Aug. 17th, 1993

The definition and some properties of spider-martingales.

Let k be an integer, and $[\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P]$ be a filtered probability space.

Definition: A process $\{M_t = (M_t^{(1)}, M_t^{(2)}, \dots, M_t^{(k)}), t \geq 0\}$ taking values in $(\mathbb{R}_+)^k$, and such that:

a) $M_0 = 0$;

b) (M_t) has continuous trajectories, and is $(\mathcal{F}_t)$ adapted;

c) the $(k+1)$ random sets:

$$\Gamma_0 = \{(t,\omega);\ \forall j,\ M_t^{(j)}(\omega) = 0\};\quad \Gamma_i = \{(t,\omega);\ M_t^{(i)}(\omega) \neq 0\},\ i = 1, 2, \dots, k$$

constitute a partition of $\mathbb{R}_+ \times \Omega$; in other words, for any given (t,ω), there is at most one of the $M_t^{(i)}(\omega)$ which is not 0.

d) for every bounded stopping time σ, the k positive reals

$$E[M_\sigma^{(i)}],\ i = 1, 2, \dots, k,$$ are equal.

will be called a spider-martingale.

Here are two basic examples of spider-martingales:

(i) In the case $k = 2$, this notion coincides with the usual notion of a martingale valued in $\mathbb{R}$, say $(M_t, t \geq 0)$, represented in the form:

$$M_t = (M_t^+, M_t^-),\quad t \geq 0,$$

with M^+ and M^- its positive and negative parts.
Assume $M_0 = 0$; then, the properties b), c), d) are satisfied;

In fact, property d) is usually presented as: $E[M_\tau] = 0$, as τ varies among bounded stopping times.
Hence, we have: $$E[M_\tau^+] = E[M_\tau^-].$$

Furthermore, it may be worth remarking that this common value of $E[M_\tau^+]$ and $E[M_\tau^-]$ is also: $\frac{1}{2}E(L_\tau)$, where $(L_t, t \geq 0)$ is the local time of M at 0.

(ii) The general notion of a spider-martingale, for k any integer, originates from Walsh's (symmetric) Brownian motion on k rays. (To be developed).

Here is a summary of the properties of a spider-martingale which I have now obtained.

<u>Proposition 1</u>: Let $(M_t; t \geq 0)$ be a spider-martingale. Furthermore, assume that:

(*) <u>one component</u> of <u>M is a semi-martingale</u>.

Then, each component is a semi-martingale; more precisely, there exists an increasing process $(A_t, t \geq 0)$ and k martingales $(N_t^{(i)}, t \geq 0)$, $i = 1, 2, \dots, k$ such that:

a) for each i, $\quad M_t^{(i)} = N_t^{(i)} + A_t, \quad t \geq 0$

b) (dA_t) is carried by $\Gamma_0 = \{(t, \omega);\ M_t(\omega) = 0\}$

c) for every $i = 1, 2, \dots, k$, $d\langle N^{(i)} \rangle_t$ is carried by $\Gamma_i = \{(t,\omega);\ M_t^{(i)}(\omega) > 0\}$

From now on, I shall assume that the hypothesis (*) is satisfied.

There certainly exists a Dubins-Schwarz representation of a general spider-martingale [We <u>must</u> be able to do this! This is somehow the "heart" of the winding story between Jim and I!!].

For the moment, I can state

<u>Proposition 2</u>: <u>If</u> $(M_t; t \geq 0)$ is <u>a spider-martingale, and if, for</u> <u>simplicity</u>, <u>we assume</u>: $\langle M^{(i)} \rangle_\infty = \infty$ a.s., for every i, <u>then</u>:

<u>there exist k independent reflecting Brownian motions</u> $(R^{(i)}(u); u \geq 0)$ <u>such that</u>:

$$M_t^{(i)} = R^{(i)}(\langle M^{(i)} \rangle_t), \quad t \geq 0, \quad i = 1, 2, \dots, k.$$

Hence, we now have the following situation:

a) the reflecting Brownian motions $(R^{(i)}(u); u \geq 0)$ are independent;

b) We have k increasing processes $(\langle M^{(i)} \rangle_t, t \geq 0)$ with disjoint supports $S^{(i)}$, which are precisely the sets:

$$\Gamma_i = \{(t, \omega); M_t^{(i)}(\omega) > 0\};$$

c) We have the common local time $(A_t, t \geq 0)$ of the $M^{(i)}$'s, which lives on $\Gamma_0 = \{(t, \omega); M_t(\omega) = 0\}$

From these 3 ingredients, we should have a pathwise construction of the most general spider-martingale. (The next set of remarks, which involves <u>transforms of a spider-martingale</u>, may/should help to reduce this question to one we are more familiar with, in the context of planar windings).

