\text{if } \sum_i u_i p_i = 0, \text{ then } \sum_i u_i M_t^{(i)} \text{ is a martingale}}$.

I have not assumed a priori that: $0 < p_i < 1$, and $\sum_i p_i = 1$, but we have the following

Lemma: Let $p = (p_1, \dots, p_k)$, with $p_i \neq 0$, for every i.

Assume that there exists a p-spider martingale (M_t) which reaches every point $x = (x_1, \dots, x_k)$. Then, the k reals $p_1, \dots, p_k$ are either all > 0, or all < 0.

Proof: From $(iv)_p$, we deduce that if: $T_x = \inf\{t : M_t = x\}$, then:

$$\sum_i u_i p_i = 0 \quad \text{implies:} \quad \sum_i u_i \left(p_i E(M_{T_x}^{(i)}) \right) = 0.$$

Consequently, as before, we must have: $p_i E(M_{T_x}^{(i)}) = C$, independent of i.
Thus, we have:

$$(p_i a_i) P(M_{T_x} = x_i) = C.$$

The constant C cannot be equal to 0, since: $\sum_i P(M_{T_x} = x_i) = 1$.
Hence, we have, for all pairs (i,j),

$$(p_i p_j)\, a_i a_j > 0,$$

and, therefore: $p_i p_j > 0$, which finishes the proof. $\square$

Once this remark has been made, the study of p- spider martingales is obviously reduced to that of spider-martingales, since:

if (M_t) is a p- spider martingale, then: $\left(\frac{1}{p_i} M_t^{(i)} ; i=1,2,\cdots,k\right)$ is a spider-martingale, and conversely.

Appendix A   On two generalizations of spider-martingales.

It appears clearly, from the developments in Section 4 of [L] that the set of hypotheses: a), b), c), d) on one hand, and the fundamental property: e) on the other hand play quite different rôles.
Hence, it seems natural to consider processes $\{M_t \equiv (M_t^{(1)}, M_t^{(2)}, \dots, M_t^{(k)}); t \geq 0\}$ which satisfy [a), b), c), d)] only, and to study to which extent they satisfy e), or not.
We will call such processes [which satisfy [a), b), c), d)] only] generalized spider-martingales of type I.
Another kind of processes (M_t), which satisfy e), but not necessarily [a), b), c), d)] seems to occur fairly naturally (e.g. from transforms by space time ~~harmonic~~ harmonic functions)! We will call such processes generalized spider martingales of type II.

We have the following results

Theorem 1: Let (M_t) be a generalized spider-martingale of type I.

a) Assume that every component $(M_t^{(i)}; t \geq 0)$ is a semimartingale, so that: $M_t^{(i)} = N_t^{(i)} + A_t^{(i)}$, with $(N_t^{(i)}, t \geq 0)$ a local martingale, and $(A_t^{(i)})$ a continuous process with bounded variation.
Then, the random measure $dA_t^{(i)}$ is carried by the set:

$$\tilde{\Gamma}^{(i)} \equiv \{(t, \omega) : M_t^{(i)}(\omega) \geq 0\};$$

hence, the process $(A_t^{(i)})$ is given by the formula: $A_t^{(i)} = \sup_{s \leq t} (-N_s^{(i)})$.

b) Assume furthermore that $(M_t, t \geq 0)$ satisfies the following property:
if z is a bounded predictable process, and $t > 0$, then:

(f) $$E\left[\int_0^t z_s \, dM_s^{(i)}\right] = 0 \text{ for one } i, \text{ if, and only if, for all } i\text{'s}.$$

This property holds iff there exists a continuous adapted process $(A_t, t \geq 0)$, which is carried by $\Gamma^0 \equiv \{(t,\omega) : M_t(\omega) = 0\}$, and strictly positive predictable processes $(z_t^{(i)}, t \geq 0)$ such that:

$$A_t^{(i)} = \int_0^t z_s^{(i)} \, dA_s \quad , \quad i = 1, 2, \dots, k .$$

Finally, there exists a spider-martingale $\{\hat{M}_t = (\hat{M}_t^{(1)}, \dots, \hat{M}_t^{(k)}), t \geq 0\}$ such that:

$$M_t^{(i)} = z_{g_t}^{(i)} \hat{M}_t^{(i)} \; , \; t \geq 0 .$$

<u>Proof:</u> a) Let $i \neq j$. We have, from our hypotheses:

$$0 = M_t^{(i)} M_t^{(j)} = \int_0^t M_s^{(i)} dM_s^{(j)} + \int_0^t M_s^{(j)} dM_s^{(i)} + \langle M^{(i)}, M^{(j)} \rangle_t$$

