Aug. 14^{th}, 1993.

Some formulae for the Brownian and Bessel spiders, I.

1)

1. The Brownian case.

Let $(B_t, t \geq 0)$ be the Brownian spider, living on k rays $I_1, I_2, \dots, I_k$. Let $x_1, x_2, \dots, x_k$ be points situated respectively on $I_1, I_2, \dots, I_k$, and define $a_i = |x_i|$, the distance of x_i to the origin. (This is preferable to the old notation, I think, where we confused x_i and a_i; $x_i \in \mathbb{C}$, say, whereas $a_i \in \mathbb{R}_+$).

Starting from formula (5·b) in "A list of formulae", Aug. 9^{th}, 1993, which I now write in the form:

$$(5\cdot b) \qquad \boxed{P\left(S^{(1)}_{\tilde{T}} \leq a_1, \dots, S^{(k)}_{\tilde{T}} \leq a_k\right) = 1 - \frac{\sum_j \frac{1}{(\sinh(a_j))}}{\left(\sum_j \coth(a_j)\right)}}$$

and remarking that the left-hand side is:

$$P\left(\tilde{T} \leq T_{(x_1, \dots, x_k)}\right) = 1 - P\left(\tilde{T} \geq T_{(x_1, \dots, x_k)}\right),$$

I obtained formula $(7)^k$ in "A list":

$$(7)^k \qquad \boxed{E\left[\exp -\frac{\lambda^2}{2} T_{(x_1, \dots, x_k)}\right] = \frac{\sum_j \frac{1}{\sinh(\lambda a_j)}}{\sum_j \coth(\lambda a_j)}}$$

(I have used the scaling property to inject λ into the formula).

Now, for simplicity, I will write T_* for $T_{(a_1,\dots,a_k)}$.

It is not difficult to refine $(7)^k$ as:

$$(7)^k_j \qquad \boxed{E\left[\exp\left(-\frac{\lambda^2}{2}T_*\right); B_{T_*}=x_j\right] = \frac{1/\sinh(\lambda a_j)}{\sum_\ell \coth(\lambda a_\ell)}.}$$

and, consequently, we get: (9) $\boxed{P\left(B_{T_*}=x_j\right) = \dfrac{(1/a_j)}{\sum_\ell (1/a_\ell)}}$

From $(7)^k_j$ and (9), we obtain:

$$(10) \qquad \boxed{E\left[\exp\left(-\frac{\lambda^2}{2}T_*\right)\middle| B_{T_*}=x_j\right] = \left(\frac{\lambda a_j}{\sinh(\lambda a_j)}\right)\frac{\sum_\ell\left(\frac{1}{\lambda a_\ell}\right)}{\left(\sum_\ell \coth(\lambda a_\ell)\right)}}$$

<u>Here is an explanation of this formula:</u>

a) Conditionally on $(B_{T_*}=x_j)$, the process $\left(B_{g_{T_*}+u};\, u\le T_*-g_{T_*}\right)$ is a 3-dimensional Bessel process, considered up to its first hitting time of a_j;

b) Moreover, the processes $\left(B_u;\, u\le g_{T_*}\right)$ and $\left(B_{g_{T_*}+u};\, u\le T_*-g_{T_*}\right)$ are independent, and we have:

$$(11) \qquad \boxed{E\left[\exp\left(-\frac{\lambda^2}{2}g_{T_*}\right)\right] = \frac{\sum_\ell\left(\frac{1}{\lambda a_\ell}\right)}{\sum_\ell \coth(\lambda a_\ell)}}$$

[To finish this discussion, we should now give a formula for :

$$E\left[\exp\left(-\frac{\lambda^2}{2} g_{T_*}\right) \Big| \ell_{T_*} ; S^{(1)}_{g_{T_*}}, \dots, S^{(k)}_{g_{T_*}}\right],$$

and also describe the process $(B_t ; t \leq g_{T_*})$ completely;

again, there is an agreement formula, and so on....].

2. The Bessel case.

Let $-1<\mu<0$, and $\nu=\mu$.

$(X_t, t\geq 0)$ now denotes the Bessel spider with index μ, and I keep the above notation concerning points and distances. P^μ denotes the law of $(X_t, t\geq 0)$.

Then, as formula (6.e) in "A list" clearly suggests, we have:

$$_\mu(5.b) \qquad P^\mu\left(S^{(1)}_{\tilde{T}} \leq a_1, \dots, S^{(k)}_{\tilde{T}} \leq a_k\right) = 1 - \frac{\sum_j \theta_\mu(a_j) h_\mu(a_j)}{\sum_j h_\mu(a_j)}$$

where: $\theta_\mu(x) = \dfrac{x^\mu}{c_\mu I_\mu(x)} \equiv E^\mu\left[\exp\left(-\frac{x^2}{2} T_\mu\right)\right]$ (as is well known)

and $\qquad h_\mu(x) = \dfrac{I_\mu(x)}{I_\nu(x)}$.

I shall now present the Bessel variants of formulae $(7)^k$, $(7)^k_j$, (9), (10) and (11) which follow from $_\mu(5.b)$ by the same arguments as above:

$$_\mu(7)^k \qquad \boxed{E^\mu\left[\exp\left(-\frac{\lambda^2}{2} T_*\right)\right] = \frac{\sum_j \theta_\mu(\lambda a_j) h_\mu(\lambda a_j)}{\sum_j h_\mu(\lambda a_j)}}$$

$$_\mu(7)^k_j \qquad \boxed{E^\mu\left[\exp\left(-\frac{\lambda^2}{2} T_*\right); X_{T_*} = x_j\right] = \frac{\theta_\mu(\lambda a_j) h_\mu(\lambda a_j)}{\sum_\ell h_\mu(\lambda a_\ell)}}$$

Letting $\lambda \to 0$, and using the equivalence: $c_\mu I_\mu(x) \sim x^\mu$ (which we may deduce from the representation of $\theta_\mu(x)$ as a Laplace transform),
we obtain:

$$\mu^{(9)} \qquad \boxed{P(X_{T_*} = x_j) = \frac{a_j^{2\mu}}{\sum_\ell (a_\ell)^{2\mu}}},$$

and then, since: $\dfrac{\theta_\mu(x)\, h_\mu(x)}{x^{2\mu}} = \theta_\nu(x) \dfrac{c_\nu}{c_\mu}$, we obtain:

$$\mu^{(10)} \qquad \boxed{E^\mu\left[\exp\left(-\frac{\lambda^2}{2} T_*\right) \Big| X_{T_*} = x_j\right] = \theta_\nu(\lambda a_j)\, \frac{c_\nu}{c_\mu} \frac{\left(\sum_\ell (\lambda a_\ell)^{2\mu}\right)}{\left(\sum_\ell h_\mu(\lambda a_\ell)\right)}}$$

This formula may be decomposed into:

$$\mu^{(11)^+} \qquad E^\mu\left[\exp -\frac{\lambda^2}{2}(T_* - g_{T_*}) \Big| X_{T_*} = x_j\right] = \theta_\nu(\lambda a_j)$$

and

$$\mu^{(11)^-} \qquad \boxed{E^\mu\left[\exp\left(-\frac{\lambda^2}{2} g_{T_*}\right)\right] = \left(\frac{c_\nu}{c_\mu}\right) \frac{\sum_\ell (\lambda a_\ell)^{2\mu}}{\sum_\ell h_\mu(\lambda a_\ell)}}$$