Transforms of a spider-martingale.

I shall now consider transforms of M, of the form: $g\left(M_t^{(i)}, \langle M^{(j)}\rangle_t ; 1 \leq i,j \leq k\right)$

for various functions g, in order to find certain invariants of the spider-martingales.

Proposition 3: If $(M_t, t \geq 0)$ is a spider martingale, and if we note:

$$|M_t| = \left(\sum_{i=1}^{k} (M_t^{(i)})^2\right)^{1/2}, \text{ and } \langle M\rangle_t = \sum_{i=1}^{k} \langle M^{(i)}\rangle_t ,$$

then, there exists a reflecting Brownian motion $(R(u), u \geq 0)$ such that:

$$|M_t| = R(\langle M\rangle_t), \quad t \geq 0.$$

The next Proposition gives a number of examples of spider martingales, built from one of them. (this has the flavor of Lévy's conformal invariance).

Proposition 4: To a function $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, we associate a function

$$\tilde{f} : (\mathbb{R}_+)^k \times \mathbb{R} \longrightarrow (\mathbb{R}_+)^k$$

defined by:

$$\tilde{f}(x_1, x_2, \dots, x_k; u) = \left(f(x_1, u); f(x_2, u); \dots ; f(x_k, u)\right).$$

Then, if f is space-time harmonic, and satisfies: $f(0,u) \equiv 0$, and if, furthermore (M_t) is a spider-martingale,

$$\left(\tilde{f}(M_t; \langle M\rangle_t), t \geq 0\right) \text{ is also a spider-martingale.}$$

August, 18th [Continuation of Aug. 17th]

Proofs

1. Proof of Proposition 1.

a) Assume that $(M_t^{(1)}, t \geq 0)$ is a semimartin

Therefore, it admits a canonical decomposition: $M_t^{(1)} = N_t^{(1)} + A_t^{(1)}$,

where $(N_t^{(1)}, t \geq 0)$ is a martingale, and $(A_t^{(1)}, t \geq 0)$ has bounded variation.

Now, we have, for $i \neq 1$, and τ a bounded stopping time:

$$E[M_\tau^{(i)}] = E[M_\tau^{(1)}] = E[A_\tau^{(1)}].$$

Hence, $E[M_\tau^{(i)} - A_\tau^{(1)}] = 0$, which is equivalent to saying that $(M_t^{(i)} - A_t^{(1)}, t \geq 0)$ is a martingale.

From now on, we shall write $(A_t, t \geq 0)$ instead of $(A_t^{(1)}, t \geq 0)$, and the k-dimensional canonical decomposition as:

$$M_t^{(i)} = N_t^{(i)} + A_t \quad , \quad t \geq 0 .$$

c) The support property of $d\langle N^{(i)} \rangle_t$ is immediate, since $(N_t^{(i)}, t \geq 0)$ is the martingale part of $(M_t^{(i)})$, and it is a general fact that:

$$\int_0^t 1_{(M_s^{(i)} = 0)} \, d\langle N^{(i)} \rangle_s = 0 \quad \text{(nothing to do with spiders).}$$

b) It remains to prove this part, for this, we remark that:

(1) $\quad 0 = M_t^{(i)} M_t^{(j)} = \int_0^t M_s^{(i)} dM_s^{(j)} + M_s^{(j)} dM_s^{(i)} + \langle M^{(i)}, M^{(j)} \rangle_t .$

However, $\langle M^{(i)}, M^{(j)} \rangle_t = \langle N^{(i)}, N^{(j)} \rangle_t = 0$, from c) above,

and the fact that, for every (t, ω), either $M_t^{(i)}(\omega) = 0$, or $M_t^{(j)}(\omega) = 0$.

Hence, (1) reduces to:

$$0 = \int_0^t M_s^{(i)} \left(dN_s^{(j)} + dA_s \right) + \int_0^t M_s^{(j)} \left(dN_s^{(i)} + dA_s \right),$$

which implies, in particular: (2) $\qquad M_s^{(i)} dA_s + M_s^{(j)} dA_s = 0$

(and also: (3) $\quad M_s^{(i)} dN_s^{(j)} + M_s^{(j)} dN_s^{(i)} = 0$, but this is a consequence of c) above, since: $dN_s^{(j)} = 1_{(M_s^{(j)} > 0)} dN_s^{(j)}$, and $dN_s^{(i)} = 1_{(M_s^{(i)} > 0)} dN_s^{(i)}$)

I now come back to (2), and multiply by $1_{(M_s^{(i)} > 0)}$, which gives:

$$0 = 1_{(M_s^{(i)} > 0)} M_s^{(i)} dA_s \quad ;$$

hence, for every i, $\quad 0 = 1_{(M_s^{(i)} > 0)} dA_s$, so that

dA_s is carried by $\Gamma_0 = \{ (t, \omega) : M_t(\omega) = 0 \}$.