Hence, both the local martingale part, and the bounded variation part of the right-hand side are equal to 0. Therefore, we have:

$$\int_0^t \left(M_s^{(i)} dA_s^{(j)} + M_s^{(j)} dA_s^{(i)} \right) + \langle M^{(i)}, M^{(j)} \rangle_t = 0 .$$

Integrating $M_s^{(j)}$ with respect to the left-hand side, we obtain:

(1) $$\int_0^t (M_s^{(j)})^2 dA_s^{(i)} + \int_0^t M_s^{(j)} d\langle M_s^{(i)}, M_s^{(j)} \rangle_s = 0 ;$$

the second integral is equal to: $\int_0^t 1_{(M_s^{(i)}=0)} M_s^{(j)} d\langle M^{(i)}, M^{(j)} \rangle_s$,

and this is equal to 0, since: $\int_0^t 1_{(M_s^{(i)}=0)} d\langle M^{(i)} \rangle_s = 0$.

Hence, we deduce from (1) that: $dA_s^{(i)} = 0$ on $\Gamma^j \equiv \{(s,\omega) : M_s^{(j)}(\omega) > 0\}$, $\forall j \neq i$.

Finally, $dA_s^{(i)}$ is carried by $\tilde{\Gamma}^{(i)}$.

The fact that $(A^{(i)}_t)$ is equal to : $\sup_{s \le t} (-N^{(i)}_s)$ then follows from the next lemma.

b) Under the hypothesis (f), all measures: $dA^{(i)}_t \, dP$, $i = 1, 2, \dots, k$, are equivalent on the predictable σ-field, hence the existence of Radon-Nikodym densities $(z^{(i)}_t, t \ge 0)$, $i = 1, 2, \dots, k$, with respect to, say:

$$dA_t \overset{\text{def}}{=} \sum_j dA^{(j)}_t .$$

The final assertion is a consequence of the balayage formula. □

Here is the promised Lemma.

<u>Lemma</u>: Let $(X_t, t \ge 0)$ be an $\mathbb{R}_+$-valued continuous semimartingale, with canonical decomposition: $X_t = M_t + A_t$; assume that dA_t is carried by $\{t : X_t = 0\}$. Then, A_t is increasing, and given by:

$$A_t = \sup_{s \le t} (-M_s).$$

<u>Proof</u>: ~~From a~~ A general formula about the jumps of local times of semimartingales in the space variable [ ] asserts that:

$$(2) \qquad L^0_t - L^{0-}_t = 2 \int_0^t 1_{(X_s = 0)} \, dA_s ,$$

where $(L^a_t, t \ge 0)$ denotes the local time of X at a (chosen to be right-continuous in a), and (L^{a-}_t) is the process of left-limits in the space variable.

From our hypothesis, we have:

$$L^{0-}_t = 0, \quad \text{and} \quad 1_{(X_s = 0)} \, dA_s = dA_s .$$

Finally, we obtain, from (2), that: $\quad L^0_t = 2A_t$.

Hence, in particular, (A_t) is increasing, and the final formula follows from Skorokhod's lemma □

Appendix C : Some space-time martingales associated with a Markov process.

[Sept. 8th, 1993].

In this Appendix, we exhibit a general family of "space-time martingales" associated with a Markov process.
The family of martingales constructed from the Hermite polynomials in Section 3 of this paper is a particular case.

$\{(X_t)_{t \geq 0} ; (P_x)_{x \in E}\}$ is a "nice" Markov process taking values in a measurable space E, and A denotes its infinitesimal extended generator (see, e.g., Kunita [ ]) :

$f : E \longrightarrow \mathbb{R}$ belongs to $D(A)$ if there exists a measurable function $g : E \longrightarrow \mathbb{R}$ such that : $f(X_t) - \int_0^t ds\, g(X_s)$ is a P_x-martingale, for every $x \in E$.

g is unique, up to sets of zero potential, and we write : $g = Af$. If, in turn, g (or, in fact, a selected representative of g) belongs to $D(A)$, we write $Ag = A^2(f)$, and so on.

