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Two remarkable subordinators

It is well known that hyperbolic functions crop up very naturally in relation with the heat equation, hence with the study of Brownian motion; in particular, let $\{B_t, t \geq 0\}$ denote a one-dimensional Brownian motion starting from 0, and $T_1 = \inf\{t : B_t = 1\}$,

$g_{T_1} = \sup\{t < T_1 : B_t = 0\}$,

$T_1^* = \inf\{t : |B_t| = 1\}$, $g_{T_1^*} = \sup\{t < T_1^* : |B_t| = 0\}$.

The following Laplace transforms are well known:

(i) $E\left[\exp\left(-\frac{\lambda^2}{2} T_1\right)\right] = \exp(-\lambda)$; $E\left[\exp - \frac{\lambda^2}{2}(T_1 - g_{T_1})\right] = \lambda / \sinh\lambda$;

$$E\left[\exp\left(-\frac{\lambda^2}{2} g_{T_1}\right)\right] = \frac{(1-\exp(-2\lambda))}{2\lambda}.$$

(ii) $E\left[\exp - \frac{\lambda^2}{2} T_1^*\right] = 1/\cosh\lambda$

$$E\left[\exp\left(-\frac{\lambda^2}{2} g_{T_1^*}\right)\right] = \frac{\tanh\lambda}{\lambda} ; E\left[\exp\left(-\frac{\lambda^2}{2}(T_1^* - g_{T_1^*})\right)\right] = \lambda/(\sinh\lambda).$$

Moreover, it is not difficult to show that each of the variables T_1^*, $g_{T_1^*}$, $T_1^* - g_{T_1^*}$, is infinitely divisible;

in the present paper, we shall study some properties of the two subordinators, which we shall denote as: $\{\Sigma^*_h, h \geq 0\}$ s)

and $\{\Sigma_h, h \geq 0\}$, whose laws are characterized by: //

$$E\left[\exp\left(-\frac{\lambda^2}{2}\Sigma^*_h\right)\right] = 1/(\cosh\lambda)^h$$

$$E\left[\exp\left(-\frac{\lambda^2}{2}\Sigma_h\right)\right] = \left(\lambda/\sinh\lambda\right)^h$$

It is well known that, for each $h \geq 0$, Σ^*_h may be realized as: $\int_0^1 ds\, R^2_{2h}(s)$, and Σ_h as $\int_0^1 ds\, r^2_{2h}(s)$,

where $(R_\delta(s), s \geq 0)$ denotes a Bessel process with dimension δ, issued from 0, and $(r_\delta(s), s \leq 1)$ is a standard Bessel bridge of dimension δ.

More general computations of Laplace transforms of quadratic functionals of Bessel processes, i.e.: $\int_0^1 d\mu(s)\, R^2_\delta(s)$, or Bessel bridges: $\int_0^1 d\mu(s)\, r^2_\delta(s)$, are found in many places in the literature, going back to Cameron–Martin [ ], Kac [ ], etc....

1. Some relations between the distributions of Σ^*_h and Σ^*_{h+2}. 3)

We shall prove, using different methods, the following

Theorem 1: (i) For any $h \geq 0$, and any $\alpha \geq 0$, one has:

$$(a^*)\quad h(h+1)\, E\left[\left(\Sigma^*_{h+2}\right)^\alpha\right] = h^2 E\left[\left(\Sigma^*_h\right)^\alpha\right] + (2\alpha+1)\, E\left[\left(\Sigma^*_h\right)^{\alpha+1}\right].$$

(ii) Let V be a standard uniform variable, taking values in $[0,1]$, and independent of ~~the~~ both Σ^*_h and Σ^*_{h+2}. Then, one has, for any Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$:

$$(b^*)\quad h(h+1)\, E\left[\varphi\left(V^2 \Sigma^*_{h+2}\right)\right] = h^2 E\left[\varphi\left(V^2 \Sigma^*_h\right)\right] + E\left[\varphi\left(\Sigma^*_h\right)\Sigma^*_h\right]$$

(iii) Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be C^1, and increasing.

