The distribution of Brownian and Bessel quantiles.

(A) /

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Seconde rédaction.

Introduction

Let $(B_t, t \geq 0)$ be a 1-dimensional Brownian motion starting from 0. The computation of the price of Asian options amounts to the computation of the following quantity:

$$C_T^{\nu}(k) \overset{def}{=} E\left[\left(\frac{1}{T}\int_0^T ds\, \exp(B_s + \nu s) - k\right)^+\right], \tag{0.a}$$

for some given time T, and some reals ν and k. A number of results have been obtained about this topic (see H. Geman - M. Yor [ ], [ ], and [ ], [ ], in particular).

In November 1993, P. Embrechts suggested that, instead of considering, as in (0.a), the time-average of $\exp(B_s + \nu s)$ on the interval $[0,T]$, it would be interesting to study the median of this (random) function of $s\ (\leq T)$, and, more generally, the α-quantiles, for every $0 < \alpha < 1$. Precisely, to simplify the discussion, we may take, without loss of generality, $T = 1$, and we define, for $0 \leq \alpha < 1$:

$$X_\alpha^{(\nu)} = \inf\left\{x : \int_0^1 du\, 1_{(B_u + \nu u \leq x)} > \alpha\right\}.$$

Restes d'une 1ère version
Intéressant à cause du programme, + ambitieux, bref
+ ambitieux

The main aim of this paper is to find the law of $X_\alpha^{(\nu)}$ explicitly, and to compute:

$$M^{\nu}(\alpha, k) \overset{def}{=} E\left[\left(X_\alpha^{(\nu)} - k\right)^+\right]. \tag{0.b}$$

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Section 1 is devoted to this problem in the case $\nu = 0$, for which the formulae are most explicit, whereas in section 2, the general case, with $\nu \in \mathbb{R}$, is studied.

In Section 3, two variants of the preceding question are studied; the first variant concerns the study of the Brownian quantiles when the probability space $([0,1]; dt)$ is being replaced by $([0,\infty[; e^{-t}\,dt)$; thus, one is interested in:

$$Y_\alpha^{(\nu)} = \inf\left\{x : \int_0^\infty dt\, e^{-t}\, 1_{(B_t + \nu t \le x)} > \alpha\right\};$$

the second variant concerns the Brownian quantiles over the probability space $\left([0,T_1]; \frac{dt}{T_1}\right)$, where $T_1 = \inf\{t : B_t + \nu t = 1\}$; hence, one is interested in:

$$Z_\alpha^{(\nu)} = \inf\left\{x : \frac{1}{T_1}\int_0^{T_1} dt\, 1_{(B_t + \nu t \le x)} > \alpha\right\}.$$

Finally, in Section 4, slightly different quantities are considered, namely:

$$\eta_\alpha^{(\delta)} = \inf\left\{x : \int_0^\infty dt\, 1_{(R_t \le x)} > \alpha\right\},$$

and

$$\mu_\alpha^{(\delta)} = \inf\left\{x : \int_0^\infty dt\, 1_{\left(|B_t| + \frac{2}{\delta} l_t \le x\right)} > \alpha\right\}.$$

2. The general case: ν real.

Keeping with the notation in the introduction, we consider here:

$$X_\alpha^{(\nu)} = \inf\left\{ u : \int_0^1 du\, 1_{(B_u^{(\nu)} \leq x)} > \alpha \right\},$$

where $B_u^{(\nu)} = B_u + \nu u$, $u \geq 0$.

The symmetry relation (1.c) may be extended as follows:

$$(2\text{-}a) \qquad X_\alpha^{(\nu)} \stackrel{(law)}{=} - X_{1-\alpha}^{(-\nu)}.$$

Hence, to derive the distribution of $X_\alpha^{(\nu)}$, it suffices to compute $P(X_\alpha^{(\nu)} > x)$, for every α, and $x \geq 0$.