Theorem : Let $N \in \mathbb{N}$, and suppose $f : E \longrightarrow \mathbb{R}$ belongs to $D(\cancel{A}\, A^{N+1})$.
Then, the process :

$$(1) \quad \sum_{n=0}^{N} \frac{(-1)^n}{n!} t^n A^n(f)(X_t) - \frac{(-1)^N}{N!} \int_0^t ds\, s^N A^{N+1}(f)(X_s)$$

is a martingale.

Proof : For every $n \leq N$, we note $(M_t^{(n)})$ the martingale defined by :

2).

$$M_t^{(n)} = A^n(f)(X_t) - \int_0^t ds\, A^{n+1}(f)(X_s).$$

We have the following equalities:

$$f(X_t) = M_t^{(0)} + \int_0^t ds\, (Af)(X_s)$$

$$-t\,(Af)(X_t) = -\int_0^t s\, d_s\big((Af)(X_s)\big) - \int_0^t ds\,(Af)(X_s)$$

$$= -\int_0^t s\,\big[dM_s^{(1)} + ds\, A^2(f)(X_s)\big] - \int_0^t ds\,(Af)(X_s)$$

$$\frac{t^2}{2}(A^2 f)(X_t) = \int_0^t \frac{s^2}{2}\big[dM_s^{(2)} + ds\, A^3(f)(X_s)\big] + \int_0^t ds\, s\, A^2(f)(X_s).$$

$$\vdots$$

$$(-1)^N \frac{t^N}{N!} A^N(f)(X_t) = \int_0^t \frac{(-1)^N s^N}{N!}\big[d_s M_s^{(N)} + ds\, A^{(N+1)}(f)(X_s)\big] + \int_0^t ds\, (-1)^N \frac{s^{N-1} A^N(f)(X_s)}{(N-1)!}$$

If we add up the left-hand sides of these successive equalities, and then the right-hand sides, we obtain that the process in (1) is equal to:

$$\sum_{n=0}^N \int_0^t \frac{(-1)^n s^n}{n!}\, d_s\big(M_s^{(n)}\big)$$; in particular, it is a martingale. □

Remarks: If we take $X_t = B_t$, and $f(x) = x^{2N}$, then: $A^{N+1}(f) = 0$ and the process in (1) is precisely: $H_{2N}(B_t, t)$, as the reader will prove without any difficulty.

August, 18th.

A working program.

1. Further stability properties of spider-martingales, i.e.: "Cauchy-Riemann type equations"

If $(M_t; t \geq 0)$ is a spider-martingale, for which general functionals F is:

$$F(M_s; s \leq t) \equiv \left(F_1(M_s; s \leq t); F_2(M_s, s \leq t); \dots\dots; F_k(M_s, s \leq t)\right)$$

still a spider-martingale?
I believe this has a lot to do with the martingales of BM which vanish on the zero set of BM. See, Azéma-Yor, Sém. XXV or XXVI.

2. Plan to write:

$$\boxed{\text{A study of some functionals of Walsh's BM as a "testing-ground" for excursion theory and stochastic calculus.}}$$

(see Intro. next page).

3. Instead of thinking about the "positive parts" $(M_t^{(i)})$, think of the vector $\left(M_t^{(i,j)}, t \geq 0\right)_{i,j}$, and its "intrinsic properties".

4. More generally than Spider-Martingales, think of Martingale systems, i.e:

Certain martingales are linked in a "canonical" way, which arises in various problems, via changes of scale and speed.
Give ~~some~~ examples, probably with planar BM.

5. The importance of spider-martingales may be even better accepted if one can show that many "systems" (which would be spider-semi-martingales) could be transformed into spider-martingales, and vice-versa).
Develop on the lines of:
<u>Feller's representation of diffusions</u> ; exs: Skew BM ⟷ BM.
Skew Walsh BM ⟷ Walsh BM.

6. <u>Intro. to : "A study .... ".</u>