Then, one has:

$$(c^*)\quad h(h+1)\, E\left[\Phi\left(\Sigma^*_{h+2}\right)\right] = h^2 E\left[\Phi\left(\Sigma^*_h\right)\right] + E\left[\Sigma^*_h\left\{\Phi\left(\Sigma^*_h\right) + 2\Sigma^*_h \Phi'\left(\Sigma^*_h\right)\right\}\right]$$

Proof: α) We assume (i) for one moment and we prove (ii):
we divide both sides of (a^*) by $2\alpha+1$, and we use: $\dfrac{1}{2\alpha+1} = E\left[V^{2\alpha}\right]$.

Then, (b^*) holds for $\varphi(x) = x^\alpha$, for any $\alpha \geq 0$, and the identity (b^*) follows in complete generality, using the uniqueness of ~~Laplace~~ Mellin transforms.

β) We now prove (iii) as a consequence of (ii):
indeed, it suffices to take: $\Phi(\sigma) = \int_0^1 dv\, \varphi(v^2 \sigma)$

We then obtain:

$$\Phi(\sigma) = \frac{1}{\sqrt{\sigma}} \int_0^\sigma \frac{dy}{2\sqrt{y}} \varphi(y) \;; \text{ hence: } \varphi(\sigma) = \Phi(\sigma) + 2\sigma \Phi'(\sigma).$$

Plugging these formulae in (b^*), we obtain (c^*).

γ) We also remark that we may prove (b^*) directly, without having to refer ourselves to (a^*) (~~and~~ so that in fact (a^*) may be considered as a consequence of (b^*), in which we take $\varphi(x) = x^\alpha$.

This direct proof of (b^*) relies upon the computation / comparison of the Laplace transforms of the laws of each of the three random variables involved there: taking $\varphi(x) = \exp(-\frac{\lambda^2}{2}x)$, and making elementary changes of variables, one sees that (b^*) is equivalent to:

$$h(h+1)\int_0^\lambda \frac{dv}{(\cosh(v))^{h+2}} \stackrel{?}{=} h^2 \int_0^\lambda \frac{dv}{(\cosh(v))^h} - \lambda \frac{d}{dx}\left(1/(\cosh(\sqrt{2x}))^h \right)\Big|_{x=\lambda^2/2}$$

which, taking derivatives with respect to λ of both sides, amounts to:

$$(h+1)\; 1/(\cosh(\lambda))^{h+2} \stackrel{?}{=} h/(\cosh\lambda)^h + \frac{d}{d\lambda}\left(\frac{\sinh\lambda}{(\cosh(\lambda))^{h+1}} \right),$$

and is easily verified — $\square$

2. Some relations between the laws of Σ_h and Σ_{h+2}.

Analogously to our development in Section 1, we shall now state and prove three relationships between the laws of Σ_h and Σ_{h+2}.

Theorem 2: (i) For any $h \geq 0$, and any $m \geq 0$, one has:

$$\text{(a)}\quad \boxed{\begin{aligned} &h(h+1)\,E\left[\Sigma_{h+2}^{m+1/2}\right] + 4\cancel{h}(m+1)(h+1)\,E\left[(\Sigma_h)^{m+1/2}\right] \\ &= \left\{(1+h)(2+h) + 2(m+1)(2m+1)\right\} E\left[(\Sigma_h)^{m+1/2}\right] + h^2(2m+1)\,E\left[(\Sigma_h)^{m-1/2}\right] \end{aligned}}$$

(ii) Let $H \overset{(\text{law})}{=} UV^2$, where U and V denote two independent uniform variables on $[0,1]$, independent from both Σ_h and Σ_{h+2}. Then, one has, for any Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\text{(b)}\quad \boxed{\begin{aligned} &h(h+1)\,E\left[\varphi(H\Sigma_{h+2})(\Sigma_{h+2})^{1/2}\right] + 4(h+1)\,E\left[\varphi(V^2\Sigma_h)(\Sigma_h)^{1/2}\right] \\ &= (1+h)(2+h)\,E\left[\varphi(H\Sigma_h)(\Sigma_h)^{1/2}\right] + 2\,E\left[\varphi(\Sigma_h)(\Sigma_h)^{1/2}\right] \\ &\quad + h^2\,E\left[\varphi(U\Sigma_h)(\Sigma_h)^{-1/2}\right]. \end{aligned}}$$