<u>Remark</u>: At the moment, I do not see how to avoid the hypothesis $(*)$. Could one prove that this hypothesis is implied by <u>the definition of a spider martingale</u>?

<u>Comments</u> / <u>Continuation of the proofs.</u>

1. The "mystery" in the remark at the bottom of p. 7 is solved in a completely elementary way:
consider $i \neq j$, and $M_t^{(i,j)} \overset{\text{def}}{=} M_t^{(i)} - M_t^{(j)}$.

Then, the hypothesis d) tells us that $(M_t^{(i,j)}, t \geq 0)$ is a martingale, and from c), we know that:

$$M_t^{(i)} = (M_t^{(i,j)})^+ \quad ; \quad M_t^{(j)} = (M_t^{(i,j)})^- .$$

Hence, $(M_t^{(i)}, t \geq 0)$ and $(M_t^{(j)}, t \geq 0)$ are semimartingales; in fact they are submartingales, and we have:

$$M_t^{(i)} = \int_0^t 1_{(M_s^{(i,j)} > 0)} \, d(M_s^{(i,j)}) + \frac{1}{2} L_t^{(i,j)}$$

$$M_t^{(j)} = -\int_0^t 1_{(M_s^{(i,j)} < 0)} \, d(M_s^{(i,j)}) + \frac{1}{2} L_t^{(i,j)}.$$

Then, with my previous notation, we have:

$$\boxed{A_t = \frac{1}{2} L_t^{(i,j)}, \quad \text{for all pairs } (i,j), \text{ with } i \neq j.}$$

In particular, the hypothesis (*) is obviously unnecessary.

2. <u>Proof of Proposition 2</u>: This Proposition follows immediately from Knight's theorem on continuous orthogonal martingales, and Skorokhod's lemma.

3. Comments on Propositions 2 and 3.

With the notation introduced in Proposition 3, it is obvious that:

$$\langle M^{(i)} \rangle_t = \int_0^t 1_{(M_s^{(i)} > 0)} \, d\langle M \rangle_s$$

Hence, we can write the representation of $M \equiv (M^{(1)}, \dots, M^{(k)})$ obtained in Proposition 2 as:

$$M_t^{(i)} = R^{(i)} \left(\int_0^t 1_{(M_s^{(i)} > 0)} \, d\langle M \rangle_s \right) \quad , \quad i = 1, 2, \dots, k,$$

which leads us immediately to the Dubins – Schwarz representation of spider-martingales:

$$\boxed{M_t = W\left(\langle M \rangle_t\right) \quad , \quad t \geq 0},$$

where $(W(u), u \geq 0)$ is Walsh's Brownian motion.

4. In my opinion, the interest of Walsh's Brownian motions, hence of spider-martingales, comes from their stability under certain transforms [See below].

Here, I would like to elaborate a little on my Note of Aug. 16th:
Some formulae for the Brownian and Bessel spiders, II,

in relation with Prop. 4, p. 5, of this Note.
I was led to Prop. 4 after remarking that my arguments to prove

$$(7)_i \qquad \boxed{E\left[\exp\left(-\frac{\lambda^2}{2} T_*\right) 1_{(B_{T_*} = x_i)}\right] = \frac{\left(1/\sinh(\lambda a_i)\right)}{\left(\sum_\ell \coth(\lambda a_\ell)\right)}}$$

were very similar to those I was making to prove:

$$(9)_i \qquad P\left(B_{T_*} = x_i\right) = \frac{(1/a_i)}{\sum_\ell (1/a_\ell)}$$

Now, Prop. 4 gives an explanation of this, as it shows in particular that:

$$\left\{ \sinh\left(\lambda B_t^{(i)}\right) \exp\left(-\frac{\lambda^2 t}{2}\right) ; \quad i = 1, 2, \dots, k \right\}$$

is a <u>spider-martingale</u>.