With obvious notation, we remark, following the arguments developed in Section 1, that:

$$\left(X_\alpha^{(\nu)} > x \right) = \left(T_x^{(\nu)} + \int_{T_x^{(\nu)}}^1 dt\, 1_{(B_t^{(\nu)} < x)} < \alpha \right)$$

$$\stackrel{(law)}{=} \left(T_x^{(\nu)} + \int_0^{1-T_x^{(\nu)}} dt\, 1_{(\hat{B}_t^{(\nu)} < 0)} < \alpha \right).$$

Hence, we obtain:

$$(2\text{-}b) \qquad P(X_\alpha^{(\nu)} > x) = \int_0^\alpha dt\; \theta_x^{(\nu)}(t)\, P\left(\int_0^{1-t} du\, 1_{(\hat{B}_u^{(\nu)} < 0)} < \alpha - t \right),$$

where:

$$\theta_x^{(\nu)}(t)\,dt = P(T_x^{(\nu)} \in dt) = E\left[T_x \in dt;\ \exp\left(\nu x - \frac{\nu^2 t}{2}\right)\right]$$

$$= \frac{dt\, x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t} + \nu x - \frac{\nu^2 t}{2}\right).$$

3. Two Variants.

The purpose of this section is to show how the arguments developed in Sections 1 and 2, although elementary, may yield results in somewhat different contexts.

(3.1) As a first variant, we shall consider:

$$Y_\alpha = \inf\left\{x : \int_0^\infty dt\, e^{-t} 1_{(B_t < x)} > \alpha\right\} \qquad (0<\alpha<1)$$

With the same notation as in (1.2) above, we find, for $x>0$:

$$(Y_\alpha > x) = \left(\int_0^{T_x} dt\, e^{-t} + \int_{T_x}^\infty e^{-t} 1_{(B_t<x)}\, dt < \alpha\right)$$

$$= \left(\int_0^{T_x} dt\, e^{-t} + e^{-T_x} A < \alpha\right),$$

where $A = \int_0^\infty du\, e^{-u} 1_{(\hat{B}_u < 0)}$.

Hence, we obtain: $$(Y_\alpha > x) = \left(T_x < \log\left(\frac{1-A}{1-\alpha}\right)\right)$$

$$\overset{(law)}{=} \left(\frac{x}{|N|} < \left(\log\left(\frac{1-A}{1-\alpha}\right)\right)^{1/2}\right).$$

Similar arguments yield, for $x<0$:

$$(Y_\alpha > x) \overset{(law)}{=} \left(\frac{-x}{|N|} > \left(\log\frac{A}{\alpha}\right)^{1/2}\right).$$

Consequently, we have obtained the following

<u>Proposition:</u> Let $0<\alpha<1$. <u>The distribution</u> of Y_α <u>is characterized by</u>:

$$P(Y_\alpha \in dx) = \begin{cases} \sqrt{\frac{2}{\pi}}\, E\left[\frac{1}{(\lambda_+)^{1/2}} \exp\left(-\frac{x^2}{2\lambda_+}\right)\right] & (x\geq 0) \\ \sqrt{\frac{2}{\pi}}\, E\left[\frac{1}{(\lambda_-)^{1/2}} \exp\left(-\frac{x^2}{2\lambda_-}\right)\right] & (x\leq 0), \end{cases}$$

<u>where:</u> $\lambda_+ = \log\left(\frac{1-A}{1-\alpha}\right)$, $\lambda_- = \left\{\log\left(\frac{A}{\alpha}\right)\right.$, <u>and</u> $A = \int_0^\infty dt\, e^{-t} 1_{(B_t<0)}$

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The distribution of Brownian quantiles. November, 16th.

Let (B_t) be a 1-dimensional Brownian motion starting from 0.
Define, for $0<\alpha<1$:

$$X_\alpha = \inf\left\{x : \int_0^1 du\, 1_{(B_u<x)} = \alpha\right\}.$$

I will show the following formula:

$$P(X_\alpha \in dx) = \begin{cases} \sqrt{\frac{2}{\pi}}\, E\left[g\le\alpha ; \sqrt{\frac{1-g}{\alpha-g}} \exp -\frac{x^2}{2}\left(\frac{1-g}{\alpha-g}\right)\right] & (x\geq 0) \\ \sqrt{\frac{2}{\pi}}\, E\left[g\geq\alpha ; \frac{1}{\sqrt{1-\frac{\alpha}{g}}} \exp\left(-\frac{x^2}{2\left(1-\frac{\alpha}{g}\right)}\right)\right] & (x\leq 0)\end{cases}$$

where g is arc sine distributed.

Proof:

I will use the following identity:

$$(X_\alpha > x) = \left(\int_0^1 ds\, 1_{(B_s<x)} < \alpha\right).$$

(i) Assume $x>0$; define $T_x = \inf\{u : B_u = x\}$.