(iii) Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded function, [with compact support]. Then, define:

$$\Phi(\sigma) = \frac{1}{2\sqrt{\sigma}} \int_0^\sigma \frac{dt}{\sqrt{t}}\,\varphi(t), \quad \text{and} \quad \hat{\varphi}(\sigma) = \int_0^\sigma dt\,\varphi(t).$$

The following relation holds:

$$\text{(c)}\quad \boxed{\begin{aligned} &h(h+1)\,E\left[\left\{\Phi(\Sigma_{h+2}) - \frac{1}{2\sqrt{\Sigma_{h+2}}}\hat{\varphi}(\Sigma_{h+2})\right\}\right] \\ &= h(h+1)\,E\left[\Phi(\Sigma_h)\right] + E\left[\varphi(\Sigma_h)\sqrt{\Sigma_h}\right] - \frac{1}{2}E\left[\frac{3h+2}{\sqrt{\Sigma_h}}\hat{\varphi}(\Sigma_h)\right]. \end{aligned}}$$

Sketch of the proof of theorem 2:

i) Again, it is not difficult to show, as for theorem 1, that the identities (a) and (b) are equivalent, using the injectivity of the Mellin transform, and the elementary facts:

$$E\left[U^m\right] = \frac{1}{m+1} \; ; \; E\left[V^{2m}\right] = \frac{1}{2m+1} \; ;$$

$$E\left[H^m\right] \equiv E\left[\left(UV^2\right)^m\right] = \frac{1}{(m+1)(2m+1)} .$$

Thus, taking $\varphi(x) = x^m$ in (b), one obtains:

$$\frac{h(h+1)}{(m+1)(2m+1)} E\left[\left(\Sigma_{h+2}\right)^{m+1/2}\right] + \frac{4(h+1)}{(2m+1)} E\left[\left(\Sigma_h\right)^{m+1/2}\right]$$

(b')

$$= \left(\frac{(1+h)(2+h)}{(m+1)(2m+1)} + 2\right) E\left[\left(\Sigma_h\right)^{m+1/2}\right] + \frac{h^2}{(m+1)} E\left[\left(\Sigma_h\right)^{m-1/2}\right]$$

It now suffices to multiply both sides of the equality (b') by $(m+1)(2m+1)$ to recover (a).
Thus, (a) and (b) are clearly equivalent.

I find that dealing with exponents of the form $(m+1/2)$ in the statement of theorem 2 is not very nice; so, I made the change of exponents: $m = \mu + 1/2 = \nu - 1/2$, which yield (: using ν for instance):

$$\text{(a')} \quad h(h+1) E\left[\Sigma_{h+2}^{\nu}\right] = u(h,\nu) E\left[\Sigma_h^{\nu}\right] + 2h^2\nu\, E\left[\Sigma_h^{\nu-1}\right]$$

where,

$$u(h,\vartheta) = (1+h)(2+h) + 2(2\vartheta+1)(\vartheta-(h+1)).$$

It is also possible to transform (b) in the same vein, that is by changing $\varphi(x)$ into: $\sqrt{x}\,\psi(x)$.
This produces a change in the laws of the variables H, U, V, i.e.: we may now refer ourselves to the three variables:

$$\tilde{H} : 3(1-\sqrt{\theta})\,d\theta \;;\quad \tilde{U} : \frac{3}{2}\sqrt{u}\,du, \quad \hat{V} : \text{uniforme}.$$

i.e.: $E[f(\tilde{H})] = 3E[\sqrt{H}\,f(H)]$.

Thus, we obtain the slightly friendlier expression of (b):

$$(\tilde{b}) : \quad h(h+1)E[\psi(\tilde{H}\Sigma_{h+2})\Sigma_{h+2}] + 6E[\psi(\hat{V}\Sigma_h)\Sigma_h]$$

$$= (1+h)(2+h)E[\psi(\tilde{H}\Sigma_h)\Sigma_h] + 6E[\psi(\Sigma_h)\Sigma_h] + 2h^2 E[\psi(\tilde{U}\Sigma_h)]$$

using which, with the exp. functions: $\psi(x) = e^{-\lambda x}$, we should be able to get a "direct" proof of these identities, similarly to what I wrote on p. 4, although here the computations are more cumbersome..... /