Note that the functions: $f(x,u) = \sinh(\lambda x)\, e^{-\frac{\lambda^2 u}{2}}$

are the extremal space time harmonic functions such that: $f(0,u) \equiv 0$, since any such space-time harmonic function may be written as:

$$f(x,u) = \int_{0+}^{\infty} d\mu(\lambda) \; \sinh(\lambda x) \exp\left(-\frac{\lambda^2 u}{2}\right).$$

Such computations as $(7)_i$ above can be generalized to spiders with any radial diffusion $(\rho_t ; t \geq 0)$, as presented in Jim's note (Aug. 16th):

<u>Poisson explanation of the spider formulae, I and II.</u>

[Computations by M.Y., to be presented in detail].

I follow Jim's notations, and I introduce the "classical" pair of functions f and g such that:

$$\begin{cases} f(|B_t|, \lambda) \exp(-\lambda t) \text{ is a martingale} & ; \; f(0,\lambda) = 0 \\ g(|B_t|, \lambda) \exp(-\lambda t) \text{ is a martingale} & ; \; g(0,\lambda) = 1 \end{cases}$$

(f is determined up to a constant multiple, depending on λ).

Then, we find that:

$$\{f(B_t^{(i)})\exp(-\lambda t);\ i=1,2,\dots,k\} \text{ is a spider-martingale,}$$

whereas $\{g(|B_t|,\lambda)\exp(-\lambda t);\ t\geq 0\}$ is an ordinary martingale.

Consequently, we have:

$$E\left[f(B_{T_*}^{(i)},\lambda)\exp(-\lambda T_*)\right] = C, \text{ independently of } i,$$

so that: (13) $\quad f(a_i,\lambda)\ E\left[\exp(-\lambda T_*);\ 1_{(B_{T_*}=x_i)}\right] = C$

Now, using the martingale associated with g, we obtain:

(14) $$E\left[g(|B_{T_*}|,\lambda)\exp(-\lambda T_*)\right] = 1,$$

and the left-hand side of (14) is equal to:

$$\sum_{i=1}^{k} \frac{g(a_i,\lambda)}{f(a_i,\lambda)}\, E\left[f(a_i,\lambda)\exp(-\lambda T_*);\ 1_{(B_{T_*}=x_i)}\right],$$

which is equal, from (13), to: $\left(\sum_{i=1}^{k} \frac{g(a_i,\lambda)}{f(a_i,\lambda)}\right) C$

So, from (14), we have:

(15) $$\left(\sum_{i=1}^{k} \frac{g(a_i,\lambda)}{f(a_i,\lambda)}\right) C = 1$$

Therefore, the constant C has now been determined, and, plugging its value in (13), we obtain:

(16) $$\boxed{E\left[\exp(-\lambda T_*);\ 1_{(B_{T_*}=x_i)}\right] = \frac{1/f(a_i,\lambda)}{\sum_{i=1}^{k}\left(\frac{g(a_i,\lambda)}{f(a_i,\lambda)}\right)}}$$

Comparing with Jin's formula:

$$(17)\qquad \boxed{E\left[\exp(-\lambda T_*)\right] = \frac{\sum_k m(a_k)\,\phi(a_k,\lambda)}{\sum_k \psi(a_k,\lambda)}},$$

where:

$$m(a) = m(\mathrm{Max} > a)\,;\quad \phi(a,\lambda) = m\left(e^{-\lambda T_a} \,\middle|\, T_a < V\right);$$

$$\psi(a,\lambda) = -\log\left(E\left[e^{-\lambda\tau_1} ; M_{\tau_1} \le a\right]\right),$$

yields the following:

$$(18)\qquad \underline{\frac{1}{f(a,\lambda)} = c(\lambda)\, m(a)\,\phi(a,\lambda)}\;;\quad \underline{\frac{g(a,\lambda)}{f(a,\lambda)} = c(\lambda)\,\psi(a,\lambda)}$$

Example 1 : The Brownian case.

We take: $f(a,\lambda) = \sinh(\sqrt{2\lambda}\,a)$; $g(a,\lambda) = \cosh(\sqrt{2\lambda}\,a)$

On the other hand, we also know that, in the Brownian case, we have:

$$m(a) = 1/a\;;\quad \phi(a,\lambda) = \frac{a\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda}\,a)}\;;\quad \psi(a,\lambda) = \sqrt{2\lambda}\,\coth(\sqrt{2\lambda}\,a),$$

so that the function $c(\lambda)$ which appears twice in (18) is equal to: $c(\lambda) = \frac{1}{\sqrt{2\lambda}}$.

Example 2: The Bessel cases ([to be developed] ; this is also interesting).