Then, we have: $P(X_\alpha > x) = P\left(T_x + \int_{T_x}^1 ds\, 1_{(B_s<x)} < \alpha ; T_x<1\right)$

Since $(B_{T_x+u} - x ; u\geq 0)$ is a Brownian motion independent of $\mathcal{F}_{T_x}$, we have:

$$P(X_\alpha > x) = P\left(T_x + (1-T_x)g < \alpha ; T_x < 1\right),$$

where g is arc sine distributed, and independent of T_x.

Therefore, we have:

$$P(X_\alpha > x) = P\left(T_x(1-g) + g < \alpha\,;\ T_x < 1\right)$$

$$= P\left(g < \alpha\,;\ x^2(1-g) < \underset{(\alpha - g)}{\uparrow} N^2\,;\ x^2 < N^2\right)$$

$$= P\left(g < \alpha\,;\ x \leq \sqrt{\frac{\alpha-g}{1-g}}\,|N|\right).$$

$$= \sqrt{\frac{2}{\pi}}\ E\left[g < \alpha\,;\ \sqrt{\frac{1-g}{\alpha-g}} \int_x^\infty d\xi\ \exp\left(-\frac{\xi^2}{2}\left(\frac{1-g}{\alpha-g}\right)\right)\right].$$

It now remains to differentiate with respect to x to obtain the formula for $x > 0$.

(ii) Assume $x < 0$.
Again, when I consider: $\left(\int_0^1 ds\ 1_{(B_s \leq x)} \leq \alpha\right)$,

I have to decompose with $(T_x \leq 1)$, and $(T_x > 1)$.
On $T_x > 1$, we have:

$$\int_0^1 ds\ 1_{(B_s \leq x)} = 0 \cdot (< \alpha)$$

Therefore, we obtain:

$$P(X_\alpha > x) = P(T_x > 1) + P\left(T_x \leq 1\,;\ \int_{T_x}^1 ds\ 1_{(B_s \leq x)} \leq \alpha\right)$$

$$= P(T_x > 1) + P\left(T_x \leq 1\,;\ (1 - T_x)\,g \leq \alpha\right)$$

$$= P(|N| \leq |x|) + P(g \leq \alpha)\ P(|x| \leq |N|) + E\left[g \geq \alpha\,;\ 1 - \frac{\alpha}{g} \leq \frac{x^2}{N^2} \leq 1\right]$$

Therefore, writing $z = -x$, we get:

$$P(X_\alpha > x) = \sqrt{\frac{2}{\pi}} \int_0^z dy\, e^{-y^2/2} + P(g \leq \alpha) \sqrt{\frac{2}{\pi}} \int_z^\infty dy\, e^{-y^2/2}$$

$$+ E\left[(g \geq \alpha) \sqrt{\frac{2}{\pi}} \int_z^{\frac{z}{\sqrt{1-\frac{\alpha}{g}}}} dy \exp\left(-\frac{y^2}{2}\right)\right].$$

Then, taking the derivative with respect to x on both sides, we get that the density, for $x < 0$, is:

$$\sqrt{\frac{2}{\pi}}\, E\left[g \geq \alpha ; \frac{1}{\sqrt{1-\frac{\alpha}{g}}} \exp\left(-\frac{x^2}{2\left(1-\frac{\alpha}{g}\right)}\right)\right],$$ which is the

second formula $\square$.

Further simplifications: 1) Using the explicit distribution of g, we obtain the following formula:

$$P(X_\alpha \in dx) = \begin{cases} \sqrt{\frac{2}{\pi}}\, E\left[\exp -\frac{x^2}{2}\left(1+\left(\frac{1-\alpha}{\alpha}\right)\frac{1}{g}\right)\right] & (x \geq 0) \\ \sqrt{\frac{2}{\pi}}\, E\left[\exp -\frac{x^2}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)\frac{1}{g}\right)\right] & (x \leq 0). \end{cases}$$

It appears clearly with this formula that:

$$-X_\alpha \overset{(law)}{=} X_{(1-\alpha)},$$ as can be shown directly from the symmetry of BM.

2) The preceding formula may be simplified again; we introduce the following notation:

$$\frac{1-\alpha}{\alpha} = \beta^2 \qquad (\beta > 0).$$

We will use the following

<u>Lemma</u>: $$E\left[\exp\left(-\frac{y^2}{2}\left(\frac{1}{g}\right)\right)\right] = \Phi(|y|),$$

where: $$\Phi(z) = \sqrt{\frac{2}{\pi}}\int_z^\infty dx\, \exp\left(-\frac{x^2}{2}\right). \equiv P(|N| > z).$$

<u>Proof</u>: We know that: $\frac{1}{g} \overset{(law)}{=} 1 + C^2$, where C is a standard Cauchy variable. Then, we have:

$$\begin{aligned} E\left[\exp\left(-\frac{y^2}{2}\left(\frac{1}{g}\right)\right)\right] &= E\left[\exp\left(-\frac{y^2}{2}(1+C^2)\right)\right] \\ &= \exp\left(-\frac{y^2}{2}\right) E\left[\exp\left(iyNC\right)\right] \\ &= \exp\left(-\frac{y^2}{2}\right) E\left[\exp\left(-|y||N|\right)\right]. \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty dx\, \exp\left(-\frac{(x+|y|)^2}{2}\right) = \Phi(|y|) \end{aligned}$$

Now, we can write:

$$P(X_\alpha \in dx) = \begin{cases} \sqrt{\frac{2}{\pi}}\exp\left(-\frac{x^2}{2}\right)\Phi(\beta x)\,dx, & x \geq 0. \\ \sqrt{\frac{2}{\pi}}\exp\left(-\frac{x^2}{2}\right)\Phi\left(\frac{|x|}{\beta}\right)dx, & x \leq 0. \end{cases}$$

In the particular case where $\alpha = 1/2$, we get: $\beta = 1$, and then the previous formula becomes:

$$P\left(X_{\frac{1}{2}} \in dx\right) = \sqrt{\frac{2}{\pi}}\left(\exp\left(-\frac{x^2}{2}\right)\right)\Phi\left(|x|\right) dx, \qquad (x \in \mathbb{R}).$$

Thus, we have:
$$P\left(|X_{\frac{1}{2}}| \geq y\right) = \left(\Phi(y)\right)^2,$$

which proves that: $|X_{\frac{1}{2}}| \overset{(law)}{=} \inf\left(|N|, |N'|\right)$.

(C) / Pour Thierry : Ceci est certainement la version la + proche de ce que fait ton élève —

On the "true" Brownian quantiles — November, 16th, 1993

Let $(B_t, t \geq 0)$ be a 1-dimensional Brownian motion starting from 0.
For any fixed t, define, for $0 < \alpha < t$:

(1) $$X_\alpha(t) = \inf\left\{x : \int_0^t du\, 1_{(B_u \leq x)} = \alpha\right\}$$

We are interested in finding the distribution of $X_\alpha(1)$, for a given $\alpha < 1$.

We will take advantage of the scaling property of Brownian motion, from which we get:

(2) $$X_\alpha(t) = \sqrt{t}\, X_{\frac{\alpha}{t}}(1)$$

Again, I keep t fixed. Since $\alpha \longrightarrow X_\alpha(t)$, $\alpha \leq t$, is the inverse of the increasing process:

$$x \longrightarrow \int_0^t du\, 1_{(B_u \leq x)} \equiv \int_{-\infty}^x dy\, \ell_t^y,$$

we have, for any $f : \mathbb{R} \longrightarrow \mathbb{R}_+$:

(3) $$\int_{-\infty}^{\infty} dx\, \ell_t^x \exp\left(-\lambda \int_{-\infty}^x dy\, \ell_t^y\right) f(x)$$

$$= \int_0^t d\alpha \exp(-\lambda\alpha) f(X_\alpha(t)) \overset{(law)}{=} t \int_0^1 d\beta \exp(-\lambda\beta t) f(\sqrt{t}\, X_\beta(1))$$

Moreover, if we assume f to have compact support, we get, by integration by parts, that (3) is equal to:

$$\frac{1}{\lambda} \int_{-\infty}^{\infty} dx \exp\left(-\lambda \int_{-\infty}^x dy\, \ell_t^y\right) f'(x).$$

Hence, this is equal, in law, to:

$$t \int_0^1 d\beta \; \exp(-\lambda \beta t) \, f\left(\sqrt{t}\, X_\beta(1)\right).$$

Now, the distribution of $(\ell_t^y \,;\, y \in \mathbb{R})$ is not so easy to manipulate – Therefore, we randomize the time t, i.e: we replace t by T_θ, an exponential random variable with parameter θ; thus, we get:

(4)
$$E\left[T \int_0^1 d\beta \; \exp(-\lambda \beta T) \, f\left(\sqrt{T}\, X_\beta(1)\right) \right]$$
$$= \frac{1}{\lambda} \int_{-\infty}^{\infty} dx \;\; f'(x) \;\; E\left[\exp - \lambda \int_0^T ds \; 1_{(B_s \leq x)} \right]$$

[!].